Journal of Computational and Applied Mathematics 231 (2009) 964–972

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Certain summation and transformation formulas for generalized hypergeometric series

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ARTICLE INFO

Article history:
Received 23 September 2008
Received in revised form 9 April 2009

MSC:
33C20
33C50

Keywords:
Generalized hypergeometric series

ABSTRACT

We derive summation formulas for generalized hypergeometric series of unit argument, one of which upon specialization reduces to Minton's summation theorem. As an application we deduce a reduction formula for a certain Kampé de Fériet function that in turn provides a Kummer-type transformation formula for the generalized hypergeometric function ${}_pF_p(x)$.

1. Introduction

The generalized hypergeometric function ${}_pF_q(x)$ may be defined by the series

$$ {}_pF_q\left(\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix}\middle| x\right) \equiv \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \dots (a_p)_k}{(b_1)_k (b_2)_k \dots (b_q)_k} \frac{x^k}{k!}, $$

where for nonnegative integers k the Pochhammer symbol or the ascending factorial $(a)_k$ is defined by $(a)_0 \equiv 1$ and for $k \geq 1$ by

$$ (a)_k \equiv a(a+1)\dots(a+k-1). $$

More succinctly $(a)_k = \Gamma(a+k)/\Gamma(a)$, but some authors [1] employ a rising factorial power $a^{\bar{k}}$ and falling factorial power $a^{\underline{k}}$ respectively defined by

$$ a^{\bar{k}} \equiv (a)_k, \qquad a^{\underline{k}} \equiv (-1)^k (-a)^{\bar{k}}. \tag{1.1} $$

When $p = q + 1$ and argument $x = 1$, the series ${}_pF_q(1)$ converges provided that $\mathrm{Re}(b_1 + \dots + b_q) > \mathrm{Re}(a_1 + \dots + a_p)$. However, when only one of the numerator parameters a_i is a negative integer or zero, then ${}_pF_q(x)$ converges for all x since it is merely a polynomial in x of degree $-a_i$.

In what follows we will denote the sequence $(a_1, \dots, a_p)$ simply by (a_p). We also define the product of p Pochhammer symbols by

$$ ((a_p))_k \equiv (a_1)_k \dots (a_p)_k, $$

where an empty product ($p = 0$) reduces to unity.

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doi:10.1016/j.cam.2009.05.013

Let (m_r) be a nonempty sequence of r positive integers $m_1, \ldots, m_r$. In 1970 Minton deduced the summation formula for a generalized hypergeometric series of unit argument

$$ {}_{r+2}F_{r+1}\left(\left.\begin{matrix} -n, a, (f_r+m_r) \\ a+1, (f_r) \end{matrix}\right| 1\right) = \frac{n!}{(a+1)_n} \frac{(f_1-a)_{m_1}}{(f_1)_{m_1}} \cdots \frac{(f_r-a)_{m_r}}{(f_r)_{m_r}}, \tag{1.2} $$

where n is an integer such that $n \geq m_1 + \cdots + m_r$. The importance of this result derives from the observation that it often appears as a solution to problems in mathematical physics [2]. Minton's summation formula has subsequently been deduced by Karlsson [3] who gave a derivation of a modified form of Eq. (1.2) (with $n!/(a+1)_n$ replaced by $\Gamma(1+a)\Gamma(1-b)/\Gamma(1+a-b)$) under the less restrictive condition that $-n$ may be replaced by the complex parameter b such that $\mathrm{Re}(-b) > m_1 + \cdots + m_r - 1$ which guarantees the convergence of ${}_{r+2}F_{r+1}(1)$. We mention here that many summation formulas for other specializations of ${}_{p+1}F_p(1)$ are recorded in [4, Section 7.10.2].

It is the purpose of the present investigation to provide other generalizations of Eq. (1.2), one of which upon specialization reduces to Minton's theorem. To this end in the next section we shall derive several preliminary lemmas some of which depend on the nonnegative integers known as Stirling numbers of the second kind. Although various definitions and notations for Stirling numbers of the second kind are used in the literature, the properties of these integers are well known (see e.g. [5, Section 24], [6, pp. 100–102]). In what follows we shall adopt the elegant notation $\left\{ {n \atop k} \right\}$ for Stirling numbers of the second kind employed by Graham et al. [1, Section 6].

