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Hochschild cohomology of incidence algebras as one-point extensions ☆

María Andrea Gatica *, María Julia Redondo

Instituto de Matemática, INMABB, Universidad Nacional del Sur, Av. Alem 1253, (8000) Bahía Blanca, Argentina

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Abstract

The aim of this paper is to compute the Hochschild cohomology groups of particular classes of algebras associated to partially ordered sets.

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Let k be an algebraically closed field and A a finite dimensional k-algebra associative with an identity. The Hoschschild cohomology groups $\mathrm{H}^i(A, X)$ of an algebra A with coefficients in an A-bimodule X were introduced by Hochschild in [9]. When $X = A$ we write $\mathrm{H}^i(A)$ instead of $\mathrm{H}^i(A, A)$ and $\mathrm{H}^i(A)$ is called the ith Hochschild cohomology group of A.

Let $P = (P, \leqslant)$ be a finite partially ordered set (called poset). We may assume that $P = \{1, \ldots, n\}$. With P we may associate its *incidence algebra* $I(P)$. This is by definition the subalgebra of the algebra of $(n \times n)$-matrices over k with elements $(x_{ij}) \in M_n(k)$ satisfying $x_{ij} = 0$ if $i \nleqslant j$. The Hochschild cohomology groups of incidence algebras have been studied in [3–5,7,8,10]. It is known [4,10] that the

☆ The first author has a fellowship from CONICET and the second author is a researcher from CONICET.
* Corresponding author.
E-mail addresses: agatica@criba.edu.ar (M.A. Gatica), mredondo@criba.edu.ar (M.J. Redondo).

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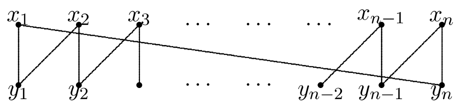
Hochschild cohomology groups of an incidence algebra vanish if the associated poset does not contain crowns, that is, full subcategories of the form



where the intersection of the convex hull of $\{x_i, y_i\}$ and the convex hull of $\{x_i, y_{i-1}\}$ is $\{x_i\}$ (resp. the intersection of the convex hulls of $\{x_i, y_i\}$ and $\{x_{i+1}, y_i\}$ is $\{y_i\}$) for $1 \leqslant i \leqslant n$, with $y_0 = y_n, x_{n+1} = x_1, n \geqslant 2$.

Therefore we are interested in those algebras whose associated posets contain crowns. In particular we consider the algebras $A^j_{qn+s}, q \geqslant 0,\ n \geqslant 3,\ 0 \leqslant s < n$ and $0 < j < n$, associated to posets P^j_{qn+s} which are defined in the following way

$$P^j_{qn+s} = P_{qn+s} \cup \{(qn + s + 1, p) :\ 1 \leqslant p \leqslant j\}$$

with $P_{qn+s} = [qn + s] \times [n - 1],$ where $[m] = \{0, \dots, m\}$, and the partial order is defined by $(l, t) < (l + 1, t)$ and $(l, t) < (l + 1, t + 1)$ with $(l, n) = (l, 0)$. Observe that the poset P_{qn+s} is a superposition of $qn + s$ crowns of the same width n.

The purpose of this paper is to compute the Hochschild cohomology groups of the incidence algebras A^j_{qn+s}, $1 \leqslant j \leqslant n - 1$. In order to do this we use an inductive method due to Happel [8] and the results obtained in [5] for the incidence algebras $A_{qn+s} = I(P_{qn+s})$ and for the algebras associated to posets with a unique maximal element.

The article is organized in two sections. In the first one, we briefly recall the definitions and results that will be needed throughout this paper. The second section is devoted to compute the Hochschild cohomology groups of the algebras A^j_{qn+s}.

1. Preliminaries

1.1. Notation

Throughout this paper k denotes a fixed algebraically closed field. By an algebra is meant an associative, finite dimensional k-algebra with an identity. We assume, without loss of generality, that A is basic and connected. For more details on this subject we refer to [1].

We recall that a quiver Q is defined by the set of vertices Q_0, its set of arrows Q_1, and two maps $s, t : Q_1 \to Q_0$ associating with each arrow its starting and ending points respectively. For an algebra A we denote Q_A the ordinary quiver of A. It is well-known that for every basic algebra A there exists a surjective k-algebra morphism $v : kQ_A \to A$ whose kernel I_v is admissible. Thus we have $A \simeq kQ_A/I_v$. The bound quiver (Q_A, I_v) is called a *presentation* of A. The algebra A is called *triangular* if Q_A has no oriented cycles.

We denote by $\operatorname{mod} A$ the category of finitely generated left A-modules. If $A = kQ/I$, then $\operatorname{mod} A$ is equivalent to the category of all bound representations of (Q, I). We may thus identify a module M with the corresponding representation $(M, f) = (M(x), f_\alpha)_{x\in Q_0,\, \alpha\in Q_1}$.

We say that an algebra B is a convex subcategory of $A = kQ/I$ if there is a path closed full subquiver Q' of Q such that $B = kQ'/I \cap kQ'$. This means that any path in Q with source and end in Q' lies entirely in Q'.

The following known fact will be necessary in the sequel.

