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Note

The chromatic uniqueness of certain complete t-partite graphs☆

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Abstract

Let G be a simple graph and $P(G,\lambda)$ denote the chromatic polynomial of G. Then G is said to be chromatically unique if for any simple graph H, $P(H,\lambda)=P(G,\lambda)$ implies that H is isomorphic to G. A class $\mathscr{L}$ of graphs is called a class of chromatically normal graphs if, for any $Y,G\in\mathscr{L}$, $P(Y,\lambda)=P(G,\lambda)$ implies that Y is isomorphic to G. Let $K(n_1,n_2,\dots,n_t)$ denote a complete t-partite graph and $\mathscr{L}_t=\{K(n_1,n_2,\dots,n_t)\,|\,0<n_1\leqslant n_2\leqslant\cdots\leqslant n_t\}$. The main results of the paper are as follows.

Let $G=K(n_1,n_2,\dots,n_t)\in\mathscr{L}_t$, $t\geqslant 3$, $\mathscr{Q}(G)=\{Y\,|\,P(Y,\lambda)=P(G,\lambda)\}$ and $a_t=\left(\sum_{1\leqslant i<j\leqslant t}(n_i-n_j)^2/(2t)\right)^{1/2}$. If

$$\sum_{i=1}^{t} n_i > ta_t^2+\sqrt{2t(t-1)}a_t, \qquad (*)$$

then $\mathscr{Q}(G)\subseteq\mathscr{L}_t$. Furthermore, if $\mathscr{L}_t$ is also a class of chromatically normal graphs, then G is chromatically unique. In particular, if G satisfies the condition (*) and one of the following conditions:

(i) $n_1=n_2=\cdots=n_t$, (ii) $n_1<n_2<\cdots<n_t$, (iii) $t=3$, (iv) $t=4$,

then G is chromatically unique.

Keywords: Complete t-partite graph; Chromatically unique graph; Chromatically normal graphs; Partition into color classes

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1. Introduction

We consider only finite, undirected and simple graphs. Notation and terminology that are not defined here may be found in [1,2,7,8].

Let $P(G,\lambda)$ denote the chromatic polynomial of a graph G. Two graphs H and G are said to be *chromatically equivalent* (in notation: $H \sim G$) if $P(H,\lambda)=P(G,\lambda)$. A graph G is said to be *chromatically unique* if, for any graph H, $H \sim G$ implies that $H \cong G$.

The notion of chromatic uniqueness was first introduced and studied by Chao and Whitehead in 1978 [4] . Koh and Teo, in their expository paper [7,8], gave a survey of most of the work done before 1997.

In this paper, let $K(n_1,n_2,\dots,n_t)$ denote a complete t-partite graph with partite sets N_i such that $|N_i|=n_i$ for $i=1,2,\dots,t$, and let $K(n_1,n_2,\dots,n_t)-A$ denote the t-partite graph obtained by deleting a set A of edges from the graph $K(n_1,n_2,\dots,n_t)$.

When $t=2$, the beautiful results are that $K(m,n)$ (for $2 \leqslant m \leqslant n$) and $K(m,n)-\{e\}$ (for $3 \leqslant m \leqslant n$) are chromatically unique [10]. On the chromatic uniqueness of $K(n_1,n_2,\dots,n_t)$ for $t \geqslant 3$, the authors pointed out in [3,5–9] that the following graphs (under certain conditions) are chromatically unique:

$K(n,n,n+k)$ (for $n \geqslant 2$ and $0 \leqslant k \leqslant 3$), $K(n-k,n,n)$ (for $n \geqslant k+2$ and $0 \leqslant k \leqslant 3$), $K(n-k,n,n+k)$ (for $n \geqslant 5$ and $0 \leqslant k \leqslant 2$) [5]; $K(n_1,n_2,\dots,n_t)$ (for $|n_i-n_j| \leqslant 1$ where $1 \leqslant i \leqslant t$ and $1 \leqslant j \leqslant t$) [3]; $K(n-1,n,\dots,n,n+1)$ and $K(n,n,\dots,n)-\{e\}$ (for $n \geqslant 3$) [6]; $K(1,n_2,\dots,n_t)$ (if and only if $\max\{n_2,\dots,n_t\} \leqslant 2$) [9].

Thus, $K(1,n_2,\dots,n_t)$ is not chromatically unique if $\max\{n_2,\dots,n_t\} \geqslant 3$.

