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Preface

Every connected graph gives rise to a metric space on its vertex set, where the metric is given by the path distance in the graph. This connection enriched in many ways both the theory of metric spaces and the graph theory. Graph metrics constitute a fundamental class of discrete metric spaces, from which the theory of metric spaces draws many of its examples. Conversely, many metric concepts and methods found their way into the graph theory and became powerful tools in the hands of graph theorists. Convexity, geodesic paths, intervals are but a few examples of these.

Many classical graph theoretical concepts have metric flavour. For example, the Cartesian product of graphs can be viewed as the realization of the sum of the graph metrics of the factors. In the same spirit, the strong product of graphs realizes the maximum of the graph metrics of the factors, which is a second natural way to define a metric on the Cartesian product of two metric spaces.

Furthermore, the metric point of view leads to distinguishing classes of graphs with particularly nice properties of the associated metrics. Examples of these are the classes of ℓ_p graphs, out of which most prominent so far is the class of ℓ_1 graphs. In the bipartite case, this class is represented by the partial cubes, which received a lot of attention in recent years.

The combination of the graph theory setup with the metric point of view found applications in other parts of mathematics and even outside of mathematics. In organic chemistry, where the structure of a molecule can be represented by a connected graph, this point of view led to the development of a number of numerical invariants (or indices) which correlate with various physical and chemical properties of the molecule. One of the better-known such invariants is the Wiener index of the graph, defined as the half of the sum of all mutual distances between the vertices, but there are also a number of other important indices defined in terms of distances and paths.

The applicable character of the metric graph theory calls for the development of effective algorithms for the computation of graph invariants and indices, product decompositions, and other metric related parameters and constructions. So the algorithmic aspect is a very important part of the metric graph theory.

The purpose of this special issue is to bring together selected and at the same time representative papers on metric aspects of contemporary graph theory. Roughly, the areas covered are: classical metric concepts, convexity issues, metrically defined classes of graphs, algorithmic aspects of graph metrics, applications in chemical graph theory, metric graph theory on graph products, and distances in graphs related to other central graph theory notions.

The paper of Koolen, Lesser, and Moulton on optimal realizations of 5-point metrics in weighted graphs covers a central, traditional area of metric graph theory. They show that the metric cone C_5 has a natural subdivision into subcones such that, in the interior of each subcone, each metric has a unique optimal realization. In particular, this means that almost all 5-point metrics have a unique optimal realization. Furthermore, it is proven that all of these optimal realizations come from the weighted versions of just three particular graphs.

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Yet another central notion of metric spaces is the interval function, and it is the focus of the paper of Mulder and Nebeský. With a novel approach, they reprove an earlier result of Nebeský on the axioms for the interval functions of connected graphs. Furthermore, it is shown that these axioms are minimal (that is, optimal) in some precise sense. They also obtain new characterizations of the interval functions of modular and median graphs.

The convexity issues are covered with two papers. Oellermann, Nielsen, and Henning investigate graphs satisfying some standard local convexity properties, where the usual convexity is substituted with the stronger 3-Steiner convexity. For two local convexity conditions, they obtain explicit characterizations of all such graphs in terms of forbidden subgraphs.

Polat and Sabidussi introduce and study the pre-hull number of a convexity space X, which measures the intrinsic nonconvexity of X. Then they apply this concept to the convexity spaces coming from bipartite graphs. They show, in particular, that median graphs have pre-hull number ≤ 1 and that partial cubes can have arbitrary large pre-hull number.

One of the most fruitful areas of metric graph theory is formed by the metrically defined classes of graphs. This area is represented here by several papers. Beaudou, Gravier, and Meslem deal with the class of partial Hamming graphs, which is a subclass of ℓ_1 graphs and a generalization of partial cubes. They determine all partial Hamming graphs among the subdivisions of the complete graphs and wheel graphs.