2. Preliminary results

We recall that Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$ represent the number of ways to partition n objects into k nonempty subsets. Thus $\left\{ {0 \atop 0} \right\} \equiv 1$ and $\left\{ {n \atop 0} \right\} = 0$ when integer $n > 0$. Moreover, for nonnegative integers n a generating relation for the $\left\{ {n \atop k} \right\}$ is given by

$$ x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}} \tag{2.1a} $$

which is readily proved by induction (see [1, p. 262, Eq. 6.10]). However, noting Eqs. (1.1), Eq. (2.1a) may be written as

$$ (-x)^n = \sum_{k=0}^{n} (-1)^k \left\{ {n \atop k} \right\} (x)_k \, . \tag{2.1b} $$

Moreover, when $x = -m$ for nonnegative integers m, since

$$ (-m)_k = (-1)^k k! \binom{m}{k} $$

we see that

$$ m^n = \sum_{k=0}^{n} k! \binom{m}{k} \left\{ {n \atop k} \right\} . \tag{2.1c} $$

Lemma 1. *For nonnegative integers m define*

$$ S_m \equiv \sum_{k=0}^{\infty} k^m \frac{\lambda_k}{k!}, \qquad S_0 \equiv \sum_{k=0}^{\infty} \frac{\lambda_k}{k!}, \tag{2.2a} $$

where the sequence (λ_k) *is such that* S_m *converges for all m. Then*

$$ S_m = \sum_{j=0}^{m} \left\{ {m \atop j} \right\} \sum_{k=0}^{\infty} \frac{\lambda_{k+j}}{k!}, \tag{2.2b} $$

where the $\left\{ {m \atop j} \right\}$ *are Stirling numbers of the second kind.*

Proof. By using Eq. (2.1c)

$$ k^m = \sum_{j=0}^{m} j! \left\{ {m \atop j} \right\} \binom{k}{j} $$

and so we have

$$\begin{aligned} S_m &= \sum_{k=0}^{\infty}\sum_{j=0}^{m} j! \left\{ {m \atop j} \right\} \binom{k}{j} \frac{\lambda_k}{k!} \\ &= \sum_{j=0}^{m} j! \left\{ {m \atop j} \right\} \sum_{k=0}^{\infty} \binom{k}{j} \frac{\lambda_k}{k!}. \end{aligned}$$

But $\binom{k}{j} = 0$ for $k < j$ so that

$$\begin{aligned} S_m &= \sum_{j=0}^{m} j! \left\{ {m \atop j} \right\} \sum_{k=j}^{\infty} \binom{k}{j} \frac{\lambda_k}{k!} \\ &= \sum_{j=0}^{m} j! \left\{ {m \atop j} \right\} \sum_{k=0}^{\infty} \binom{k+j}{j} \frac{\lambda_{k+j}}{(k+j)!} \end{aligned}$$

which yields Eq. (2.2b) since $\binom{k+j}{j} = (k+j)!/k!j!$. □

Lemma 2. *Suppose j, m, n are nonnegative integers such that $n \geq j$, $m \geq j$. Then*

$$ {}_2F_1\left(\left. {-n+j, a+j \atop b+j} \right| 1 \right) = \frac{(\lambda)_n}{(b)_n} \frac{(\lambda+n)_{m-j}(b)_j}{(\lambda)_m}, \tag{2.3}$$

where $\lambda \equiv b - a - m$.

Proof. Because $j - n \leq 0$ the series terminates and therefore converges. Thus employing Gauss' summation theorem we have

$$\begin{aligned} {}_2F_1\left(\left. {-n+j, a+j \atop b+j} \right| 1 \right) &= \frac{\Gamma(b+j)\Gamma(\lambda+n+m-j)}{\Gamma(b+n)\Gamma(\lambda+m)} \\ &= \frac{(b)_j}{(b)_n} \frac{(\lambda+n)_{m-j}}{(\lambda)_m} \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \end{aligned}$$

which yields Eq. (2.3) since $\Gamma(\lambda+n)/\Gamma(\lambda) = (\lambda)_n$. □

Lemma 3. *For nonnegative integers j we have*

$$\sum_{k=0}^{n} k^j \frac{(-n)_k (a)_k}{(b)_k k!} = \frac{(\lambda)_n}{(b)_n (\lambda)_m} \sum_{\ell=0}^{j} \left\{ {j \atop \ell} \right\} (a)_\ell (-n)_\ell (n+\lambda)_{m-\ell}, \tag{2.4}$$

where $\lambda = b - a - m$ and m is an integer such that $m \geq j$.