Lemma 1.1. *Let B be a convex subcategory of A, X, Y in* $\operatorname{mod} B$. *Then* $\operatorname{Ext}^i_A(X, Y) \simeq \operatorname{Ext}^i_B(X, Y)$ *for all $i \geqslant 1$.*

For $x \in Q_0$, we denote by S_x the corresponding simple A-module and by P_x the projective cover of S_x. It can be seen that $\operatorname{Hom}_A(P_x, M) \simeq M(x)$.

An incidence algebra A is a subalgebra of the algebra $M_n(k)$ of square matrices over k with elements $(x_{ij}) \in M_n(k)$ satisfying $x_{ij} = 0$ if $i \nleqslant j$, for some partial order $\leqslant$ defined in the poset $P = \{1, \ldots, n\}$. In this case, we denote $A = I(P)$.

The ordinary quiver associated to an incidence algebra $I(P)$ is given as follows: the set of vertices $Q_0 = P$, and there is an arrow $i \to j$ in Q_1 whenever $i > j$ and there is no $s \in P$ such that $i > s > j$. We say that two paths are parallel if they have the same starting and ending points. Then $I(P) = kQ/I$, where I is the ideal generated by differences of parallel paths.

Given an A-module M, the support $\operatorname{supp} M$ of M is the set $\{x \in Q_0 : M(x) \neq 0\}$. Consider the following conditions:

(S1) for any $x \in \operatorname{supp} M$, $M(x) = k$;
(S2) if $\alpha \in Q_1$ and $s(\alpha), t(\alpha) \in \operatorname{supp} M$ then $M(\alpha) = \mathrm{id}_k$.

Proposition 1.2 [5]. *Let M_1, M_2 be two A-modules satisfying conditions* (S1) *and* (S2). *If $\emptyset \neq \operatorname{supp} M_1 \subseteq \operatorname{supp} M_2$ and $\operatorname{supp} M_1$ is connected then* $\operatorname{Hom}_A(M_1, M_2) \simeq k$.

1.2. Hochschild cohomology

We recall the construction of the Hochschild cohomology groups $\mathrm{H}^i(A)$ of an algebra A. Consider the A-bimodule A and the complex $C^\bullet = (C^i, d^i)$ defined by: $C^i = 0$, $d^i = 0$ for $i < 0$, $C^0 = A$, $C^i = \operatorname{Hom}_k(A^{\otimes i}, A)$ for $i > 0$, where $A^{\otimes i}$ denotes the i-fold tensor product $A \otimes_k \cdots \otimes_k A$, $d^0 : A \to \operatorname{Hom}_k(A, A)$ the map $d^0(x)(a) = ax - xa$ and $d^i : C^i \to C^{i+1}$ defined by

$$\begin{aligned}(d^i f)(a_1 \otimes \cdots \otimes a_{i+1}) = {} & a_1 f(a_2 \otimes \cdots \otimes a_{i+1}) \\ & + \sum_{j=1}^{i} (-1)^j f(a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{i+1}) \\ & + (-1)^{i+1} f(a_1 \otimes \cdots \otimes a_i) a_{i+1}\end{aligned}$$

for $f \in C^i$ and $a_1, \ldots, a_{i+1} \in A$. Then $\mathrm{H}^i(A) = \mathrm{H}^i(C^\bullet) = \mathrm{Ker}\, d^i / \mathrm{Im}\, d^{i-1}$ is the ith Hochschild cohomology group of A with coefficients in A, see [2,9,11].

The Hochschild cohomology groups of a given algebra are generally hard to compute by using the definition. For this reason, one often tries to find alternative methods for computing these groups. For example, we can use an inductive method to compute the Hochschild cohomology groups when we are dealing with triangular algebras.

Let $A = kQ_A/I$ be an algebra. A vertex i in Q_A is called a *sink* if there is no arrow α in Q_A starting at i and a *source* if there is no arrow α ending at i.

If A is a triangular algebra then the quiver has sinks and sources, and this allows us to describe A as a one-point extension (co-extension) algebra.

Let x be a source in Q_A. The full convex subcategory B of A consisting of all objects except x has as quiver Q_B obtained from Q_A by deleting x and all arrows starting at x. Any presentation (Q_A, I_v) yields (by restriction) an induced presentation $(Q_B, I_{v'})$ of B. The A-module $M = \mathrm{rad} P_x$ has a canonical B-module structure, and A is isomorphic to the one–point extension algebra

$$B[M] = \begin{pmatrix} k & 0 \\ M & B \end{pmatrix}$$

where the operations are the usual addition of matrices and the multiplication is induced by the B-module structure of M.

The next theorem due to Happel [8] is useful for the computation of the Hochschild cohomology groups of the algebras considered in this article.

Theorem 1.3 [8]. *Let $A = B[M]$ be a one-point extension of B by a B-module M. Then there exists a long exact sequence of k-vector spaces connecting the Hochschild cohomology of A and B*:

$$0 \to \mathrm{H}^0(A) \to \mathrm{H}^0(B) \to \mathrm{End}_B(M)/k \to \mathrm{H}^1(A) \to \mathrm{H}^1(B) \to \mathrm{Ext}^1_B(M, M) \to \ldots$$

$$\ldots \to \mathrm{H}^i(A) \to \mathrm{H}^i(B) \to \mathrm{Ext}^i_B(M, M) \to \mathrm{H}^{i+1}(A) \to \ldots$$

Remark 1.4. Let $A = B[M]$. If $\mathrm{H}^i(A) = 0$ for all $i > 0$ and $\mathrm{End}_B(M) = k$, then $\mathrm{H}^0(A) = \mathrm{H}^0(B)$ and $\mathrm{H}^i(B) = \mathrm{Ext}^i_B(M, M)$ $\forall i > 0$.

