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Journal of Algebra 256 (2002) 414–432

JOURNAL OF Algebra

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Hereditary abelian categories with tilting object over arbitrary base fields

Dieter Happel [a,1] and Idun Reiten [b,*]

[a] *Fakultät für Mathematik, TU Chemnitz, PSF 964, D-09107 Chemnitz, Germany*
[b] *Department of Mathematical Sciences, Norwegian University of Science and Technology, 7491 Trondheim, Norway*

Received 22 March 2001

Communicated by Kent R. Fuller

Dedicated to Daniel Simson on the occasion of his 60th birthday

Introduction

Let R be a commutative Artin ring and $\mathcal{H}$ a hereditary abelian R-category which is Ext-finite, that is $\mathrm{Hom}_{\mathcal{H}}(X,Y)$ and $\mathrm{Ext}^1_{\mathcal{H}}(X,Y)$ are finitely generated R-modules for all X and Y in $\mathcal{H}$. Assume also that $\mathcal{H}$ has a tilting object, that is, some object T such that $\mathrm{Ext}^1_{\mathcal{H}}(T,T)=0$, and whenever $\mathrm{Hom}(T,X)=0=\mathrm{Ext}^1_{\mathcal{H}}(T,X)$ for X in $\mathcal{H}$, then $X=0$.

Such hereditary categories with tilting object were the basis for the definition of the class of quasitilted algebras in [HRS], generalizing the classes of tilted and canonical algebras, as well as containing other classes of algebras. The quasitilted algebras are those of the form $\mathrm{End}_{\mathcal{H}}(T)^{\mathrm{op}}$, where T is a tilting object in an Ext-finite hereditary abelian R-category. Equivalently, an Artin R-algebra Λ is quasitilted if and only if the global dimension of Λ is at most two, and for any indecomposable finitely generated Λ-module either the projective or the injective dimension is at most one [HRS].

* Corresponding author.
E-mail addresses: happel@mathematik.tu-chemnitz.de (D. Happel), idunr@math.ntnu.no (I. Reiten).

[1] These results were obtained during a visit of the first author at NTNU, supported by the Norwegian Research Council. He would like to thank the second author for her kind hospitality.

PII: S0021-8693(02)00088-1

The main examples of Ext-finite hereditary abelian R-categories are $\operatorname{mod} H$ for an Artin R-algebra H and also the category $\operatorname{coh}\mathbb{X}$ of coherent sheaves over a weighted projective line $\mathbb{X}$ in the sense of Geigle–Lenzing, in the case when $R = k$ is an algebraically closed field [GL]. Actually, in this case it was shown in [H2] that any connected Ext-finite hereditary abelian k-category is derived equivalent to some $\operatorname{mod} H$ or to some $\operatorname{coh}\mathbb{X}$, in particular to some noetherian hereditary abelian category. It was already shown in [L] that the $\operatorname{mod} H$ and $\operatorname{coh}\mathbb{X}$ are the only noetherian examples.

In this paper we generalize this result from R being an algebraically closed field to the case of an arbitrary commutative Artin ring, which actually easily can be reduced to the case of R being a field. Since there is at present no "geometric" definition available for coherent sheaves on a weighted projective line over an arbitrary field, the formulation of our main result will be somewhat different from the formulation for an algebraically closed field k. We prove that an Ext-finite hereditary abelian R-category with tilting object is derived equivalent to $\operatorname{mod} H$ for a hereditary Artin R-algebra H or to $\operatorname{mod} \Lambda$ for a canonical Artin R-algebra Λ. Note that over algebraically closed fields the derived equivalence classes of the canonical algebras and the categories $\operatorname{coh}\mathbb{X}$ are known to coincide [GL]. For the canonical algebras there is a definition over arbitrary fields k [R2]. Also a related class of algebras, called squid algebras, are defined in the general case [R2]. The canonical algebras and squid algebras belong to the same derived equivalence class [R2]. We also show that an Ext-finite hereditary abelian R-category $\mathcal{H}$ with tilting object is derived equivalent to a noetherian hereditary abelian category.

Most of the results for Ext-finite hereditary abelian categories with tilting object over an algebraically closed field carry over to the case of arbitrary fields, and some are already formulated in the more general setting in the literature. In [H2] the main classification result cited above is reduced to considering three main cases:

(i) $\mathcal{H}$ has some directing object;
(ii) $\mathcal{H}$ has some simple object;
(iii) there exists an indecomposable exceptional object E of infinite length, which is a factor of a finite number of copies of a tilting object, such that the perpendicular category $E^{\perp}$ is equivalent to $\operatorname{mod} H$ for a tame hereditary algebra H.

For the reduction to these three cases we can use the work in [H2], together with proving that $\operatorname{Hom}(\tau E, E) = 0$ when E is quasisimple exceptional of infinite length. Case (i) is taken care of using [HRe1], where the desired result is already proved in the generality we want. For (ii) the relevant result is taken from [HRe2]. The first four sections of [HRe2], dealing with $\mathcal{H}$ with simple objects, remain valid in the larger generality. When showing that the derived equivalence class of some $\mathcal{H}$ with simple objects contains $\operatorname{mod} \Lambda$ for a canonical k-algebra Λ, the assumption that k is algebraically closed is used in an essential way. So here we take a completely different approach, which is also more streamlined than the proof in the algebraically closed case, taking advantage of results from [H2]. For case (iii) we extend the proof from the algebraically closed case to the more general setting, and here some additional work has to be done.

We now describe the content of this paper section by section. In Section 1 we recall some background material, and in Section 2 we consider the case of $\mathcal{H}$ having nonzero objects of finite length (and no directing objects). Without loss of generality we can assume that there is no nonzero map from an object of finite length to an indecomposable object of infinite length. We prove that then $\mathcal{H}$ is derived equivalent to a noetherian hereditary abelian category, and also to the category of finitely generated modules for a canonical algebra or hereditary algebra and for a squid algebra or hereditary algebra. In Section 3 we prove the lemma giving the basis for getting the same reduction to the three cases as in [H2], and provide the proof of case (iii).

1. Preliminaries

Let $\mathcal{H}$ be a hereditary abelian category over a commutative Artin ring R, and assume that $\mathcal{H}$ is Ext-finite and has a tilting object. In this section we give some background material on such categories.

We start by pointing out that without loss of generality we can assume that the commutative Artin ring R is a field k. The idea of proof is taken from [AP], and we include the proof for the convenience of the reader.

Lemma 1.1. *Let $\mathcal{H}$ be a connected* Ext-*finite hereditary abelian R-category with tilting object, for a commutative Artin ring R. Then $\mathcal{H}$ is an* Ext-*finite hereditary abelian k-category for a field k.*

Proof. Let T be a tilting object in $\mathcal{H}$. Then by definition $\Lambda = \mathrm{End}_{\mathcal{H}}(T)^{\mathrm{op}}$ is a quasitilted algebra, which is indecomposable since $\mathcal{H}$ is connected. Here we use that $\mathcal{H}$ can be constructed from Λ [HRS]. Since the quiver of Λ has no oriented cycles [HRS], it follows that $\mathrm{End}_{\Lambda}(P)$ is a division algebra for any indecomposable projective Λ-module P. Since Λ is indecomposable, the center $Z(\Lambda)$ of Λ is a local ring, and it is known that Λ is an Artin $Z(\Lambda)$-algebra. Let c be a nonzero element in $Z(\Lambda)$, and consider the Λ-homomorphism $f : \Lambda \to \Lambda$ which is multiplication by c. Since the map f is nonzero, there is some indecomposable projective Λ-module P, where $\Lambda = P \oplus P'$, such that $f|_P \neq 0$. Since f is multiplication by c, we clearly have $f(P) \subset P$, and since $\mathrm{End}(P)$ is a division ring, $f : P \to P$ is an isomorphism. Hence it follows that c is invertible in $Z(\Lambda)$, and consequently $Z(\Lambda)$ is a field k. Hence Λ is a k-algebra, and so the bounded derived category $D^b(\mathrm{mod}\,\Lambda)$ is a k-category. Then also $\mathcal{H}$ is a k-category, since $\mathrm{mod}\,\Lambda$ and $\mathcal{H}$ are derived equivalent, and $\mathcal{H}$ is Ext-finite over k. $\square$

Denote by $\mathcal{H}_0$ the full subcategory of $\mathcal{H}$ consisting of the objects of finite length, and by $\mathcal{H}_\infty$ the full subcategory where the indecomposable summands of all objects have infinite length. For future reference we collect the following known basic properties (see [H1,L]).