Proof. Employing Lemma 1 with $\lambda_k \equiv (-n)_k (a)_k / (b)_k$ gives

$$\sum_{k=0}^{\infty} k^j \frac{(-n)_k (a)_k}{(b)_k k!} = \sum_{\ell=0}^{j} \left\{ {j \atop \ell} \right\} \sum_{k=0}^{\infty} \frac{(-n)_{k+\ell} (a)_{k+\ell}}{(b)_{k+\ell} k!}.$$

Since $(-n)_k = 0$ when $k > n$ and $(a)_{k+\ell} = (a)_\ell (a+\ell)_k$ the latter may be written as

$$\sum_{k=0}^{n} k^j \frac{(-n)_k (a)_k}{(b)_k k!} = \sum_{\ell=0}^{j} \left\{ {j \atop \ell} \right\} \frac{(-n)_\ell (a)_\ell}{(b)_\ell} \sum_{k=0}^{\infty} \frac{(-n+\ell)_k (a+\ell)_k}{(b+\ell)_k k!}, \tag{2.5}$$

where the infinite k-summation is just a Gaussian series of unit argument. We can always assume that $\ell - n \leq 0$ for if $\ell - n > 0$ the right side of Eq. (2.5) vanishes since $(-n)_\ell = 0$. Thus employing Lemma 2 we have

$$ {}_2F_1\left(\left. {-n+\ell, a+\ell \atop b+\ell} \right| 1 \right) = \frac{(\lambda)_n}{(b)_n} \frac{(\lambda+n)_{m-\ell}(b)_\ell}{(\lambda)_m}, \tag{2.6}$$

where $\lambda = b - a - m$ and $m \geq \ell$ for $\ell = 0, 1, \ldots, j$. Eq. (2.4) then follows from Eqs. (2.5) and (2.6). □

Lemma 4. *Consider the polynomial in n of degree $\mu \geq 1$ given by $a_0 n^\mu + a_1 n^{\mu-1} + \cdots + a_{\mu-1} n + a_\mu$, where $a_0 \neq 0$, $a_\mu \neq 0$ and n is a nonnegative integer. Then we may write*

$$a_0 n^\mu + a_1 n^{\mu-1} + \cdots + a_{\mu-1} n + a_\mu = a_\mu \frac{(1+\xi_1)_n}{(\xi_1)_n} \cdots \frac{(1+\xi_\mu)_n}{(\xi_\mu)_n}, \tag{2.7}$$

where $\xi_1, \ldots, \xi_\mu$ are nonvanishing zeros of the polynomial $Q_\mu(t)$ defined by

$$Q_\mu(t) \equiv a_0(-t)^\mu + a_1(-t)^{\mu-1} + \cdots + a_{\mu-1}(-t) + a_\mu. \tag{2.8}$$

Proof. Factor the polynomial in n into a unique product of μ linear terms so that

$$a_0 n^\mu + a_1 n^{\mu-1} + \cdots + a_{\mu-1} n + a_\mu = a_0(n+\xi_1)\ldots(n+\xi_\mu),$$

where $a_\mu = a_0 \xi_1 \ldots \xi_\mu$. Since for $j = 1, \ldots, \mu$

$$n + \xi_j = \frac{\Gamma(1+\xi_j+n)}{\Gamma(\xi_j+n)} = \xi_j \frac{(1+\xi_j)_n}{(\xi_j)_n}$$

we immediately obtain Eq. (2.7). Moreover, it is evident that $\xi_1, \ldots, \xi_\mu$ must be nonvanishing zeros of $Q_\mu(t)$ defined by Eq. (2.8). □

3. Summation theorems

Suppose m is a positive integer and $f \neq 0$. Since $(f+k)_m$ is a polynomial in k of degree m we may write

$$(f+k)_m = \sum_{j=0}^{m} s_{m-j} k^j,$$

where $s_0 = 1$ and $s_m = (f)_m$. Accordingly, we define for the nonempty sequence of positive integers $m_1, \ldots, m_r$

$$\begin{aligned} (f_1+k)_{m_1} &= \sum_{j_1=0}^{m_1} s^{(1)}_{m_1-j_1} k^{j_1} \\ &\vdots \\ (f_r+k)_{m_r} &= \sum_{j_r=0}^{m_r} s^{(r)}_{m_r-j_r} k^{j_r}, \end{aligned} \tag{3.1}$$

where each $f_i \neq 0$. Thus defining

$$\sigma(j_1, \ldots, j_r) \equiv s^{(1)}_{m_1-j_1} \ldots s^{(r)}_{m_r-j_r} \tag{3.2a}$$

and

$$j \equiv j_1 + \cdots + j_r \tag{3.2b}$$

we may write

$$(f_1+k)_{m_1} \ldots (f_r+k)_{m_r} = \sum_{j_1=0}^{m_1} \cdots \sum_{j_r=0}^{m_r} \sigma(j_1, \ldots, j_r) k^j. \tag{3.2c}$$