We now recall some known results about Hochschild cohomology of incidence algebras that will be very useful for our computations.

Theorem 1.5 [5]. *If P is a poset with a unique maximal (minimal) element, then* $\mathrm{H}^i(I(P)) = 0$ *for all* $i \geqslant 1$.

The following theorem computes the Hochschild cohomology groups of incidence algebras associated to posets P_{qn+s}, for $q \geqslant 0, n \geqslant 3, 0 \leqslant s < n$, where $P_{qn+s} = [qn+s] \times [n-1]$ with $[m]=\{0,\ldots,m\}$, $(l,j) < (l+1,j)$, $(l,j) < (l+1, j+1)$ and $(l,n) = (l,0)$. These posets are superposition of $qn+s$ crowns of the same width n.

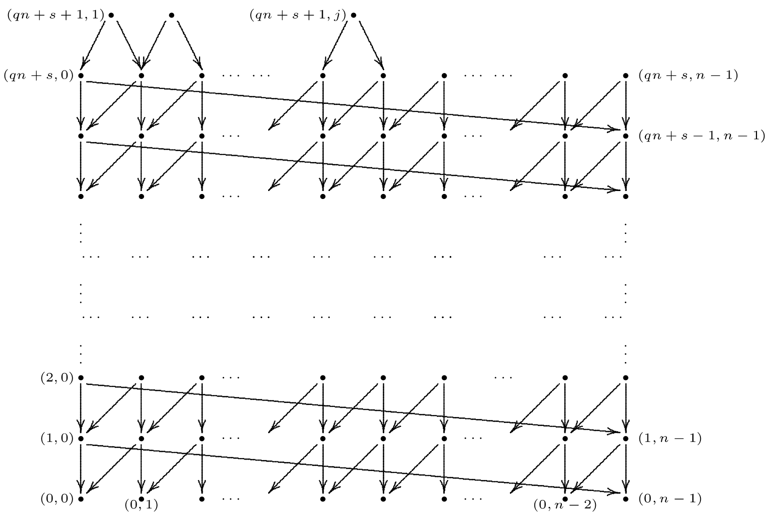
Theorem 1.6 [5]. *Let A_{qn+s} $(n \geqslant 3,\ q \geqslant 0,\ 0 \leqslant s < n)$ be the incidence algebra associated to the poset P_{qn+s}. Then*

$$\mathrm{H}^i(A_{qn+s}) = \begin{cases} k & \text{if } i = 0, \\ k & \text{if } i = 2q+1, \ \ s \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathrm{H}^i(A_{qn}) = \begin{cases} k & \text{if } i = 0, \\ k^{n-1} & \text{if } i = 2q, \\ 0 & \text{otherwise.} \end{cases}$$

2. Hochschild cohomology computations

Let A^j_{qn+s} be the incidence algebra associated to the poset $P^j_{qn+s} = P_{qn+s} \cup \{(qn+s+1,p) : 1 \leqslant p \leqslant j\}$ with partial order defined by $(l,t) < (l+1,t)$ and $(l,t) < (l+1,t+1)$, where $(l,n) = (l,0)$. Then A^j_{qn+s} has the following quiver



We compute the Hochschild cohomology groups of this particular classes of incidence algebras using the long exact sequence in Theorem 1.3. Hence we have to compute $\mathrm{Ext}^i_B(M, M)$ for some algebras B and some B-modules M, and we do this by constructing a projective resolution of M. In order to do this we need the following notation. Given x in a poset P we define the subset $x^{\downarrow}$ of P as follows

$$x^{\downarrow} = \{y \in P : y \leqslant x\}$$

In particular, the elements in the poset P_{qn+s} associated to the incidence algebra A_{qn+s} satisfy

$$(x, y) \leqslant (x + m, y)$$

and

$$(x, y) \leqslant (x + m, t) \quad \text{if } t \equiv y + m (\mathrm{mod}\, n) \text{ and } 0 \leqslant t \leqslant n - 1$$

for all $m \in \mathbf{N}$ such that $x + m \leqslant qn + s$. Moreover, the partial order in P_{qn+s} can be described as follows.

Lemma 2.1. *Let* $(x, y), (l, t) \in P_{qn+s}$. *Then* $(x, y) \leqslant (l, t)$ *if and only if*

(a) $l - x \geqslant n - 1$ *or*
(b) $0 \leqslant l - x < n - 1$ *and* $t \equiv y + u (\mathrm{mod}\, n)$ *for some* $u = 0, \ldots, l - x$.

Proof. If $l - x \geqslant n - 1$, let r be such that $t \equiv y + r (\mathrm{mod}\, n)$ with $0 \leqslant r < n$. Then $(x, y) \leqslant (x + r, t) \leqslant (l, t)$ since $r \leqslant n - 1 \leqslant l - x$.