Proposition 1.2. *Let $\mathcal{H}$ be a connected hereditary abelian* Ext*-finite k-category with tilting object, for any field k, and assume that $\mathcal{H}$ is not equivalent to* $\operatorname{mod} H$ *for a finite-dimensional hereditary k-algebra H.*

Then we have the following statements.

(a) *$\mathcal{H}$ has no nonzero projective objects.*
(b) *The AR-quiver of $\mathcal{H}_0$ is a union of stable tubes, with all but a finite number of rank one.*
(c) *Each tube corresponds to a uniserial abelian category.*
(d) *The tubes are pairwise orthogonal.*
(e) *The quasisimple objects of a tube of rank greater than one are pairwise orthogonal.*

The following normalization result from [HRe2] for the case where k is an algebraically closed field holds with the same proof for any field k.

Proposition 1.3. *Let $\mathcal{H}$ be an* Ext*-finite hereditary abelian k-category with tilting object, for any field k. Assume that $\mathcal{H}_0 \neq (0)$ and that $\mathcal{H}$ is not equivalent to* $\operatorname{mod} \Lambda$ *for a finite-dimensional hereditary k-algebra Λ. Then up to derived equivalence we can assume $(\mathcal{H}_0, \mathcal{H}_\infty) = 0$.*

A central class of objects in our categories $\mathcal{H}$ are the exceptional objects. We say that an object E in $\mathcal{H}$ is *exceptional* if it is indecomposable and $\operatorname{Ext}^1_{\mathcal{H}}(E, E) = 0$. The indecomposable summands of a tilting object are examples of exceptional objects. We say that an object is *torsionable* if it is a factor object of a finite direct sum of copies of some tilting object. Associated with an exceptional object E is the perpendicular category $E^\perp$, the full subcategory of $\mathcal{H}$ whose objects are the X in $\mathcal{H}$ with $\operatorname{Hom}(E, X) = 0 = \operatorname{Ext}^1(E, X)$.

We have the following result from [HRe2], where the proof is valid for any field k.

Proposition 1.4. *Let $\mathcal{H}$ be an* Ext*-finite hereditary abelian k-category with tilting object, not equivalent to* $\operatorname{mod} \Lambda$ *for a finite-dimensional hereditary k-algebra Λ. Assume $\mathcal{H}_0 \neq 0$, and $(\mathcal{H}_0, \mathcal{H}_\infty) = 0$. Let E be an exceptional torsionable object in $\mathcal{H}$, and let $0 \to \tau E \to M \to E \to 0$ be the almost split sequence in $\mathcal{H}$ with right hand term E.*

(a) *The perpendicular category $E^{\perp}$ is an* Ext-*finite hereditary abelian k-category with tilting object.*
(b) *If M is indecomposable, then $E^{\perp}$ is connected.*
(c) *If $E^{\perp}$ is equivalent to* $\operatorname{mod} H$ *for some finite-dimensional basic hereditary k-algebra H, then $T = H \oplus E$ is a tilting object in $\mathcal{H}$, and* $\operatorname{End}_{\mathcal{H}}(T)^{\mathrm{op}} \simeq H[M]$, *the one-point extension algebra*

$$\begin{pmatrix} \operatorname{End}(E)^{\mathrm{op}} & 0 \\ M & H \end{pmatrix}.$$

(d) *If $X \subset E^t$ for some $t > 0$ and X is in $E^{\perp}$, then X is projective in $E^{\perp}$.*
(e) *Let Z be in $\mathcal{H}$, and let $f : E^t \to Z$ be a minimal right* $\operatorname{add} E$*-approximation. Then* $\operatorname{Ker} f$ *is a projective object in $E^{\perp}$.*

The next result is stated and proved in [HRe2] without the assumption that M is indecomposable. However, the proof is much simpler when M is indecomposable, and, combined with the results from [H2], it is the only case needed. This represents a simplification also in the case of algebraically closed fields. We include the proof (taken from [HRe2]). We say that an object E in $\mathcal{H}$ is quasisimple if the middle term of the almost split sequence with right hand term E is indecomposable.

Proposition 1.5. *Let $\mathcal{H}$ be a connected* Ext-*finite hereditary abelian k-category with tilting object, such that $(\mathcal{H}_0, \mathcal{H}_\infty) = 0$ and $\mathcal{H}$ has no nonzero projective objects. Let E be an exceptional torsionable quasisimple object in $\mathcal{H}_\infty$.*

Then the perpendicular category $E^{\perp}$ is equivalent to $\operatorname{mod} H$ *for some finite-dimensional hereditary k-algebra H, and* $\operatorname{Hom}(E, \mathcal{R}) \neq 0$ *if $\mathcal{R}$ is a tube in H_0.*

Proof. Since E is in $\mathcal{H}_\infty$, there is some proper epimorphism $E \to Z$ with $Z \neq 0$ and Z indecomposable. Let $g : E^t \to Z$ be a minimal right $\operatorname{add} E$-approximation. Then g is an epimorphism because there already is some epimorphism $E \to Z$ and E is in $\operatorname{add} E$. Then $P = \operatorname{Ker} g$ is nonzero, and is a projective object in $E^{\perp}$ by Proposition 1.4(e). Since the middle term M in the almost split sequence $0 \to \tau E \to M \to E \to 0$ is indecomposable, $E^{\perp}$ is connected by Proposition 1.4(b). Hence it follows that $E^{\perp}$ is equivalent to $\operatorname{mod} H$ for some finite-dimensional hereditary k-algebra H (see [H1, Theorem 4.2]).

Let now $\mathcal{R}$ be a tube in $\mathcal{H}_0$, and assume that $\operatorname{Hom}(E, \mathcal{R}) = 0$. Since $E \in \mathcal{H}_\infty$, we have $\operatorname{Ext}^1_{\mathcal{H}}(E, R) \simeq D \operatorname{Hom}(\tau^{-1} R, E) = 0$ for $R \in \mathcal{R}$, where $D = \operatorname{Hom}_k(\cdot, k)$ (see [ARS]). Hence we get $\mathcal{R} \subset E^{\perp}$, and clearly $\mathcal{R}$ is a tube also in $E^{\perp}$. Therefore, the hereditary k-algebra H must be tame (see [DR]). We then know that each tube is sincere, that is, there is a nonzero map to $\mathcal{R}$ from each indecomposable projective H-module. In particular, we have $\operatorname{Hom}(P, \mathcal{R}) \neq 0$. Consider again the exact sequence $0 \to P \to E^t \to Z \to 0$. Note that $Z \notin \mathcal{R}$, since $\mathcal{R} \subset E^{\perp}$ and $Z \notin E^{\perp}$. Hence it follows that $\operatorname{Hom}(\mathcal{R}, Z) = 0$. Consider the induced exact sequence $0 \to \mathrm{Hom}(Z, \mathcal{R}) \to \mathrm{Hom}(E^t, \mathcal{R}) \to \mathrm{Hom}(P, \mathcal{R}) \to \mathrm{Ext}^1(Z, \mathcal{R}) \simeq D\,\mathrm{Hom}(\mathcal{R}, Z)$. Since $\mathrm{Hom}(\mathcal{R}, Z) = 0$ and $\mathrm{Hom}(P, \mathcal{R}) \neq 0$, we conclude that $\mathrm{Hom}(E, \mathcal{R}) \neq 0$, and we are done. □

The proof of the following result from [HS] is valid for any field k.

Proposition 1.6. *Let $\mathcal{H}$ be an* Ext-*finite hereditary abelian k-category with tilting object and with no directing object. Let E be an exceptional torsionable object of infinite length. Let $0 \to \tau E \to M \to E \to 0$ be the associated almost split sequence, and let H be the basic hereditary k-algebra such that $E^{\perp}$ is equivalent to* $\mathrm{mod}\,H$.

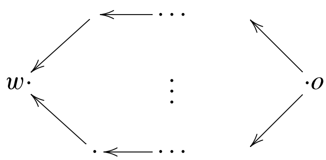
Then M is either indecomposable or the direct sum of two indecomposable objects. In the second case one of the indecomposable summands is a projective H-module.