Now consider the generalized hypergeometric series of unit argument

$${}_{r+2}F_{r+1}\left(\left.\begin{matrix} -n, a, (f_r+m_r) \\ b, (f_r) \end{matrix}\right| 1\right) = \sum_{k=0}^{n} \frac{(-n)_k (a)_k}{(b)_k k!} \frac{((f_r+m_r))_k}{((f_r))_k}, \tag{3.3a}$$

where $((f_r))_k = (f_1)_k \ldots (f_r)_k$. Since

$$\frac{(f+m)_k}{(f)_k} = \frac{(f+k)_m}{(f)_m} \tag{3.3b}$$

upon using Eq. (3.2c) we have for the right side of Eq. (3.3a)

$$\frac{1}{(f_1)_{m_1} \ldots (f_r)_{m_r}} \sum_{j_1=0}^{m_1} \cdots \sum_{j_r=0}^{m_r} \sigma(j_1, \ldots, j_r) \sum_{k=0}^{n} k^j \frac{(-n)_k (a)_k}{(b)_k k!}. \tag{3.3c}$$

In Lemma 3 let $m = m_1 + \cdots + m_r$ so that $m \geq j_1 + \cdots + j_r = j$. Thus we have finally the following Lemma 5.

Lemma 5. *For nonnegative integer n and positive integers (m_r) the generalized hypergeometric series of unit argument*

$$ {}_{r+2}F_{r+1}\left(\left.\begin{matrix} -n, a, (f_r+m_r) \\ b, (f_r) \end{matrix}\right| 1\right) = \frac{(\lambda)_n}{(b)_n} \frac{P_m(n)}{(\lambda)_m (f_1)_{m_1} \dots (f_r)_{m_r}}. \tag{3.4a} $$

The polynomial in n of degree m is defined by

$$ P_m(n) \equiv \sum_{j_1=0}^{m_1} \dots \sum_{j_r=0}^{m_r} \sigma(j_1, \dots, j_r) \sum_{\ell=0}^{j} \left\{ \begin{matrix} j \\ \ell \end{matrix} \right\} (a)_\ell (-n)_\ell (n+\lambda)_{m-\ell}, \tag{3.4b} $$

where $m = m_1 + \cdots + m_r$, $\lambda = b - a - m$, $j = j_1 + \cdots + j_r$ and $\sigma(j_1, \dots, j_r)$ is given by Eq. (3.2a).

It is easy to see that the constant term in $P_m(n)$ is given by $P_m(0)$. Since the only contributions to this constant come from the indices $\ell = 0$ and $j_1 = \ldots = j_r = 0$ upon noting that $\sigma(0, \dots, 0) = s_{m_1}^{(1)} \dots s_{m_r}^{(r)} = (f_1)_{m_1} \dots (f_r)_{m_r}$ we see that

$$ P_m(0) = (\lambda)_m (f_1)_{m_1} \dots (f_r)_{m_r}. $$

Furthermore, the coefficient of n^m in the polynomial $P_m(n)$ is readily seen to be given by

$$ \begin{aligned} \sum_{j_1=0}^{m_1} \dots \sum_{j_r=0}^{m_r} s_{m_1-j_1}^{(1)} \dots s_{m_r-j_r}^{(r)} \sum_{\ell=0}^{j} (-1)^\ell \left\{ \begin{matrix} j \\ \ell \end{matrix} \right\} (a)_\ell &= \sum_{j_1=0}^{m_1} s_{m_1-j_1}^{(1)} (-a)^{j_1} \dots \sum_{j_r=0}^{m_r} s_{m_r-j_r}^{(r)} (-a)^{j_r} \\ &= (f_1 - a)_{m_1} \dots (f_r - a)_{m_r}, \end{aligned} $$

where we have used Eq. (2.1b) and Eqs. (3.1) with k replaced by $-a$.