On the other hand, if the elements $(x, y), (l, t)$ verify (b) then $(x, y) \leqslant (x + u, t) \leqslant (l, t)$ since $u \leqslant l - x$.

Conversely, suppose that $(x, y) \leqslant (l, t)$. If $(x, y), (l, t)$ do not verify (a) then $0 \leqslant l - x < n - 1$. Let $r = l - x$. By induction over r we will prove that $t \equiv y + u (\mathrm{mod}\, n)$ for some $u = 0, \ldots, r$. If $r = 0$ it is clear. Assume that $r > 0$ and that the result holds for any r' such that $0 \leqslant r' < r < n - 1$. By hypothesis $r = l - x > 0$, so $x < l$. Therefore $(x + 1, y) \leqslant (l, t)$ or $(x + 1, y + 1) \leqslant (l, t)$ since $(x + 1, y)$ and $(x + 1, y + 1)$ are the immediate successors of (x, y). Note that $l - (x + 1) = r - 1 < r$. Then using the induction thesis we infer that $t \equiv y + u (\mathrm{mod}\, n)$ or $t \equiv y + 1 + u (\mathrm{mod}\, n)$ for some $u = 0, \ldots, r - 1$. □

Now we are in a position to prove the next lemma, that will be used to construct the projective resolutions we need.

Lemma 2.2. *In* P_{qn+s}, *for every* p, j *such that* $0 \leqslant p \leqslant q$ *and* $1 \leqslant j \leqslant n - 1$, *we have*

(i) $(pn+s, j-1)^{\downarrow} \cap (pn+s, j)^{\downarrow} = (pn+s-1, j-1)^{\downarrow} \cup ((p-1)n+s+1, j)^{\downarrow}$,
(ii) $(pn+s-1, j-1)^{\downarrow} \cap ((p-1)n+s+1, j)^{\downarrow} = ((p-1)n+s, j-1)^{\downarrow} \cup ((p-1)n+s, j)^{\downarrow}$.

Proof. We only consider (i) since (ii) can be proved in an analogous way. Suppose that $(x, y) \in (pn+s, j-1)^{\downarrow} \cap (pn+s, j)^{\downarrow}$. Then it is clear that $x < pn+s$. We will consider three cases:

(1) $(p-1)n+s+1 < x \leqslant pn+s-1$,
(2) $x = (p-1)n+s+1$,
(3) $x < (p-1)n+s+1$.

(1) Observe that in this case $0 \leqslant pn+s-1-x < n-2$. Using the fact that $(x, y) \leqslant (pn+s, j-1)$ and $(x, y) \leqslant (pn+s, j)$, and applying Lemma 2.1 we can conclude that $j-1 \equiv y+u (\mathrm{mod}\, n)$ for some $u = 0, \ldots, pn+s-x-1$. Hence $(x, y) \leqslant (pn+s-1, j-1)$.
(2) If $y = j$ then $(x, y) = ((p-1)n+s+1, j)$ and we are done. Otherwise $j-1 \equiv y+u (\mathrm{mod}\, n)$ for some $u = 0, \ldots, n-2$, and Lemma 2.1 implies that $(x, y) \in (pn+s-1, j-1)^{\downarrow}$.
(3) It follows directly from Lemma 2.1 that $(x, y) \in (pn+s-1, j-1)^{\downarrow}$ since $pn+s-1-x \geqslant n-1$.

Then, it follows from cases (1)–(3) that

$$\begin{aligned}&(pn+s, j-1)^{\downarrow} \cap (pn+s, j)^{\downarrow}\\ &\quad \subseteq (pn+s-1, j-1)^{\downarrow} \cup ((p-1)n+s+1, j)^{\downarrow}.\end{aligned}$$

The other inclusion follows directly applying Lemma 2.1. □

Now we are in a position to compute $\mathrm{Ext}^i_B(M, M)$ for some convenient algebras B and B-modules M.

Lemma 2.3. *Let $M = M^{(j)}$, $1 \leqslant j < n$, be the following A_{qn+s}-module*

$$M(l, t) = \begin{cases} k & \text{if } (l, t) \in (qn+s, j-1)^{\downarrow} \cup (qn+s, j)^{\downarrow},\\ 0 & \text{otherwise},\end{cases}$$

$$M(\alpha) = \begin{cases} \mathrm{id}_k & \text{if } s(\alpha) \in (qn+s, j-1)^{\downarrow} \cup (qn+s, j)^{\downarrow},\\ 0 & \text{otherwise}.\end{cases}$$

Then

(i) *if $s = 0$ and $q \neq 0$*

$$\mathrm{Ext}^i_{A_{qn}}(M, M) = \begin{cases} k & \text{if } i = 0,\\ k & \text{if } i = 2q,\\ 0 & \text{otherwise};\end{cases}$$

(ii) *if $0 < s < n-1$*

$$\mathrm{Ext}^i_{A_{qn+s}}(M, M) = \begin{cases} k & \text{if } i = 0,\\ 0 & \text{otherwise};\end{cases}$$