The central role of the quasisimple objects amongst the exceptional torsionable objects is given by the following [H2].

Proposition 1.7. *Let $\mathcal{H}$ be an* Ext-*finite hereditary abelian k-category with tilting object and no directing object. Then any torsionable exceptional object of infinite length has a filtration by quasisimple torsionable exceptional objects.*

We shall also need some background material related to canonical algebras.

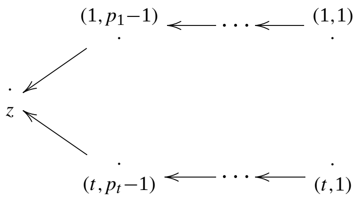
Recall that for an algebraically closed field k a canonical algebra is defined on the basis of a finite set of positive integers $(p_1, \ldots, p_t)$ with $t \geqslant 3$ and a corresponding set of distinct elements $(\lambda_1, \ldots, \lambda_t)$ from $\mathbb{P}^1(k)$. The associated canonical algebra Λ is given by the quiver



with t arms from o to w, each having $p_1, \ldots, p_t$ arrows, respectively, and with relations depending on the λ_i (see [R1]).

In [R2] a definition is given also for the case where k is an arbitrary field, in which case it is much more complicated. The starting point is a tame bimodule ${}_F M_G$, that is, F and G are division algebras over k and $(\dim_F M)(\dim_G M) = 4$. The objects of the category $\mathrm{rep}({}_F M_G)$ of representations of ${}_F M_G$ are triples (A, B, f) where A is in $\mathrm{mod}\,G$, B is in $\mathrm{mod}\,F$ and $f : M \otimes_G A \to B$ is a map in $\mathrm{mod}\,F$. When k is algebraically closed, $M = k^2$ is the only choice. Then $\mathrm{rep}(M)$ is equivalent to $\mathrm{mod}\,k\Gamma$, where Γ is the Kronecker quiver $\cdot \rightrightarrows \cdot$. The elements $(\lambda_1, \ldots, \lambda_t)$ in the definition of a canonical k-algebra correspond to distinct tubes (which are all of rank one), and hence also to simple regular modules for $k\Gamma$. In the general case a canonical algebra is defined on the basis of a finite set of positive integers $(p_1, \dots, p_t)$ and a set $(N_1, \dots, N_t)$ of quasisimple regular representations of the tame bimodule ${}_F M_G$.

There is a related class of algebras, the *squid* algebras, whose precise definition for an arbitrary field is more easily explained [R2]. We start with the same data. Then $F_i = \mathrm{End}(N_i)^{\mathrm{op}}$ is a division algebra over k, for $i = 1, \dots, t$. The associated squid algebra is then the tensor algebra associated with



where the tensor algebra of ${}_F M_G$ is associated with the vertex z, the division algebras F_i are associated with vertices (i, j) for $j = 1, \dots, p_i - 1$, the bimodule N_i with the arrow $\underset{z}{\cdot} \leftarrow \underset{(i,p_i-1)}{\cdot}$, and the bimodule ${}_{F_i}(F_i)_{F_i}$ with the arrows $\underset{(i,j)}{\cdot} \leftarrow \underset{(i,j+1)}{\cdot}$ for $1 \leqslant j < p_i - 1$.

It is proved in [R2] that for any squid algebra Λ there is some (co)tilting module T such that $\mathrm{End}_\Lambda(T)^{\mathrm{op}}$ is a canonical algebra, and consequently the derived equivalence classes of squid algebras and canonical algebras are the same.

Also recall from [GL] that the category $\mathrm{coh}\,\mathbb{X}$ of coherent sheaves over a weighted projective line $\mathbb{X}$ is defined, when k is algebraically closed, depending on points $(\lambda_1, \dots, \lambda_t)$ and associated integers $(p_1, \dots, p_t)$ where $p_i > 1$. For each $\mathrm{coh}\,\mathbb{X}$ there is some tilting object T such that $\mathrm{End}(T)^{\mathrm{op}}$ is a canonical algebra, and all canonical algebras occur this way. In particular, the derived equivalence classes of the $\mathrm{coh}\,\mathbb{X}$ and of the categories of finitely generated modules over canonical algebras coincide, when k is an algebraically closed field.

For a canonical algebra Λ over a field k there is the following structure of the indecomposable modules [R2]. They are divided into three groups: $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{I}$, where $\mathrm{Hom}(\mathcal{Q}, \mathcal{P}) = 0 = \mathrm{Hom}(\mathcal{I}, \mathcal{Q}) = \mathrm{Hom}(\mathcal{I}, \mathcal{P})$, $\mathcal{Q}$ is a family of (stable) tubes, and any map $f : P \to I$ with P in $\mathcal{P}$ and I in $\mathcal{I}$ factors through any tube in $\mathcal{Q}$. Consider the additive subcategory $\mathcal{L}$ generated by $\mathcal{P}$ and $\mathcal{Q}$, and $\mathcal{R}$ generated by $\mathcal{I}$. Then $(\mathcal{R}, \mathcal{L})$ is a split torsion pair, and when tilting with respect to this torsion pair, we obtain a hereditary abelian category $\mathcal{C}_\Lambda$, which is derived equivalent to $\mathrm{mod}\,\Lambda$ [HRS]. When k is an algebraically closed field, it follows by using [GL,HRS] that $\mathcal{C}_\Lambda$ is equivalent to $\mathrm{coh}\,\mathbb{X}$ for the associated weighted projective line $\mathbb{X}$. Hence $\mathcal{C}_\Lambda$ is the natural replacement for $\mathrm{coh}\,\mathbb{X}$ in the general case. When starting with Λ tame hereditary, we define the hereditary abelian category $\mathcal{C}_\Lambda$ in the same way as above.

2. The case with objects of finite length

Let as usual $\mathcal{H}$ be a connected Ext-finite hereditary abelian k-category with tilting object, over an arbitrary field k. When k is algebraically closed, we know that $\mathcal{H}$ is noetherian if and only if $\mathcal{H}$ is equivalent to some category $\operatorname{coh}\mathbb{X}$ of coherent sheaves on a weighted projective line or to some category $\operatorname{mod}\Lambda$ for a finite-dimensional hereditary k-algebra Λ [L]. Further any $\mathcal{H}$ is derived equivalent to some $\operatorname{coh}\mathbb{X}$ or to some $\operatorname{mod}\Lambda$ [H2].

In this section we characterize the $\mathcal{H}$ which are noetherian when k is an arbitrary field, and we show that when $\mathcal{H}$ has some simple object, then it is derived equivalent to some $\operatorname{mod}\Lambda$ where Λ is a finite-dimensional hereditary algebra or squid algebra (or equivalently to a hereditary abelian k-category $\mathcal{C}_\Lambda$ where Λ is canonical or to $\operatorname{mod}\Lambda$ for a finite-dimensional hereditary k-algebra).

For these questions it is no restriction to assume that $\mathcal{H}$ is not equivalent to $\operatorname{mod}\Lambda$ for some finite-dimensional hereditary k-algebra Λ. Then we know that if the subcategory $\mathcal{H}_0$ of objects of finite length is nontrivial, it is given by a union of tubes (Lemma 1.2), and up to derived equivalence we can also assume that $(\mathcal{H}_0, \mathcal{H}_\infty) = 0$ by Proposition 1.3.

Rank functions have also previously played an important role in the investigation of hereditary abelian categories with tilting object, when there are nonzero objects of finite length [GL,L,HRe2]. Here they are important for getting criteria for $\mathcal{H}$ to be noetherian. We follow the idea from [L] for the definition.