The paper of Polat continues the study of netlike partial cubes, which generalize the classes of median graphs, benzenoid graphs, even cycles, and some other well-known classes of partial cubes. In these he includes infinite graphs as well. Assuming that the netlike partial cube contains no isometric ray, it is shown that the graph contains a convex cycle or a finite subhypercube that is fixed by every symmetry of the graph. As a consequence, it is also shown that every self-contraction of a netlike partial cube fixes a convex cycle or a finite subhypercube.

Brešar and Tepeh-Horvat study graphs that can be obtained by a sequence of gated amalgamations from Cartesian products of chordal graphs. They give various characterizations of this new class of graphs and prove several tree-like equalities generalizing similar results for chordal and median graphs.

Finally, Deza and Shpectorov classify finite polyhexes (that is, trivalent surface graphs with hexagonal faces) that are ℓ_1 graphs. It turns out that there is a single infinite series of such polyhexes coming from the n-prisms, together with just four sporadic examples.

This issue has two papers dealing with algorithmic aspects. Imrich and Kovše describe an efficient (with linear complexity) algorithm that computes the isometric embedding of a (finite) tree into the integer lattice of the smallest dimension. Note that the graphs allowing such isometric embeddings are precisely the partial cubes.

Motivated by problems from theoretical biology, Hellmuth, Imrich, Klöckl and Stadler introduce and study approximate graph products, which are graphs which are close (in a suitable sense) to the Cartesian and strong products. The main part of the paper is concerned with factorization algorithms for approximate products.

We have already mentioned that the metric graph theory found a very rich field of applications in chemical graph theory where distance-based graph invariants play a leading role. This application aspect is represented in our issue by two papers.

Heydari and Taeri obtain explicit formulas for the value of the Szeged index of a particular two-parameter class of nanotube graphs. The Szeged index of a graph and its variations are currently among the central topics in the studies of topological indices in mathematical chemistry.

Khalifeh, Yousefi-Azari, Ashrafi, and Wagner discuss a new interesting topological index, called the edge Wiener index. They relate this index to other known indices and find inequalities for the value of the edge Wiener index. They further provide counterexamples to a recent conjecture by Gutman and Ashrafi and establish explicit formulas for the new index for several classes of chemically relevant graphs.

We have already mentioned one paper, by Hellmuth, Imrich, Klöckl and Stadler, that deals with graph products. There are three further papers that belong in the same area. Hammack proves that if the Cartesian product of two bipartite graphs has isomorphic connected components then at least one of the factors must possess an automorphism switching the parts of its bipartition. This solves the 10+-year-old conjecture of Jha, Klavžar, and Zmazek.

A connected graph is strongly distance-balanced if for every edge uv and every $i \geq 0$ the number of vertices y with $d(y, u) = i = d(y, v) - 1$ is equal to the number of vertices with $d(y, u) - 1 = i = d(y, v)$. Balakrishnan, Changat, Peterin, Špacapan, Šparl, and Subhamathi prove that the strong product of two graphs is strongly distance-balanced if and only if so are both factors. A similar result on the direct product of bipartite graphs is also obtained.

Cartesian graph bundles generalize Cartesian products as well as covering graphs. Banič, Erveš, and Žerovnik show that the edge connectivity of a graph bundle is bounded from below by the sum of edge connectivities of the two graphs, the base and the fiber, involved in the bundle condition. This natural result is then used to study edge fault-diameter (this concept comes from the theory of computer networks) of Cartesian graph bundles.

Graph colourings and distances interact in the paper of Czabarka, Dankelmann, and Szekely, in which the diameter of a 4-colourable graph is bounded from above in terms of the number of vertices and minimum degree.

A related concept of a packing colouring of a graph was recently introduced with motivation from frequency planning. This concept, asking for a partition of the vertex set of the graph into disjoint classes X_i, where vertices in X_i are pairwise at distance more than i, is studied on infinite product graphs by Fiala, Klavžar, and Lidecky.

In conclusion, we hope that the special issue reflects the state of the art of the metric graph theory and highlights the active areas of research in the field.

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