Thus

$$ P_m(n) = (f_1 - a)_{m_1} \dots (f_r - a)_{m_r} n^m + \cdots + (\lambda)_m (f_1)_{m_1} \dots (f_r)_{m_r}, \tag{3.4c} $$

where the remaining intermediate coefficients of powers of n in $P_m(n)$ (when $m > 1$) are determined by the expression on the right of Eq. (3.4b). We shall neither need nor be concerned with these coefficients. Now assuming $a \neq f_i$ $(1 \leq i \leq r)$ and $(\lambda)_m \neq 0$ we may invoke Lemma 4 thus obtaining

$$ P_m(n) = (\lambda)_m (f_1)_{m_1} \dots (f_r)_{m_r} \frac{(1+\xi_1)_n}{(\xi_1)_n} \dots \frac{(1+\xi_m)_n}{(\xi_m)_n}, \tag{3.5} $$

where the $\xi_1, \dots, \xi_m$ are nonvanishing zeros of the polynomial in t of degree $m = m_1 + \cdots + m_r$ defined by

$$ Q_m(t) \equiv \sum_{j_1=0}^{m_1} \dots \sum_{j_r=0}^{m_r} s_{m_1-j_1}^{(1)} \dots s_{m_r-j_r}^{(r)} \sum_{\ell=0}^{j} \left\{ \begin{matrix} j \\ \ell \end{matrix} \right\} (a)_\ell (t)_\ell (\lambda - t)_{m-\ell}. \tag{3.6} $$

The $s_{m_i-j_i}^{(i)}$, $1 \leq i \leq r$ are determined by Eqs. (3.1) where the index k may be replaced by the variable x. Thus the $s_{m_i-j_i}^{(i)}$ are generated by the relations

$$ (f_i + x)_{m_i} = \sum_{j_i=0}^{m_i} s_{m_i-j_i}^{(i)} x^{j_i}, \tag{3.7} $$

where $1 \leq i \leq r$. Since $Q_m(t)$ ultimately depends on the parameters of ${}_{r+2}F_{r+1}(1)$ we shall call it the *associated parametric polynomial.*

Eqs. (3.4a) and (3.5)–(3.7) and the above discussion may now be combined in the following summation theorem for the generalized hypergeometric series ${}_{r+2}F_{r+1}(1)$.

Theorem 1. *For nonnegative integer n and positive integers (m_r)*

$$ {}_{r+2}F_{r+1}\left(\left.\begin{matrix} -n, a, (f_r+m_r) \\ b, (f_r) \end{matrix}\right| 1\right) = \frac{(\lambda)_n}{(b)_n} \frac{(1+\xi_1)_n}{(\xi_1)_n} \dots \frac{(1+\xi_m)_n}{(\xi_m)_n}, $$

where

$$ m = m_1 + \cdots + m_r, \quad \lambda = b - a - m, \quad (\lambda)_m \neq 0, \quad a \neq f_i \quad (1 \leq i \leq r). $$

The $\xi_1, \ldots, \xi_m$ are nonvanishing zeros of the associated parametric polynomial of degree m given by

$$Q_m(t) = \sum_{j_1=0}^{m_1} \cdots \sum_{j_r=0}^{m_r} s^{(1)}_{m_1-j_1} \ldots s^{(r)}_{m_r-j_r} \sum_{\ell=0}^{j} \begin{Bmatrix} j \\ \ell \end{Bmatrix} (a)_\ell (t)_\ell (\lambda - t)_{m-\ell},$$

where

$$j = j_1 + \cdots + j_r$$

and the $s^{(i)}_{m_i-j_i}$ $(1 \le i \le r)$ are determined by the generating relations

$$(f_1 + x)_{m_1} = \sum_{j_1=0}^{m_1} s^{(1)}_{m_1-j_1} x^{j_1}$$

$$\vdots$$

$$(f_r + x)_{m_r} = \sum_{j_r=0}^{m_r} s^{(r)}_{m_r-j_r} x^{j_r}.$$

In Section 4 we shall employ a modified form of Theorem 1 (recorded as Corollary 1 below) with the specialization $m_1 = \cdots = m_r = 1$ to obtain a Kummer-type transformation formula for the generalized hypergeometric function ${}_pF_p(x)$. To this end we note that in deriving Lemma 5 and Theorem 1 we used Eqs. (3.2)

$$(f_1 + k)_{m_1} \ldots (f_r + k)_{m_r} = \sum_{j_1=0}^{m_1} \cdots \sum_{j_r=0}^{m_r} s^{(1)}_{m_1-j_1} \ldots s^{(r)}_{m_r-j_r} k^j,$$

where $j = j_1 + \cdots + j_r$. For $m_1 = \cdots = m_r = 1$ this becomes

$$(f_1 + k) \ldots (f_r + k) = \sum_{j_1=0}^{1} \cdots \sum_{j_r=0}^{1} s^{(1)}_{1-j_1} \ldots s^{(r)}_{1-j_r} k^j.$$

However, the left side of the latter is a polynomial in k of degree r and so we may write

$$(f_1 + k) \ldots (f_r + k) = \sum_{j=0}^{r} s_{r-j} k^j,$$

where $s_0 = 1$ and the s_i $(1 \le i \le r)$ are sums of all possible products of i distinct elements from the set $\{f_1, \ldots, f_r\}$. Thus in Eqs. (3.3a) and (3.3c) letting $m_1 = \cdots = m_r = 1$, replacing the multiple j_i-summations by the latter j-summation and in Lemma 3 letting $m = r$ so that $m \ge j$ we have mutatis mutandis the following corollary of Theorem 1.