Let $\mathcal{H}$ be as usual, with $\mathcal{H}$ not equivalent to some $\operatorname{mod}\Lambda$ with Λ hereditary, and $(\mathcal{H}_0, \mathcal{H}_\infty) = 0$ and $\mathcal{H}_0 \neq 0$. Note that by Proposition 1.2 there is only a finite number of tubes in $\mathcal{H}_0$ of rank greater than one. For each tube $\mathcal{T}$ in $\mathcal{H}_0$ let $S_\mathcal{T}$ be the sum of the quasisimple objects in $\mathcal{T}$. The subgroup generated by the $S_\mathcal{T}$ in $K_0(\mathcal{H})$ is finitely generated, by a finite number of the $S_\mathcal{T}$, including the $S_\mathcal{T}$ when $\mathcal{T}$ has rank greater than one. Denote by S their direct sum. Denote by $r : \mathcal{H} \to \mathbb{Z}$ the function given by $r(X) = \dim_k \operatorname{Hom}(X, S) - \dim_k \operatorname{Ext}^1(X, S)$ for X in $\mathcal{H}$. This gives an additive function on $\mathcal{H}$, that is, if $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\mathcal{H}$, then $r(Y) = r(X) + r(Z)$. Hence there is induced a group homomorphism $r : K_0(\mathcal{H}) \to \mathbb{Z}$. Since $\tau S \simeq S$, we have $\operatorname{Ext}^1(X, S) \simeq D\operatorname{Hom}(S, X)$, and hence $\operatorname{Ext}^1(X, S) = 0$ for X in $\mathcal{H}_\infty$. In particular, $r(X) \geqslant 0$ for $X \in \mathcal{H}_\infty$, and it is also easy to see that $r(X) \geqslant 0$ for X in $\mathcal{H}_0$. We want to show that in fact $r(X) > 0$ when X is indecomposable of infinite length. It is convenient to first note the following special case.

Proposition 2.1. *Assume that $\mathcal{H}$ is not derived equivalent to some* $\operatorname{mod}\Lambda$ *for a finite-dimensional hereditary k-algebra Λ, that $\mathcal{H}_0 \neq 0$ and that $(\mathcal{H}_0, \mathcal{H}_\infty) = 0$.*

If E is an exceptional torsionable object of infinite length, then $r(E) > 0$.

Proof. If E is in addition quasisimple, we know from Proposition 1.5 that $\operatorname{Hom}(E, \mathcal{T}) \neq 0$ for any tube $\mathcal{T}$, and consequently $r(E) > 0$. Since it follows from Proposition 1.7 that E has a filtration by quasisimple exceptional torsionable objects, the proof is done. □

In order to extend this result beyond the case of exceptional objects, we need to investigate the case with all tubes having rank one more closely. The idea of the proof is taken from [HRe3].

Proposition 2.2. *Assume that $\mathcal{H}$ is not equivalent to some* $\operatorname{mod}\Lambda$ *for a finite-dimensional hereditary k-algebra Λ, that $\mathcal{H}_0 \neq 0$, and that $(\mathcal{H}_0, \mathcal{H}_\infty) = 0$. Assume in addition that all tubes in $\mathcal{H}_0$ have rank one. Then* $\operatorname{rk} K_0(\mathcal{H}) = 2$, *and $\mathcal{H}$ is derived equivalent to* $\operatorname{mod}\Lambda$, *where Λ is a finite-dimensional hereditary k-algebra associated with a tame bimodule.*

Proof. Assume that $\mathcal{H}$ contains no directing object. Let E be an exceptional torsionable object in $\mathcal{H}$. Since all tubes in $\mathcal{H}_0$ have rank one, no exceptional object lies in $\mathcal{H}_0$, and consequently E has infinite length. Then by Proposition 1.4 the perpendicular category $E^\perp$ is equivalent to $\operatorname{mod} H$ for a basic finite-dimensional hereditary k-algebra H, and $T = H \oplus E$ is a tilting object in $\mathcal{H}$. We have $r(E) > 0$ by Proposition 2.1, and we assume that $r(E) = a > 0$ is smallest possible amongst the exceptional torsionable objects.

Consider the almost split sequence $0 \to \tau E \to M \to E \to 0$ in $\mathcal{H}$. Since $T = E \oplus H$ with E indecomposable, we have $\operatorname{rk} K_0(\mathcal{H}) = n = \operatorname{rk} K_0(\operatorname{mod} H) + 1$. For the H-module M we then have $[M] = t_1[S_1] + \cdots + t_{n-1}[S_{n-1}]$ in $K_0(\operatorname{mod} H)$, where $S_1, \ldots, S_{n-1}$ are the nonisomorphic simple H-modules.

Since $\mathcal{H}$ is not derived equivalent to any $\operatorname{mod}\Lambda$ for Λ a finite-dimensional hereditary k-algebra, the quasitilted algebra $H[M]$ is not tilted, and hence M is a sincere H-module [H1, Proof of Theorem 7.10]. It follows that all t_i are positive. Each S_i is an exceptional H-module, and is hence an exceptional object in $\mathcal{H}$. Further S_i is clearly torsionable in $\mathcal{H}$, since $H \oplus E$ is a tilting object in $\mathcal{H}$. Then we have $r(S_i) \geqslant a$ for $i = 1, \ldots, n-1$, and since clearly $r(M) = 2a$, we get $2a \geqslant t_1 a + \cdots + t_{n-1} a \geqslant (n-1)a$. It follows that $n \leqslant 3$. Since the quiver of a quasitilted algebra has no oriented cycles [HRS], $H[M]$ would be hereditary if $n \leqslant 2$, contradicting the fact that $\mathcal{H}$, which is derived equivalent to $\operatorname{mod} H[M]$, is not derived equivalent to $\operatorname{mod}\Lambda$ for a finite-dimensional hereditary k-algebra Λ. If $n = 3$, the inequalities $2a \geqslant t_1 a + t_2 a \geqslant 2a$ give $t_1 = t_2 = 1$. Hence we have an exact sequence $0 \to S \to M \to T \to 0$ where S and T are the two nonisomorphic simple H-modules and S is projective. Since $r(M) = 2a$ and S and T are exceptional torsionable objects (of infinite length) in $\mathcal{H}$, we have $r(S) = r(T) = r(E) = a$, by the minimality of $r(E) = a$. Applying $(S, \cdot)$ to the exact sequences $0 \to S \to M \to T \to 0$ and $0 \to \tau E \to M \to E \to 0$, we obtain the exact sequences $0 \to (S, S) \to (S, M) \to (S, T)$ and $0 \to (S, \tau E) \to (S, M) \to (S, E) \to \operatorname{Ext}^1(S, \tau E)$. Since $(S, T) = 0$, $(S, \tau E) \simeq D\operatorname{Ext}^1(E, S) = 0$ and $\operatorname{Ext}^1(S, \tau E) \simeq D(E, S) = 0$, we have isomorphisms $(S, S) \tilde{\to} (S, M) \tilde{\to} (S, E)$.

Since $\mathrm{Ext}^1(E, S) = 0$, we know that any nonzero map $f : S \to E$ is either a monomorphism or an epimorphism [HRi].

If $f : S \to E$ is an epimorphism, we have an exact sequence $0 \to K \to S \to E \to 0$. Applying $(S, \cdot)$ to this exact sequence, we get the exact sequence $0 \to (S, K) \to (S, S) \to (S, E) \to \mathrm{Ext}^1(S, K) \to \mathrm{Ext}^1(S, S) = 0$, and hence we see that K is in $S^\perp$ since $(S, S) \to (S, E)$ is an isomorphism. Then K is projective in $S^\perp$ by Proposition 1.4(d), and K is hence torsionable and exceptional. Since also K has infinite length, it follows from Proposition 2.1 that $r(K) > 0$. But this contradicts $r(K) = r(S) - r(E) = a - a = 0$.

Assume now that $f : S \to E$ is a monomorphism. Then we apply $(\cdot, Q)$ to the exact sequence $0 \to S \to E \to Q \to 0$ to get the exact sequence $(S, Q) \to \mathrm{Ext}^1(Q, Q) \to \mathrm{Ext}^1(E, Q)$. Then $\mathrm{Ext}^1(E, Q) = 0$ since $\mathrm{Ext}^1(E, E) = 0$, and it follows from the exact sequence $0 \to (S, S) \tilde{\to} (S, E) \to (S, Q) \to \mathrm{Ext}^1(S, S)$ that $(S, Q) = 0$. Hence we have $\mathrm{Ext}^1(Q, Q) = 0$, so that Q is exceptional. Since Q is a factor of E, it is clearly torsionable. Because Q has infinite length, it follows that $r(Q) > 0$ by Proposition 2.1, contradicting $r(Q) = r(E) - r(S) = 0$. Hence $n = 3$ is also impossible. Then we conclude that $\mathcal{H}$ has a directing object, so that $\mathcal{H}$ is derived equivalent to $\mathrm{mod}\,\Lambda$ for a finite-dimensional hereditary algebra Λ.