(iii) *if* $s = n - 1$

$$\operatorname{Ext}^i_{A_{qn+n-1}}(M, M) = \begin{cases} k & \textit{if } i = 0, \\ k & \textit{if } i = 2q + 1, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. In order to compute the groups $\operatorname{Ext}^i_{A_{qn+s}}(M, M)$ we consider, for all $0 \leqslant p \leqslant q$, the following short exact sequences in $\operatorname{mod} A_{qn+s}$

$$0 \to K_{pn+s} \xrightarrow{\text{inc}} P_{pn+s,j} \amalg P_{pn+s,j-1} \xrightarrow{f_{pn+s}} M_{pn+s} \to 0,$$

$$0 \to M_{(p-1)n+s} \xrightarrow{\Delta} P_{(p-1)n+s+1,j} \amalg P_{pn+s-1,j-1} \xrightarrow{g_{pn+s}} K_{pn+s} \to 0,$$

where $M_{-n+s} = 0$. The corresponding representations of these modules are

$$M_{pn+s}(l, t) = \begin{cases} k & \text{if } (l, t) \in (pn + s, j - 1)^{\downarrow} \cup (pn + s, j)^{\downarrow}, \\ 0 & \text{otherwise,} \end{cases}$$

$$P_{pn+s,j}(l, t) = \begin{cases} k & \text{if } (l, t) \in (pn + s, j)^{\downarrow}, \\ 0 & \text{otherwise,} \end{cases}$$

is the projective cover of the simple module corresponding to the vertex $(pn + s, j)$,

$$K_{pn+s}(l, t) = \begin{cases} S & \text{if } (l, t) \in (pn + s, j - 1)^{\downarrow} \cap (pn + s, j)^{\downarrow}, \\ 0 & \text{otherwise,} \end{cases}$$

where $S = \{(x, -x) : x \in k\}$. In particular, if $p = 0$, $K_s = P_{s-1,j-1}$ if $s \neq n - 1$ and $K_{n-1} = P_{n-2,j-1} \amalg P_{0,j}$. The corresponding linear maps are induced by the identity. The morphisms f_{pn+s}, Δ and g_{pn+s} are induced respectively by the linear maps

$$f_{pn+s}(l, t)(x, y) = x + y,$$

$$\Delta(x) = (x, x),$$

$$g_{pn+s}(l, t)(x, y) = (x - y, y - x).$$

The exactness of the sequences constructed above follows directly from Lemma 2.2. Observe that $M = M_{qn+s}$. Hence, applying the functor $\operatorname{Hom}_{A_{qn+s}}(-, M)$ to the short exact sequences we get that for $i \geqslant 0$,

$$\operatorname{Ext}^{2i+1}_{A_{qn+s}}(M, M) = \begin{cases} \operatorname{Ext}^1_{A_{qn+s}}(M_{(q-i)n+s}, M) & \text{if } i \leqslant q, \\ \\ \operatorname{Ext}^{2(i-q)+1}_{A_{qn+s}}(M_s, M) & \text{if } i > q, \end{cases}$$

and for $i > 0$,

$$\mathrm{Ext}^{2i}_{A_{qn+s}}(M, M) = \begin{cases} \mathrm{Ext}^1_{A_{qn+s}}(K_{(q-i+1)n+s}, M) & \text{if } i \leqslant q, \\ \\ \mathrm{Ext}^{2(i-q)+1}_{A_{qn+s}}(K_{n+s}, M) & \text{if } i > q. \end{cases}$$

The short exact sequences above allow us to construct projective resolutions for M_s and K_{n+s}. We observe that $\mathrm{pdim} M_s \leqslant 1$ and $\mathrm{pdim} K_{n+s} \leqslant 2$, and hence $\mathrm{Ext}^{2(i-q)+1}_{A_{qn+s}}(M_s, M) = 0$ and $\mathrm{Ext}^{2(i-q)+1}_{A_{qn+s}}(K_{n+s}, M) = 0$ for all $i > q$. To finish the proof we have to compute $\mathrm{Ext}^1_{A_{qn+s}}(M_{(q-i)n+s}, M)$ and $\mathrm{Ext}^1_{A_{qn+s}}(K_{(q-i+1)n+s}, M)$ for all $i \leqslant q$. It follows from Proposition 1.2 that $\mathrm{Hom}_{A_{qn+s}}(M_{pn+s}, M) \simeq k$ if $pn + s > 0$, $\mathrm{Hom}_{A_{qn+s}}(M_0, M) \simeq k^2$, $\mathrm{Hom}_{A_{qn+s}}(K_{pn+s}, M) \simeq k$ if $p > 0$ or $p = 0$ and $s \neq 0$, $n - 1$, and $\mathrm{Hom}_{A_{qn+s}}(K_{n-1}, M) \simeq k^2$ (recall that $K_0 = 0$). Applying the functor $\mathrm{Hom}_{A_{qn+s}}(-, M)$ to the short exact sequences we get

$$\mathrm{Ext}^1_{A_{qn+s}}(M_{(q-i)n+s}, M) = \begin{cases} k & \text{if } q = i \text{ and } s = n-1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathrm{Ext}^1_{A_{qn+s}}(K_{(q-i+1)n+s}, M) = \begin{cases} k & \text{if } q = i \text{ and } s = 0, \\ 0 & \text{otherwise.} \quad \square \end{cases}$$

In the remaining theorems we compute the Hochschild cohomology groups of A^j_{qn+s}, $1 \leqslant j \leqslant n-1$, using the description of the cohomology groups of convenient algebras.