Corollary 1. *For nonnegative integer n*

$${}_{r+2}F_{r+1}\left(\begin{matrix} -n, a, (f_r + 1) \\ b, (f_r) \end{matrix} \middle| 1 \right) = \frac{(\lambda)_n}{(b)_n} \frac{(1+\xi_1)_n}{(\xi_1)_n} \cdots \frac{(1+\xi_r)_n}{(\xi_r)_n},$$

where

$$\lambda = b - a - r, \quad (\lambda)_r \ne 0, \quad a \ne f_i \quad (1 \le i \le r).$$

The $\xi_1, \ldots, \xi_r$ are nonvanishing zeros of the associated parametric polynomial of degree r given by

$$Q_r(t) = \sum_{j=0}^{r} s_{r-j} \sum_{\ell=0}^{j} \begin{Bmatrix} j \\ \ell \end{Bmatrix} (a)_\ell (t)_\ell (\lambda - t)_{r-\ell},$$

where the s_{r-j} $(0 \le j \le r)$ are determined by the generating relation

$$(f_1 + x) \ldots (f_r + x) = \sum_{j=0}^{r} s_{r-j} x^j.$$

Note that when all of the $f_j = f$, then $s_{r-j} = \binom{r}{j} f^{r-j}$.

Setting $r=1, f_1=f, \xi_1=\xi$ in Corollary 1 we have $\lambda=b-a-1, s_0=1, s_1=f$ and so the associated parametric polynomial is

$$Q_1(t)=(a-f)t+f(b-a-1), \tag{3.8a}$$

where $a\neq f$ and $b-a-1\neq 0$. Thus we deduce the summation formula for Clausen's series of unit argument

$${}_3F_2\left(\left.\begin{matrix}-n, a, f+1\\ b, f\end{matrix}\right| 1\right)=\frac{(b-a-1)_n}{(b)_n}\,\frac{(1+\xi)_n}{(\xi)_n},$$

where

$$\xi=\frac{f(1+a-b)}{a-f} \tag{3.8b}$$

is the nonvanishing zero of $Q_1(t)$. This result has previously been obtained [7] by other methods.

It is evident from Eq. (3.4c) that $Q_r(t)$ will always have the form

$$Q_r(t)=(-1)^r(f_1-a)\ldots(f_r-a)t^r+R_{r-1}(t)+f_1\ldots f_r(\lambda)_r,$$

where $\lambda=b-a-r$ and $R_{r-1}(t)$ is some polynomial of degree $r-1$ such that $R_{r-1}(0)=0$ for all $r\geq 1$. Thus when $r=1, R_0(t)=0$ and we immediately have Eq. (3.8a) where $f_1=f$. When $r=2$ and $f_1=f, f_2=g$, then employing the representation for $Q_r(t)$ in Corollary 1 yields

$$Q_2(t)=\alpha t^2-((\alpha+\beta)\lambda+\beta)t+fg\lambda(\lambda+1),$$

where

$$\begin{aligned}\lambda&=b-a-2\\ \alpha&=(f-a)(g-a)\\ \beta&=fg-a(a+1).\end{aligned}$$

Moreover, it is apparent that the intermediate coefficients of $Q_r(t)$ as functions of the parameters of ${}_{r+2}F_{r+1}(1)$ become ever more complex as r increases.

We conclude this section by proving that Minton's summation theorem given by Eq. (1.2) is a consequence of Lemma 5. To this end we write Eqs. (3.4a) and (3.4b) for nonnegative integer n and positive integers (m_r) as

$$\begin{aligned}{}_{r+2}F_{r+1}\left(\left.\begin{matrix}-n, a, (f_r+m_r)\\ b, (f_r)\end{matrix}\right| 1\right)&=\frac{1}{(b)_n}\frac{1}{(f_1)_{m_1}\ldots(f_r)_{m_r}}\\ &\quad\times\sum_{j_1=0}^{m_1}\cdots\sum_{j_r=0}^{m_r}s^{(1)}_{m_1-j_1}\ldots s^{(r)}_{m_r-j_r}\sum_{\ell=0}^{j}\left\{\begin{matrix}j\\ \ell\end{matrix}\right\}(a)_\ell(-n)_\ell\frac{(\lambda)_n(n+\lambda)_{m-\ell}}{(\lambda)_m},\end{aligned} \tag{3.9}$$