Since there are tubes in $\mathcal{H}$, and hence in $D^{\mathrm{b}}(\mathcal{H}) \sim D^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$, we conclude that Λ must be tame. It follows from the classification of tame hereditary k-algebras in [DR] that since all the tubes have rank one, there are exactly two nonisomorphic simple Λ-modules. Hence we must have $\mathrm{rk}\, K_0(\mathcal{H}) = 2$, and consequently Λ is given by a tame bimodule. □

Now we are in the position to prove the desired result on positivity of the rank.

Proposition 2.3. *Assume that $\mathcal{H}$ is not equivalent to some* $\mathrm{mod}\,\Lambda$ *for a finite-dimensional hereditary k-algebra Λ, that $\mathcal{H}_0 \neq 0$ and that $(\mathcal{H}_0, \mathcal{H}_\infty) = 0$. Then $r(X) > 0$ for any indecomposable object X of infinite length in $\mathcal{H}$.*

Proof. Assume first that all tubes in $\mathcal{H}_0$ have rank one. Then it follows from Proposition 2.2 that $\mathcal{H}$ is derived equivalent to $\mathrm{mod}\,\Lambda$ for a finite-dimensional tame hereditary k-algebra Λ. From the structure of such algebras we know that the indecomposable Λ-modules which are not in a tube are preprojective or preinjective [DR]. Further there is a nonzero map from any indecomposable preprojective module to any tube, and from any tube to any nonzero preinjective module. From this it is easy to see that $r(X) > 0$ when X is in $\mathcal{H}_\infty$.

We shall prove our claim by induction on $n = (t_1 - 1) + \cdots + (t_r - 1)$, where $t_1, \ldots, t_r$ are the ranks of the tubes of rank greater than one. The case $n = 0$ has already been taken care of, so assume $n > 0$.

Let X be an indecomposable object in $\mathcal{H}_\infty$. We have that $r(X) \geqslant 0$, and we want to show that $r(X) > 0$. Assume to the contrary that $r(X) =$

$0 = \dim_k \operatorname{Hom}(X, S) - \dim_k \operatorname{Ext}^1(X, S) = \dim_k \operatorname{Hom}(X, S)$. Then $\operatorname{Ext}^1(S, X) \simeq D \operatorname{Hom}(X, S) = 0$ and $\operatorname{Hom}(S, X) = 0$. Let E be an exceptional quasisimple object in a tube $\mathcal{T}$ of $\mathcal{H}_0$. By our assumption on S, we have that $S_{\mathcal{T}}$, and hence E, is a summand of S, and hence $\operatorname{Ext}^1(E, X) = 0 = \operatorname{Hom}(E, X)$. Then X is in $E^{\perp}$, which is an Ext-finite hereditary abelian k-category with tilting object. The tubes in $\mathcal{H}_0$ different from $\mathcal{T}$ all lie in $E^{\perp}$, and $\mathcal{T}$ is replaced by a tube $\mathcal{T}'$ in $E^{\perp}$ with rank $t-1$, where t is the rank of the tube $\mathcal{T}$. The quasisimple objects in $\mathcal{T}'$ are $\tau^2 E, \tau^3 E, \ldots, \tau^{t-1} E, F$, where F is given by a nonsplit exact sequence $0 \to \tau E \to F \to E \to 0$. The other indecomposable objects in $E^{\perp}$ have infinite length in $\mathcal{H}$, and we claim that they also have infinite length in $E^{\perp}$. To see this, assume that Z is of infinite length in $\mathcal{H}$, such that Z lies in $E^{\perp}$ and has finite length in $E^{\perp}$. Then Z has a proper quotient Y which is of infinite length in $\mathcal{H}$ and does not lie in $E^{\perp}$. Hence $\operatorname{Ext}^1(E, Y) \neq 0$ since $\operatorname{Hom}(E, Y) = 0$, but then also $\operatorname{Ext}^1(E, Z) \neq 0$, a contradiction.

Since $\operatorname{rk} K_0(E^{\perp}) = \operatorname{rk} K_0(\mathcal{H}) - 1$, the induction assumption gives $r_{E^{\perp}}(X) > 0$. Then there is some quasisimple object V in a tube of $E^{\perp}$ of objects of finite length such that $\operatorname{Hom}(X, V) \neq 0$. If V is in a tube different from $\mathcal{T}'$, we have a contradiction to $r(X) = 0$. If V is in $\mathcal{T}'$, we get by considering the structure of the quasisimple objects in $\mathcal{T}'$ that $\operatorname{Hom}(X, \mathcal{T}) \neq 0$. So in any case we get a contradiction, and we are done. □

As a consequence of the above we get the following result on noetherianness.

Proposition 2.4. *Let $\mathcal{H}$ be an* Ext-*finite hereditary abelian k-category with tilting object, and assume that $\mathcal{H}_0 \neq 0$ and $(\mathcal{H}_0, \mathcal{H}_\infty) = 0$. Then $\mathcal{H}$ is noetherian.*

Proof. We can clearly assume that $\mathcal{H}$ is not equivalent to $\operatorname{mod} \Lambda$ for a finite-dimensional hereditary k-algebra Λ.

Let X be an indecomposable object in $\mathcal{H}_\infty$. We want to show that X is noetherian by induction on $r(X)$, which is positive by Proposition 2.3. So assume first that $r(X)$ is smallest possible. Assume that $X_0 \subset X_1 \subset \cdots \subset X_i \subset \cdots \subset X$ is an infinite proper chain of subobjects of X. Since $(\mathcal{H}_0, \mathcal{H}_\infty) = 0$, all indecomposable summands of X_i have infinite length, for all i. Hence $r(X_i) = r(X)$ for all i, so that $r(X/X_i) = 0$. Since we have seen that $r \geqslant 0$, it follows that $r(Y_i) = 0$ for each indecomposable summand Y_i of X/X_i, and hence X/X_i has finite length by Proposition 2.3. Since $X/X_0 \to X/X_1 \to \cdots \to X/X_i \to \cdots$ is an infinite chain of proper epimorphisms, we get a contradiction, and hence X is noetherian.

Assume now that X is an indecomposable object in $\mathcal{H}_\infty$ where $r(X) = a$ is not minimal, and assume that if X' is indecomposable in $\mathcal{H}_\infty$ with $r(X') < r(X)$, then X' is noetherian. Assume that $X_0 \subset X_1 \subset \cdots \subset X_i \subset \cdots \subset X$ is a proper ascending chain of subobjects of X. If there is some i_0 such that $r(X_i) = r(X)$ for all $i \geqslant i_0$, it follows as above that X is noetherian. If there is no such i_0, there is some i_1 and some b with $0 < b < a$ such that $r(X_i) = b$ for all $i \geqslant i_1$. We have the proper ascending chain $X_{i_1+1}/X_{i_1} \subset X_{i_1+2}/X_{i_1} \subset \cdots \subset X/X_{i_1}$ of subobjects of X/X_{i_1}. Then $r(X/X_{i_1}) = a - b < a$, so that X/X_{i_1} is noetherian by the induction assumption, and we have a contradiction. □

We shall give a characterization of the noetherian $\mathcal{H}$ at the end of the section. Before we go on we point out the following consequence.

Corollary 2.5. *Let $\mathcal{H}$ be an* Ext*-finite hereditary abelian k-category with tilting object. If $\mathcal{H}_0 \neq (0)$, then $\mathcal{H}$ is derived equivalent to a noetherian hereditary abelian k-category.*

Proof. We can clearly assume that $\mathcal{H}$ is not equivalent to $\operatorname{mod}\Lambda$ for some finite-dimensional hereditary k-algebra Λ. Then up to derived equivalence we can assume $(\mathcal{H}_0, \mathcal{H}_\infty) = 0$ by Proposition 1.3, and hence we are done by Proposition 2.4. □

Now we get our main result in the case that $\mathcal{H}$ has some simple object, or equivalently $\mathcal{H}_0 \neq 0$.