Theorem 2.4. *Let* $n \geqslant 3$, $1 \leqslant j \leqslant n-1$, $q \geqslant 0$ *and* $0 < s < n-1$. *Then*

$$\mathrm{H}^i(A^j_{qn+s}) = \begin{cases} k & \textit{if } i = 0, \\ k & \textit{if } i = 2q+1, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Clearly, $A^j_{qn+s} = A^{j-1}_{qn+s}[M]$, where $M = \mathrm{rad} P_{qn+s+1,j}$ and $A^0_{qn+s} = A_{qn+s}$. Observe that $M = M^{(j)}$ is the A_{qn+s}-module considered in Lemma 2.3(ii). Since A_{qn+s} is a convex subcategory of A^{j-1}_{qn+s} we have that for all $i \geqslant 1$, $\mathrm{Ext}^i_{A_{qn+s}}(M, M) \simeq \mathrm{Ext}^i_{A^{j-1}_{qn+s}}(M, M)$ by Lemma 1.1. Since $\mathrm{Ext}^i_{A^{j-1}_{qn+s}}(M, M) = 0$ for all $i \neq 0$ and $\mathrm{Hom}_{A^{j-1}_{qn+s}}(M, M) \simeq k$, we get $\mathrm{H}^i(A^j_{qn+s}) = \mathrm{H}^i(A^{j-1}_{qn+s})$ by Theorem 1.3. Thus, by recurrence over j, we obtain $\mathrm{H}^i(A^j_{qn+s}) = \mathrm{H}^i(A_{qn+s})$. The statement now follows from Theorem 1.6. $\square$

We want to mention that in the previous theorem we compute easily $\mathrm{H}^i(A^j_{qn+s})$ for $0 < s < n-1$ since we have a description of the groups $\mathrm{H}^i(A_{qn+s})$ and the groups $\mathrm{Ext}^i_{A^{j-1}_{qn+s}}(M, M)$ vanish for all $i \neq 0$. These last equalities don't occur when $s = 0$ or $s = n - 1$. In computing the Hochschild cohomology groups of these two cases we need to calculate the Hochschild cohomology groups of A^{n-1}_{qn} and A^1_{qn+n-1} respectively.

Theorem 2.5. *Let* $n \geqslant 3, q \geqslant 0$ *and* $1 \leqslant j \leqslant n-1$. *Then*

$$\mathrm{H}^i(A^j_{qn+n-1}) = \begin{cases} k & \text{if } i = 0, \\ k^{j-1} & \text{if } i = 2q+2, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. Note that $A^j_{qn+n-1} = A^{j-1}_{qn+n-1}[M]$, where $M = \mathrm{rad} P_{qn+n,j}$ and $M = M^{(j)}$ is the A_{qn+n-1}-module considered in Lemma 2.3(iii). Then, we obtain that for all $i \neq 2q+1, 2q+2$, $\mathrm{H}^i(A^j_{qn+n-1}) = \mathrm{H}^i(A^{j-1}_{qn+n-1})$ and that the following sequence

$$\begin{aligned} 0 &\to \mathrm{H}^{2q+1}\left(A^j_{qn+n-1}\right) \to \mathrm{H}^{2q+1}\left(A^{j-1}_{qn+n-1}\right) \\ &\to k \to \mathrm{H}^{2q+2}\left(A^j_{qn+n-1}\right) \to \mathrm{H}^{2q+2}\left(A^{j-1}_{qn+n-1}\right) \to 0 \end{aligned}$$

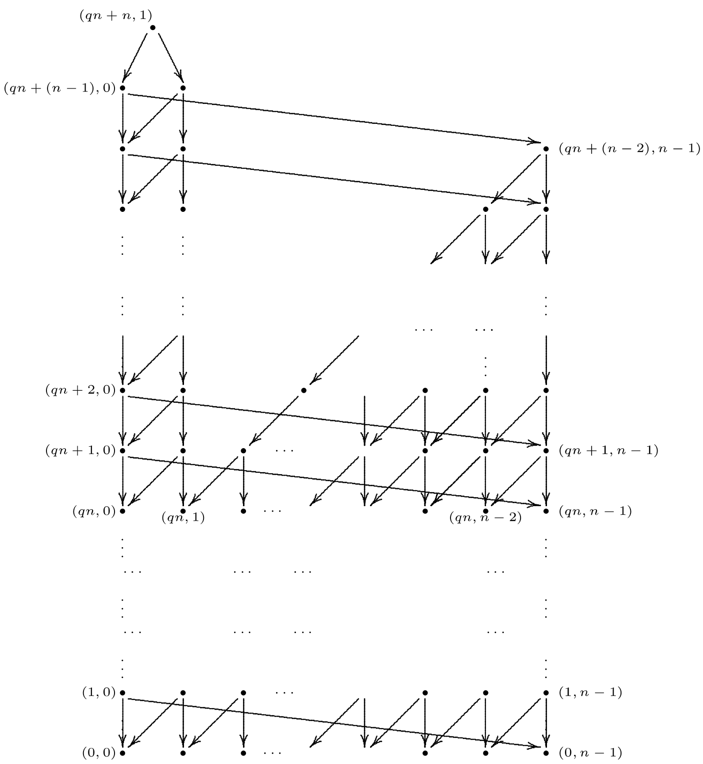
is exact.