where $\lambda=b-a-m,\ m=m_1+\cdots+m_r,\ j=j_1+\cdots+j_r$. However, we have

$$\begin{aligned}\frac{(\lambda)_n(n+\lambda)_{m-\ell}}{(\lambda)_m}&=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda+m)}\frac{\Gamma(n+\lambda+m-\ell)}{\Gamma(\lambda+n)}\\ &=\frac{\Gamma(b-a+n-\ell)}{\Gamma(b-a)}=\frac{\Gamma(b-a+n)}{\Gamma(b-a)}(b-a+n)_{-\ell}\\ &=(-1)^\ell\frac{(b-a)_n}{(1+a-b-n)_\ell}.\end{aligned}$$

Thus Eq. (3.9) yields the following lemma upon recalling the generating Eq. (3.7).

Lemma 6. *For nonnegative integer n and positive integers (m_r)*

$$\begin{aligned}{}_{r+2}F_{r+1}\left(\left.\begin{matrix}-n, a, (f_r+m_r)\\ b, (f_r)\end{matrix}\right| 1\right)&=\frac{(b-a)_n}{(b)_n}\frac{1}{(f_1)_{m_1}\ldots(f_r)_{m_r}}\\ &\quad\times\sum_{j_1=0}^{m_1}\cdots\sum_{j_r=0}^{m_r}s^{(1)}_{m_1-j_1}\ldots s^{(r)}_{m_r-j_r}\sum_{\ell=0}^{j}(-1)^\ell\left\{\begin{matrix}j\\ \ell\end{matrix}\right\}\frac{(a)_\ell(-n)_\ell}{(1+a-b-n)_\ell},\end{aligned} \tag{3.10a}$$

where $j=j_1+\cdots+j_r$ and $s^{(i)}_{m_i-j_i}$ are generated by the relations

$$(f_i+x)_{m_i}=\sum_{j_i=0}^{m_i}s^{(i)}_{m_i-j_i}x^{j_i}\quad(1\leq i\leq r). \tag{3.10b}$$

Now suppose $n \geq m_1 + \cdots + m_r$. Thus

$$n \geq m \geq j \geq \ell$$

and so $(-n)_\ell \neq 0$. Furthermore if $b = a + 1$, Eq. (3.10a) reduces to

$${}_{r+2}F_{r+1}\left(\left.\begin{matrix}-n, a, (f_r + m_r)\\ a + 1, (f_r)\end{matrix}\right| 1\right) = \frac{n!}{(a+1)_n} \frac{1}{(f_1)_{m_1} \dots (f_r)_{m_r}} \sum_{j_1=0}^{m_1} \dots \sum_{j_r=0}^{m_r} s^{(1)}_{m_1-j_1} \dots s^{(r)}_{m_r-j_r} \sum_{\ell=0}^{j} (-1)^\ell \begin{Bmatrix} j \\ \ell \end{Bmatrix} (a)_\ell.$$

But by Eq. (2.1b)

$$\sum_{\ell=0}^{j} (-1)^\ell \begin{Bmatrix} j \\ \ell \end{Bmatrix} (a)_\ell = (-a)^j, \tag{3.11}$$

where $j = j_1 + \cdots + j_r$ and by Eq. (3.10b)

$$\sum_{j_1=0}^{m_1} s^{(1)}_{m_1-j_1} (-a)^{j_1} \dots \sum_{j_r=0}^{m_r} s^{(r)}_{m_r-j_r} (-a)^{j_r} = (f_1 - a)_{m_1} \dots (f_r - a)_{m_r}$$

and so we have finally Minton's result given by Eq. (1.2).

In [8] we provide a simpler more direct derivation of Minton's summation formula which essentially utilizes elementary properties of Stirling numbers of the second kind, Eq. (3.11) and a hypergeometric identity that may be proved by induction.