Theorem 2.6. *Let $\mathcal{H}$ be an* Ext*-finite hereditary abelian k-category with tilting object and assume that $\mathcal{H}_0 \neq (0)$ and $(\mathcal{H}_0, \mathcal{H}_\infty) = 0$.*

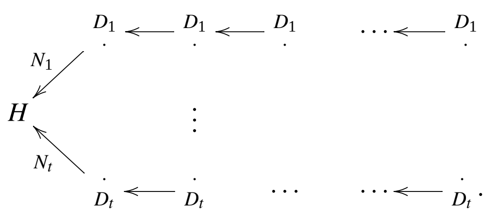
(a) *There is some tilting object T in $\mathcal{H}$ such that* $\operatorname{End}_{\mathcal{H}}(T)^{\mathrm{op}}$ *is a squid algebra or a hereditary algebra.*
(b) *$\mathcal{H}$ is derived equivalent to the category of finitely generated modules for a squid algebra or a hereditary algebra (and to some hereditary k-category $\mathcal{C}_\Lambda$ associated to a canonical k-algebra Λ or to* $\operatorname{mod}\Lambda$ *for a finite-dimensional hereditary k-algebra Λ).*

Proof. We can clearly assume that $\mathcal{H}$ is not equivalent to $\operatorname{mod}\Lambda$ for some finite-dimensional hereditary k-algebra Λ, so that $\mathcal{H}_0$ is given by a union of tubes. Let $(t_1, \ldots, t_r)$ be the ranks of the tubes in $\mathcal{H}_0$ of rank greater than one. We claim that $\mathcal{H}$ has a tilting object of the form $T_0 \oplus T_1 \oplus \cdots \oplus T_r$, where add T_0 is in $\mathcal{H}_\infty$ and $\operatorname{End}(T_0)^{\mathrm{op}}$ is a tame hereditary k-algebra given by a tame bimodule. Further T_i has $t_i - 1$ indecomposable summands for $1 \leqslant i \leqslant r$ and $T_i = C_i^{(1)} \oplus \cdots \oplus C_i^{(t_i-1)}$, where there is a chain of irreducible epimorphisms $C_i^{(1)} \to \cdots \to C_i^{(t_i-1)}$, with $C_i^{(t_i-1)}$ quasisimple.

$\mathcal{H}$ is derived equivalent to $\operatorname{mod}\Lambda$ for some quasitilted algebra Λ [HRS]. If $K_0(\mathcal{H}) \simeq K_0(\operatorname{mod}\Lambda)$ had rank one, Λ would be a simple algebra, and hence there would be no tubes for $\mathcal{H}$. We prove the claim by induction on $\operatorname{rk} K_0(\mathcal{H}) = n \geqslant 2$. If $n = 2$, then Λ is hereditary since the quiver of Λ has no oriented cycles [HRS], and since there are tubes, Λ must be tame. Since $\operatorname{rk} K_0(\operatorname{mod}\Lambda) = 2$, all tubes in $\mathcal{H}_0$ have rank one, and we have a tilting object $T = T_0$ in $\mathcal{H}_\infty$ with $\operatorname{End}(T_0)^{\mathrm{op}} \simeq \Lambda$.

Assume that $n > 2$, and let E be a quasisimple object in one of the tubes for $\mathcal{H}_0$ of rank greater than 1, say the first one (of rank t_1). Then E is exceptional, so that $E^\perp$ is a hereditary abelian k-category with tilting object, by Proposition 1.3, and $\operatorname{rk} K_0(E^\perp) = \operatorname{rk} K_0(\mathcal{H}) - 1$. The family $\mathcal{H}'_0$ of objects of finite length in $E^\perp$ consists of the same tubes as $\mathcal{H}_0$ except for the tube containing E which is replaced by a smaller tube of rank $t_1 - 1$. (See the proof of Proposition 2.3.) We also still have $(\mathcal{H}'_0, \mathcal{H}'_\infty) = 0$. By the induction assumption we have a tilting object $T' = T'_0 \oplus T'_1 \oplus \cdots \oplus T'_r$, where T'_1 is 0 if $t_1 = 2$, with the desired properties. If necessary, replace T' by some $\tau^j_{E^\perp} T'$ in order to have a chain of irreducible epimorphisms $C_1^{(1)} \to \cdots \to C_1^{(t_1-2)} \to E$ in $\mathcal{H}$. We know that $T' \oplus E$ is a tilting object, so we are done with the claim.

We now want to show that for our special choice of tilting object T in $\mathcal{H}$ we have that $\operatorname{End}(T)^{\mathrm{op}}$ is a squid algebra or a hereditary algebra. We have that $\operatorname{End}(T_0)^{\mathrm{op}} = H$ is a tame hereditary algebra given by a tame bimodule. Let $N_i = \operatorname{Hom}(T_0, C_i^{(t_i-1)})$, which is a H–D_i bimodule, where $D_i = \operatorname{End}(C_i^{(t_i-1)})^{\mathrm{op}}$ is a division algebra since $C_i^{(t_i-1)}$ is quasisimple. We have $\operatorname{Hom}(C_i^{(j)}, C_i^{(j+1)} \simeq D_i$ as a D_i-bimodule, and $\operatorname{Hom}(T_i, T_{i'}) = 0$ for i, i' not 0. Further $\operatorname{Hom}(T_i, T_0) = 0$ for $i \neq 0$. Hence we get a squid algebra given by



This finishes the proof of (a), and (b) is a direct consequence of (a). □

Observe also the following direct consequence of the above proof.

Corollary 2.7. *Let $\mathcal{H}$ be a hereditary abelian* Ext-*finite k-category with tilting object and $\mathcal{H}_0 \neq 0$ and $\mathcal{H}_\infty \neq 0$. Let $(t_1, \ldots, t_r)$ be the ranks of the tubes in $\mathcal{H}_0$ of rank greater than one. Then* $\operatorname{rk} K_0(\mathcal{H}) = 2 + \sum_{i=1}^{r}(t_i - 1)$.

Using Theorem 2.6, we now obtain characterizations of noetherian Ext-finite hereditary abelian k-categories with tilting object.

Theorem 2.8. *Let $\mathcal{H}$ be a hereditary abelian* Ext-*finite k-category with tilting object. Then the following are equivalent.*

(a) $\mathcal{H}$ *is noetherian.*
(b) $\mathcal{H}_0 \neq 0$ *and* $(\mathcal{H}_0, \mathcal{H}_\infty) = 0$.
(c) $\mathcal{H}$ *is equivalent to* $\mathcal{C}_\Lambda$ *for some canonical* k*-algebra* Λ*, or to* $\operatorname{mod} \Lambda$ *for some finite-dimensional hereditary* k*-algebra* Λ.

Proof. (b) $\Rightarrow$ (a). Follows from Proposition 2.4.

(b) $\Rightarrow$ (c). Since $\mathcal{H}_0 \neq 0$, $\mathcal{H}$ is derived equivalent to $\mathcal{C}_\Lambda$, where Λ is canonical or tame hereditary, by Theorem 2.6. If $\mathcal{H}$ is not equivalent to some $\operatorname{mod} \Lambda$ where Λ is hereditary, then $\mathcal{H}_0$ is given by a union of tubes, with $(\mathcal{H}_0, \mathcal{H}_\infty) = 0$. Also $\mathcal{C}_\Lambda$ has the same property. Since $\mathcal{C}_\Lambda$ and $\mathcal{H}$ are derived equivalent, there is some split torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{C}_\Lambda$, such that we obtain $\mathcal{H}$ when tilting with respect to this pair in the sense of [HRS] (see [H1]). In view of the above properties of $\mathcal{H}$ and Proposition 2.3, it is easy to see that the pair $(\mathcal{T}, \mathcal{F})$ is trivial, so that $\mathcal{H}$ and $\mathcal{C}_\Lambda$ are equivalent.

(c) $\Rightarrow$ (b). For $\mathcal{H} = \mathcal{C}_\Lambda$ we have $\mathcal{H}_0 \neq 0$ and $(\mathcal{H}_0, \mathcal{H}_\infty) = 0$, and for $\mathcal{H} = \operatorname{mod} \Lambda$ for a finite-dimensional hereditary k-algebra Λ we have $\mathcal{H}_0 = \mathcal{H}$.