We consider the case when $j = 1$ and we prove that $\mathrm{H}^i(A^1_{qn+n-1}) = 0$ for all $i > 0$. Then, using this result we can compute, by recurrence over j, all the groups $\mathrm{H}^i(A^j_{qn+n-1})$.

In order to prove that $\mathrm{H}^i(A^1_{qn+n-1}) = 0$ for all $i > 0$ we consider the algebra A^1_{qn+n-1} as one-point extension of a factor algebra and we iterate this procedure until we get a subcategory of A^1_{qn+n-1} whose ordered quiver has a unique maximal element. We do this by eliminating all the vertices $(l, t) \in (Q^1_{qn+n-1})_0$ such that $(qn+2, 2) \leqslant (l, t)$, and we do it in the following order

$$\begin{aligned} &(qn+n-1, 2), (qn+n-1, 3), \ldots, (qn+n-1, n-1), \\ &(qn+n-2, 2), \ldots, (qn+n-2, n-2), \\ &\ldots \\ &(qn+3, 2), (qn+3, 3), \\ &(qn+2, 2). \end{aligned}$$

In this way we get the algebras $B_0 = A^1_{qn+n-1}$, $B_{r-1} = B_r[M_r]$, $0 < r \leqslant N = ((n-1)(n-2))/2$. In each step, the B_r-module M_r is the A_{qn+s}-module M considered in Lemma 2.3(ii), for some j and s, and B_r is a convex subcategory of A_{qn+s}. Thus $\mathrm{Hom}_{B_r}(M_r, M_r) = k$ and $\mathrm{Ext}^i_{B_r}(M_r, M_r) = 0$ for all $i > 0$. Therefore $\mathrm{H}^i(B_r) = \mathrm{H}^i(B_{r-1})$ for all $i \geqslant 0$ and for all $0 < r \leqslant N$. Observe that the incidence algebra B_N has the following ordered quiver.



Therefore the poset associated to B_N has a unique maximal element, and by Theorem 1.5, we have $\mathrm{H}^i(B_N)=0$ for all $i>0$. Hence

$$\mathrm{H}^i\left(A^1_{qn+n-1}\right)=0 \quad \text{for all } i>0.$$

Using this result in the exact sequence

$$\begin{aligned}0 &\to \mathrm{H}^{2q+1}\left(A^j_{qn+n-1}\right)\to \mathrm{H}^{2q+1}\left(A^{j-1}_{qn+n-1}\right)\\ &\to k\to \mathrm{H}^{2q+2}\left(A^j_{qn+n-1}\right)\to \mathrm{H}^{2q+2}\left(A^{j-1}_{qn+n-1}\right)\to 0\end{aligned}$$

we get that $\mathrm{H}^{2q+1}\left(A^j_{qn+n-1}\right)$ is a subspace of $\mathrm{H}^{2q+1}\left(A^{j-1}_{qn+n-1}\right)$ for all $j\geqslant 1$ and $\mathrm{H}^{2q+1}\left(A^1_{qn+n-1}\right)=0$. Hence $\mathrm{H}^{2q+1}\left(A^j_{qn+n-1}\right)=0$ for all $j\geqslant 1$. This implies that

$$\dim_{\mathrm{k}}\mathrm{H}^{2q+2}\left(A^j_{qn+n-1}\right)=\dim_{\mathrm{k}}\mathrm{H}^{2q+2}\left(A^{j-1}_{qn+n-1}\right)+1.$$

This completes the proof of our theorem. □

In the final part of this section we consider the case $s = 0$, that is, we compute the Hochschild cohomology groups of A^j_{qn}.

Observe that $A^j_{qn} = A^{j-1}_{qn}[M]$, where $M = \operatorname{rad} P_{qn+1,j}$. By Lemma 2.3(i) and Theorem 1.3 we have that for all $i \neq 2q, 2q+1$, $\mathrm{H}^i(A^j_{qn}) = \mathrm{H}^i(A^{j-1}_{qn})$ and the following sequence

$$0 \to \mathrm{H}^{2q}(A^j_{qn}) \to \mathrm{H}^{2q}(A^{j-1}_{qn}) \to k \to \mathrm{H}^{2q+1}(A^j_{qn}) \to \mathrm{H}^{2q+1}(A^{j-1}_{qn}) \to 0$$

is exact. As in the previous case ($s = n-1$), even using the fact that we know the groups $\mathrm{H}^i(A_{qn})$, we can not compute directly the groups $\mathrm{H}^{2q}(A^j_{qn})$ and $\mathrm{H}^{2q+1}(A^j_{qn})$.