4. Reduction and transformation formulas

The Kampé de Fériet function is a generalized hypergeometric function in two variables that may be defined by double infinite series (see e.g. [9])

$$\mathrm{F}^{p:r;u}_{q:s;v}\left(\left.\begin{matrix}(a_p) & : & (c_r) & ; & (f_u)\\ (b_q) & : & (d_s) & ; & (g_v)\end{matrix}\right| x, y\right) \equiv \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{((a_p))_{m+n}}{((b_q))_{m+n}} \frac{((c_r))_m}{((d_s))_m} \frac{((f_u))_n}{((g_v))_n} \frac{x^m}{m!} \frac{y^n}{n!}.$$

When one of the independent variables $x = 0$ or $y = 0$, the latter reduces respectively to a generalized hypergeometric function in one variable

$${}_{p+u}F_{q+v}\left(\left.\begin{matrix}(a_p), & (f_u)\\ (b_q), & (g_v)\end{matrix}\right| y\right), \qquad {}_{p+r}F_{q+s}\left(\left.\begin{matrix}(a_p), & (c_r)\\ (b_q), & (d_s)\end{matrix}\right| x\right).$$

In [7] we showed that

$$\mathrm{F}^{p:r+1;0}_{q:s+1;0}\left(\left.\begin{matrix}(a_p) & : & (c_{r+1}) & ; & \text{—}\\ (b_q) & : & (d_{s+1}) & ; & \text{—}\end{matrix}\right| -y, y\right) = \sum_{n=0}^{\infty} \frac{((a_p))_n}{((b_q))_n} {}_{r+2}F_{s+1}\left(\left.\begin{matrix}-n, (c_{r+1})\\ (d_{s+1})\end{matrix}\right| 1\right) \frac{y^n}{n!},$$

where the horizontal line indicates an empty parameter sequence. In the above result setting $s = r$, $c_{r+1} = a$, $d_{r+1} = b$, $(c_r) = (f_r + 1)$, $(d_r) = (f_r)$ we obtain

$$\mathrm{F}^{p:r+1;0}_{q:r+1;0}\left(\left.\begin{matrix}(a_p) & : & a, (f_r + 1) & ; & \text{—}\\ (b_q) & : & b, (f_r) & ; & \text{—}\end{matrix}\right| -y, y\right) = \sum_{n=0}^{\infty} \frac{((a_p))_n}{((b_q))_n} {}_{r+2}F_{r+1}\left(\left.\begin{matrix}-n, a, (f_r + 1)\\ b, (f_r)\end{matrix}\right| 1\right) \frac{y^n}{n!}$$

which we use together with Corollary 1 to obtain the following.

Theorem 2. *Suppose $a \neq f_i$ $(1 \leq i \leq r)$ and $(b - a - r)_r \neq 0$. Then we have the reduction formula for the Kampé de Fériet function*

$$\mathrm{F}^{p:r+1;0}_{q:r+1;0}\left(\left.\begin{matrix}(a_p) & : & a, (f_r + 1) & ; & \text{—}\\ (b_q) & : & b, (f_r) & ; & \text{—}\end{matrix}\right| -y, y\right) = {}_{p+r+1}F_{q+r+1}\left(\left.\begin{matrix}b - a - r, (a_p), (\xi_r + 1)\\ b, (b_q), (\xi_r)\end{matrix}\right| y\right). \tag{4.1}$$

The (ξ_r) are nonvanishing zeros of the associated parametric polynomial of degree r given by

$$Q_r(t) = \sum_{j=0}^{r} s_{r-j} \sum_{\ell=0}^{j} \begin{Bmatrix} j \\ \ell \end{Bmatrix} (a)_\ell (t)_\ell (b - a - r - t)_{r-\ell},$$

where the s_{r-j} $(0 \leq j \leq r)$ are determined by the generating relation

$$(f_1 + x) \dots (f_r + x) = \sum_{j=0}^{r} s_{r-j} x^j.$$

In addition upon letting $y \mapsto -y$ in Eq. (4.1), the specialization $p = q = 0$ of the latter reduces to the transformation formula

$$ {}_{r+1}F_{r+1}\left(\left.\begin{matrix} a, (f_r + 1) \\ b, (f_r) \end{matrix}\right| y\right) = \mathrm{e}^y {}_{r+1}F_{r+1}\left(\left.\begin{matrix} b - a - r, (\xi_r + 1) \\ b, (\xi_r) \end{matrix}\right| - y\right). \tag{4.2} $$

Eq. (4.2) provides the generalized analogue of Kummer's first transformation formula for the confluent hypergeometric function

$$ {}_1F_1\left(\left.\begin{matrix} a \\ b \end{matrix}\right| y\right) = \mathrm{e}^y {}_1F_1\left(\left.\begin{matrix} b - a \\ b \end{matrix}\right| - y\right). $$

The specialization $r = 1$ of Eq. (4.2) with $f_1 = f$, $\xi_1 = \xi$ given by Eq. (3.8b) has previously been obtained by Miller [7] and Paris [10].

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