(a) $\Rightarrow$ (b). When $\mathcal{H}$ is noetherian, then clearly $\mathcal{H}_0 \neq 0$. Assume that $(\mathcal{H}_0, \mathcal{H}_\infty) \neq 0$. Then there is some nonzero map $f : X \to Y$ with X in $\mathcal{H}_0$ and Y in $\mathcal{H}_\infty$, and X and Y indecomposable. We can clearly assume that X is quasisimple. If f was not a monomorphism, then $\operatorname{Ker} f$ had smaller length than X, with $\operatorname{Hom}(\operatorname{Ker} f, X) \neq 0$, so there would be some quasisimple object Y in a tube, with $\operatorname{Hom}(Y, X) \neq 0$, which is impossible. Hence f is a monomorphism. Consider the almost split sequence $0 \to X \to E \to \tau^{-1}X \to 0$. Then E is indecomposable, and there is some nonzero map $f' : E \to Y$ extending f. If f' is not a monomorphism, there is a proper nonzero monomorphism $\operatorname{Ker} f' \to \tau^{-1}X$, which is impossible since $\tau^{-1}X$ is quasisimple. We can continue this way to get a proper ascending chain of subobjects of Y, so that $\mathcal{H}$ is not noetherian. □

3. The main result

In this section we finish the proof of the main result. We first show, as in [H2] for the case of algebraically closed fields, that we can reduce to the three cases:

(1) $\mathcal{H}$ has a simple object;
(2) $\mathcal{H}$ has a directing object;
(3) there exists an indecomposable torsionable exceptional object $E \in \mathcal{H}_\infty$ with $E^\perp$ tame hereditary.

Then we consider each of the three cases.

Most of the proof in [H2] carries over with no change. It is important to note that the result from [B] that if H is a wild hereditary finite-dimensional k-algebra and M and N are indecomposable regular H-modules, then there is some i_0 such that $\operatorname{Hom}(M, \tau^i N) \neq 0$ for $i \geqslant i_0$, holds for an arbitrary field k. Also Proposition 1.6 (see [HS]) is used.

The following lemma, whose proof is trivial in the case of algebraically closed fields, remains valid for arbitrary fields with some additional argument.

Lemma 3.1. *Let E be a quasisimple exceptional object of infinite length in $\mathcal{H}$, and assume that $\mathcal{H}$ has no directing objects. Then we have* $\operatorname{Hom}(\tau E, E) = 0$.

Proof. Since E is exceptional, $\operatorname{End}(E) = F$ is a division algebra. Assume that there is some nonzero map $f : \tau E \to E$. Since $\operatorname{Ext}^1(E, \tau E) \simeq D \operatorname{Hom}(E, E) \simeq F$, there is only one nontrivial extension $0 \to \tau E \to L \to E \to 0$ up to isomorphism, with the given end terms, and this is the almost split sequence with right hand term E. If f is neither a monomorphism nor an epimorphism, there is an exact sequence $0 \to \tau E \to \operatorname{Im} f \oplus K \to E \to 0$ [HRi], which is a contradiction.

So we can assume that $f : \tau E \to E$ is a monomorphism or an epimorphism. If f is an epimorphism, we get by the exactness of the functor τ epimorphisms $\tau^{i+1} E \to \tau^i E$ for $i \geqslant 1$, and hence an epimorphism $f_j : \tau^j E \to E$ for all $j \geqslant 1$. We then get by applying $(\tau^j E, \cdot)$ an epimorphism $\operatorname{Ext}^1(\tau^j E, \tau^j E) \to \operatorname{Ext}^1(\tau^j E, E)$, and hence $\operatorname{Ext}^1(\tau^j E, E) = 0$ for $j \geqslant 1$. Since also $\operatorname{Ext}^1(E, E) = 0$, we get $\operatorname{Hom}(E, \tau^i E) = 0$ for $i \geqslant 1$.

If $f : \tau E \to E$ is a monomorphism, we get a monomorphism $f_j : \tau^j E \to E$ for $j \geqslant 1$. Applying $(\cdot, E)$ we get an epimorphism $\operatorname{Ext}^1(E, E) \to \operatorname{Ext}^1(\tau^j E, E)$, so that $\operatorname{Ext}^1(\tau^j E, E) = 0$ for $j \geqslant 1$, since $\operatorname{Ext}^1(E, E) = 0$. Hence we conclude that $\operatorname{Hom}(E, \tau^i E) = 0$ for all $i \geqslant 1$. Then we know that in any case there is a chain of irreducible epimorphisms $E_i \to \cdots \to E_0 = E$ between exceptional objects [H2, proof of Corollary 2.11], which gives a contradiction by [H2, Lemma 2.1]. □

We shall also need the following proposition.

Proposition 3.2. *Let H be a tame hereditary k-algebra, and M a simple regular H-module. Then $\Lambda = H[M]$ is derived equivalent to a squid algebra or a hereditary algebra.*

Proof. It is not hard to see that there exists a tilting module T in $\operatorname{mod} H$ such that $\operatorname{End}_H(T)^{\mathrm{op}}$ is a squid algebra or a hereditary algebra. One can here use arguments similar to those given in the proof of Theorem 2.6. In particular we use induction of the rank n of $K_0(\operatorname{mod} H)$. If $n = 2$, we have the tilting module H, so that $H \simeq \operatorname{End}(H)^{\mathrm{op}}$ is tame hereditary. Also note that in this setting we choose T_0 to be preprojective (replacing that all summands are in $\mathcal{H}_\infty$ in the proof of Theorem 2.6). Denote by w the extension vertex for the one-point extension algebra $H[M]$. Let $P(w)$ be the corresponding indecomposable projective Λ-module, and let $\widetilde{T} = T \oplus P(w)$. We want to show that $\widetilde{T}$ is a tilting module in mod Λ, and that $\mathrm{End}(\widetilde{T})^{\mathrm{op}}$ is a squid algebra (or a hereditary algebra).

We divide the investigation into two different cases. Assume first that M lies in a tube of rank one. Since the indecomposable summands of T are preprojective or lie in tubes of rank greater than one, we have $\mathrm{Ext}^1(T, M) \simeq D\overline{\mathrm{Hom}}(M, DTrT) = 0$, and hence M is in $\mathrm{Fac}_H T$. We claim that $\mathrm{Ext}^1(T, P(w)) = 0$. We have the exact sequence $0 \to M \to P(w) \to S(w) \to 0$, where $S(w)$ is the simple Λ-module associated with the vertex w. Applying $(T, \cdot)$ to this exact sequence we get the exact sequence

$$\big(T, S(w)\big) \to \mathrm{Ext}^1(T, M) \to \mathrm{Ext}^1\big(T, P(w)\big) \to \mathrm{Ext}^1\big(T, S(w)\big).$$

Since $S(w)$ is a simple injective Λ-module, we have $\mathrm{Ext}^1(T, S(w)) = 0$, and since M is in $\mathrm{Fac}\, T$, we have $\mathrm{Ext}^1(T, M) = 0$. It follows that $\mathrm{Ext}^1(T, P(w)) = 0$, and hence clearly $\mathrm{Ext}^1(\widetilde{T}, \widetilde{T}) = 0$.

We have $T = T_0 \oplus T_1 \oplus \cdots \oplus T_r$ with $\mathrm{End}(T_0)^{\mathrm{op}}$ corresponding to a tame bimodule and the T_i corresponding to the r arms of the squid. We have $\mathrm{Hom}(P(w), T) = 0$ and $\mathrm{Hom}(T, P(w)) = \mathrm{Hom}(T, M)$. Since M is in a homogeneous tube, we have $\mathrm{Hom}(T_i, M) = 0$ for $i = 1, \ldots, r$. Hence for the quiver of $\mathrm{End}(\widetilde{T})^{\mathrm{op}}$ we get a new arrow to the vertex corresponding to $\mathrm{End}(T_0)^{\mathrm{op}}$, equipped with the bimodule $\mathrm{Hom}(T_0, P(w)) = \mathrm{Hom}(T_0, M)$, and $\mathrm{End}(M)^{\mathrm{op}}$ is associated with the new vertex. It follows that $\mathrm{End}(\widetilde{T})^{\mathrm{op}}$ is a squid algebra in this case.

Assume now that M is quasisimple in a tube of rank greater than one. By possibly applying a power of τ we can assume that M is a summand of T, say of T_i, for some i with $1 \leqslant i \leqslant r$. As before, let $\widetilde{T} = T \oplus P(w)$. The proof is as above, observing that now $\mathrm{Ext}^1(T, M) = 0$ since M is a summand of T. We have $\mathrm{Hom}(M, P(w)) = \mathrm{Hom}(M, M)$, a division algebra, so we get a squid algebra with the arm corresponding to T_i prolonged with one arrow. □

We shall also need the following lemma.