Theorem 2.6. *Let $n \geqslant 3$, $q > 0$ and $1 \leqslant j \leqslant n-1$. Then*

$$\mathrm{H}^i(A^j_{qn}) = \begin{cases} k & \text{if } i = 0, \\ k^{n-1-j} & \text{if } i = 2q, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. As we mentioned before the statement of the present theorem, we know that $\mathrm{H}^i(A^j_{qn}) = \mathrm{H}^i(A^{j-1}_{qn})$ for all $i \neq 2q, 2q+1$ and

$$0 \to \mathrm{H}^{2q}(A^j_{qn}) \to \mathrm{H}^{2q}(A^{j-1}_{qn}) \to k \to \mathrm{H}^{2q+1}(A^j_{qn}) \to \mathrm{H}^{2q+1}(A^{j-1}_{qn}) \to 0$$

is an exact sequence. First, we are going to prove that $\mathrm{H}^i(A^{n-1}_{qn}) = 0$ for all $i > 0$. Secondly, with this result, we can compute, by recurrence over j, all the groups $\mathrm{H}^i(A^j_{qn})$. In order to prove that $\mathrm{H}^i(A^{n-1}_{qn}) = 0$ for all $i > 0$ we construct a sequence of algebras by adding vertices, and considering each new algebra as one–point extension of the previous one. That is, algebras B_r for $0 \leqslant r < N = ((n-2)(n-1))/2$ such that $B_0 = A^{n-1}_{qn}$, $B_{r+1} = B_r[M_r]$, where the extension vertices are

$$\begin{array}{l} (qn+2, 2), (qn+2, 3), \ldots, (qn+2, n-1), \\ (qn+3, 3), \ldots, (qn+3, n-1), \\ \ldots \\ (qn+n-2, n-2), (qn+n-2, n-1), \\ (qn+n-1, n-1). \end{array}$$

In each step, the B_r-module M_r is the A_{qn+s}-module M considered in Lemma 2.3(ii), for some j and s, and B_r is a convex subcategory of A_{qn+s}. Thus $\operatorname{Hom}_{B_r}(M_r, M_r) = k$ and $\operatorname{Ext}^i_{B_r}(M_r, M_r) = 0$ for all $i > 0$. Therefore $\mathrm{H}^i(B_r) = \mathrm{H}^i(B_{r+1})$ for all $i \geqslant 0$ and for all $0 \leqslant r < N$. Observe that the poset associated to the incidence algebra B_N has a unique maximal element $(qn+n-1, n-1)$. By Theorem 1.5, we have $\mathrm{H}^i(B_N) = 0$ for all $i > 0$. Hence

$$\mathrm{H}^i(A^{n-1}_{qn}) = 0 \quad \text{for all } i > 0.$$

Using this result in the exact sequence

$$0 \to \mathrm{H}^{2q}(A^j_{qn}) \to \mathrm{H}^{2q}(A^{j-1}_{qn}) \to k \to \mathrm{H}^{2q+1}(A^j_{qn}) \to \mathrm{H}^{2q+1}(A^{j-1}_{qn}) \to 0$$

we get that $\mathrm{H}^{2q+1}(A_{qn}^{j-1})$ is a factor space of $\mathrm{H}^{2q+1}(A_{qn}^{j})$ for all $j \geqslant 1$ and $\mathrm{H}^{2q+1}(A_{qn}^{n-1}) = 0$. Hence $\mathrm{H}^{2q+1}(A_{qn}^{j}) = 0$ for all $j \geqslant 1$. This implies that

$$\dim_k \mathrm{H}^{2q}(A_{qn}^{j-1}) = \dim_k \mathrm{H}^{2q}(A_{qn}^{j}) + 1.$$

Hence we get the desired result. □

An algebra A is said to be *rigid* if any one-parameter deformation is isomorphic over k to the trivial one, see [6]. It is known that if $\mathrm{H}^2(A) = 0$ then A is rigid. Moreover, if $\mathrm{H}^3(A) = 0$, the converse is also true.

Remark 2.7. From the computations we have done we can conclude that all the algebras A_{qn+s}^{j} are rigid, except A_{n-1}^{j} for $j \neq 0, 1$ and A_{n}^{j} for $j \neq n-1$.

References

[1] M. Auslander, I. Reiten, S. Smalo, Representation theory of artin algebras, Cambridge Studies in Advanced Mathematics 36 (1995).

[2] H. Cartan, S. Eilenberg, Homological Algebra, Princeton University Press, Princeton, NJ, 1956.

[3] C. Cibils, Cohomology of incidence algebras and simplicial complexes, J. Pure Appl. Algebra 56 (1989) 221–232.

[4] P. Dräxler, Completely separating algebras, J. Algebra 165 (1994) 550–565.

[5] M.A. Gatica, M.J. Redondo, Hochschild cohomology and fundamental groups of incidence algebras, Comm. Algebra 29 (5) (2001) 2269–2283.

[6] M. Gerstenhaber, On the deformations of rings and algebras, Ann. Math. 79 (1964) 59–103.

[7] M. Gerstenhaber, S.P. Schack, Simplicial homology is Hochschild cohomology, J. Pure Appl. Algebra 30 (1983) 143–156.

[8] D. Happel, Hochschild cohomology of finite–dimensional algebras, in: M.P. Malliavin (Ed.), Séminaire (Paris, 1987–88), Lecture Notes in Mathematics, vol. 1404, Springer-Verlag, 1989, pp. 108–126.

[9] G. Hochschild, On the cohomology groups of an associative algebra, Ann. Math. 46 (1945) 58–67.

[10] K. Igusa, D. Zacharia, On the cohomology of incidence algebras of partially ordered sets, Comm. Algebra 18 (3) (1990) 873–887.

[11] C.A. Weibel, An introduction to homological algebra, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1994.