Lemma 3.3. *Let E be an exceptional object in $\mathcal{H}_\infty$, and let E_0 be quasisimple exceptional such that there is a chain of irreducible monomorphisms $E_0 \to \cdots \to E$. If $E^\perp$ is equivalent to* mod H *for H tame hereditary, then $E_0^\perp$ is also tame.*

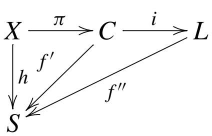
Proof. We have $E^\perp = \mathrm{mod}\, H' \times \mathrm{mod}\, A$, where A is a hereditary algebra of type A_n for some n. In fact, $\mathrm{mod}\, A$ corresponds to the wing in $\mathcal{H}$ determined by the objects $E_0, \ldots, E_{n-1}$ in the chain of irreducible monomorphisms $E_0 \to \cdots \to E_{n-1} \to E$. For details we refer to [HS] or [H2]. We have that E_0 is in $\mathrm{mod}\, \Lambda$, and hence $\mathrm{mod}\, H' \subset E_0^\perp$. Let S be simple regular in a homogeneous tube of $\operatorname{mod} H$, not containing M, where $0 \to \tau E \to M \to E \to 0$ is almost split. We then have an almost split sequence

$$0 \to S \to \begin{matrix} S \\ S \end{matrix} \to S \to 0$$

in $\operatorname{mod} H$. We claim that this sequence is also almost split in $E_0^{\perp}$, hence $E_0^{\perp}$ is tame.

To see this, let $h : X \to S$ be a nonisomorphism where X is indecomposable in $E_0^{\perp}$. Consider the minimal right add E-approximation $f : \widetilde{E} \to X$. We have the exact sequence $\widetilde{E} \to X \xrightarrow{\pi} C \to 0$. Then clearly $\operatorname{Hom}(E, C) \simeq \operatorname{Ext}^1(E, \operatorname{Im} f) = 0$ since E is exceptional and $\mathcal{H}$ is hereditary. If $\operatorname{Ext}^1(E, C) = 0$, then C is in $E^{\perp}$. If $\operatorname{Ext}^1(E, C) \neq 0$, consider the universal extension $0 \to C \to L \to E' \to 0$, with $E' \in \operatorname{add} E$. Then we have the exact sequence $0 \to (E, C) \to (E, L) \to (E, E') \to \operatorname{Ext}^1(E, C) \to \operatorname{Ext}^1(E, L) \to \operatorname{Ext}^1(E, E') \to 0$, where $\alpha : (E, E') \to \operatorname{Ext}^1(E, C)$ is an epimorphism by construction of the universal extension. Then $\operatorname{Ext}^1(E, L) = 0$ since $\operatorname{Ext}^1(E, E') = 0$. Since $\operatorname{End}(E)$ is a division ring, because E is exceptional, it follows that α is an isomorphism. Then $(E, L) = 0$ since $(E, C) = 0$. Hence L is in $E^{\perp}$.

Applying $(\cdot, S)$ to the exact sequence $0 \to C \xrightarrow{i} L \xrightarrow{j} E' \to 0$ gives the exact sequence $0 \to (E', S) \to (L, S) \to (C, S) \to \operatorname{Ext}^1(E', S)$. Here we have $\operatorname{Ext}^1(E', S) = 0$ (and $(E', S) = 0$). Since $(E, S) = 0$ we then have a commutative diagram



We claim that $f'' : L \to S$ is not a split epimorphism.

Assume to the contrary that there is a homomorphism $g : S \to L$ with $f''g = 1_S$.

First we show that $(S, E) = 0$. In fact, the almost split sequence $0 \to \tau E \to N \oplus E_{n-1} \to E \to 0$ gives by applying $(S, \cdot)$, and using that $S \not\simeq E$, rise to the exact sequences $0 \to (S, \tau E) \to (S, N) \oplus (S, E_{n-1}) \to (S, E) \to 0$ and $0 \to \operatorname{Ext}^1(S, \tau E) \to \operatorname{Ext}^1(S, N) \oplus \operatorname{Ext}^1(S, E_{n-1}) \to \operatorname{Ext}^1(S, E) \to 0$. Since S and E_{n-1} lie in different connected components of $E^{\perp}$, we have $(S, E_{n-1}) = 0 = \operatorname{Ext}^1(S, E_{n-1})$. Since S and M lie in different tubes of $\operatorname{mod} H$, we have $(S, M) = 0 = \operatorname{Ext}^1(S, M)$. Then it follows that $(S, E) = 0$ and $\operatorname{Ext}^1(S, E) = 0$.

Thus there is some $g' : S \to C$ with $ig' = g$, and hence $f'g' = 1_S$. The exact sequence $0 \to \operatorname{Im} f \to X \to C \to 0$ gives rise to the exact sequence $0 \to (S, \operatorname{Im} f) \to (S, X) \to (S, C) \to \operatorname{Ext}^1(S, \operatorname{Im} f)$, where $\operatorname{Ext}^1(S, \operatorname{Im} f) = 0$ since $\operatorname{Ext}^1(S, E) = 0$. So if $f'' : L \to S$ is a split epimorphism, then $f' : C \to S$ is a split epimorphism since $(S, E) = 0$, and $h : X \to S$ is a split epimorphism since

$\operatorname{Ext}^1(S, \operatorname{Im} f) = 0$. This is a contradiction, so we conclude that $f'' : L \to S$ is not split epimorphism, and hence

$$0 \to S \to \begin{matrix} S \\ S \end{matrix} \to S \to 0$$

is almost split in $E_0^{\perp}$. □

The proof of the following crucial result is the same as in [H2].

Theorem 3.4. *Let $\mathcal{H}$ be a hereditary abelian* Ext-*finite k-category with tilting object. Then one of the following cases occurs.*

(i) *$\mathcal{H}$ has some simple object.*
(ii) *$\mathcal{H}$ has some directing object.*
(iii) *There exists a torsionable exceptional object E in $\mathcal{H}_\infty$ with $E^{\perp}$ equivalent to* $\operatorname{mod} H$ *for a finite-dimensional tame hereditary k-algebra H.*

Putting things together we get the main result.

Theorem 3.5. *Let $\mathcal{H}$ be a hereditary abelian* Ext-*finite k-category with tilting object. Then $\mathcal{H}$ is derived equivalent to the category of finitely generated modules over a hereditary algebra or a squid algebra.*

Proof. By Theorem 3.4 we only have to consider the cases (i)–(iii). If $\mathcal{H}$ has some simple object, we use Theorem 2.6, and if $\mathcal{H}$ has some directing object, then $\mathcal{H}$ is derived equivalent to a hereditary algebra [HRe1]. Assume now that cases (i) and (ii) do not occur, and there is some indecomposable torsionable exceptional object E in $\mathcal{H}_\infty$ with $E^{\perp}$ equivalent to $\operatorname{mod} H$ for a finite-dimensional tame hereditary k-algebra H. By Lemma 3.3 we can assume that E is quasisimple. We know that $T = H \oplus E$ is a tilting object in $\mathcal{H}$, and $\mathcal{H}$ is derived equivalent to $\operatorname{mod} H[M]$ for the one-point extension $H[M]$, where $0 \to \tau E \to M \to E \to 0$ is an almost split sequence in $\mathcal{H}$. Since M is indecomposable and $H[M]$ is quasitilted and not tilted, M is a simple regular H-module [HRS]. The $H[M]$, and consequently $\mathcal{H}$, is derived equivalent to a squid algebra by Proposition 3.2. □

In [H3] some consequences were drawn of the main theorem in the algebraically closed case. Most of these generalize to the case of arbitrary fields. Here we just include a sample of these results.

For a quasitilted algebra Λ, denote as usual by $\mathcal{L}$ the additive subcategory of $\operatorname{mod} \Lambda$ whose indecomposable objects have the property that all predecessors have projective dimension at most one. Dually $\mathcal{R}$ denotes the additive subcategory of $\operatorname{mod} \Lambda$ whose indecomposable objects have the property that all successors have injective dimension at most one.

Corollary 3.6. *For an arbitrary quasitilted algebra Λ we have that* $\operatorname{ind}\mathcal{L} \cap \operatorname{ind}\mathcal{R}$ *is not empty.*

Corollary 3.7. *Let Λ be a quasitilted algebra. Then there is always an indecomposable Λ-module M such that $\Lambda[M]$ is quasitilted.*

Acknowledgment

We thank the referee for helpful and constructive comments.

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