The authors in [5,7] also put forward the following problem and conjecture:

Problem A (Koh and Teo [7]). For each $t \geqslant 2$, is the graph $K(n_1,n_2,\dots,n_t)$ chromatically unique if $|n_i-n_j| \leqslant 2$ where $1 \leqslant i \leqslant t$ and $1 \leqslant j \leqslant t$, and if $\min\{n_1,n_2,\dots,n_t\}$ is sufficiently large?

Conjecture B (Chia et al. [5]; Koh and Teo [7]). The graph $K(n-k,n,n)$ is chromatically unique for all n,k with $n \geqslant k+2$.

In this paper, we discuss the chromatic uniqueness of more general graphs $K(n_1,n_2,\dots,n_t)$ (for $t \geqslant 3$) and give a partial solution to the above problem and conjecture.

2. The main results

First, we define a class of graphs as follows.

A class $\mathscr{L}$ of graphs is called a class of *chromatically normal graphs* if, for any $Y,G \in \mathscr{L}$, $Y \sim G$ implies that $Y \cong G$.

Clearly, if a graph G is chromatically unique, then $\mathscr{Q}(G)=\{Y \mid Y \sim G\}$ is a class of chromatically normal graphs. Thus the following property holds.

Property 1. A graph G is chromatically unique if and only if there exists a class $\mathscr{L}$ of chromatically normal graphs such that $\mathscr{Q}(G) \subseteq \mathscr{L}$.

Let $\mathscr{L}_t = \{K(n_1, n_2, \ldots, n_t) \mid 0 < n_1 \leqslant n_2 \leqslant \cdots \leqslant n_t\}$.

Property 2 (Zou [15]). $\mathscr{L}_3$ and $\mathscr{L}_4$ are classes of chromatically normal graphs.

Our main results are as follows.

Theorem 1. *Let* $G = K(n_1, n_2, \ldots, n_t) \in \mathscr{L}_t$, $t \geqslant 3$, $a_t = \left(\sum_{1 \leqslant i < j \leqslant t} (n_i - n_j)^2/(2t)\right)^{1/2}$ *and* $\mathscr{Q}(G) = \{Y \mid Y \sim G\}$. *If*

$$\sum_{i=1}^{t} n_i > t a_t^2 + \sqrt{2t(t-1)}\, a_t, \qquad (*)$$

then $\mathscr{Q}(G) \subseteq \mathscr{L}_t$.

Furthermore, if $\mathscr{L}_t$ *is also a class of chromatically normal graphs, then G is chromatically unique.*

Corollary 1. *Let* $G = K(n_1, n_2, \ldots, n_t) \in \mathscr{L}_t$ *where* $t \geqslant 3$. *If G satisfies condition* (*) *of Theorem* 1 *and also satisfies one of the following conditions*:

(i) $n_1 = n_2 = \cdots = n_t$, (ii) $n_1 < n_2 < \cdots < n_t$, (iii) $t = 3$, (iv) $t = 4$,

then G is chromatically unique.

Theorem 2. *Let* $G = K(m, n, r)$ *where* $m \leqslant n \leqslant r$ *and* $r - m = u \geqslant 0$. *If* $m > (u + u^2)/3$, *then G is chromatically unique.*

Theorem 3. *If* $n > k + k^2/3$ *and* $k \geqslant 0$, *then* $K(n-k, n, n)$ *is chromatically unique.*

Theorem 4. *Let* $G = K(h, m, n, r)$ *where* $h \leqslant m \leqslant n \leqslant r$ *and* $r - h = u \geqslant 0$. *If* $h > (2\sqrt{3} - 1)u/4 + u^2/2$, *then G is chromatically unique.*

Remark 1. 1. Theorems 2 and 4 give a partial solution to Problem A for $t = 3$ and 4, respectively. Theorem 3 gives a partial solution to Conjecture B. When $k = 4$, Conjecture B is true [11].

2. Let $K(n_1, n_2, n_3) = K(n-k, n, n+i)$, where k and i are non-negative integers. By Theorem 2 if $3n > k - i + (k^2 + i^2 + ki) + 2(k^2 + i^2 + ki)^{1/2}$, then $K(n-k, n, n+i)$ is chromatically unique. In [5,7,8], it was shown that $K(n-k, n, n+i)$ is chromatically unique for some particular cases such as $k = 0$, $0 \leqslant i \leqslant 3$ and $n \geqslant 2$; or $i = 0$, $0 \leqslant k \leqslant 3$ and $n \geqslant k + 2$; or $0 \leqslant i = k \leqslant 2$ and $n \geqslant 5$ (see also Section 1).

3. If $n_1 + n_2 + n_3 \leqslant 3a_3^2 + 2\sqrt{3}a_3$, then $K(n_1, n_2, n_3)$ might not be chromatically unique. For example, $K(1, n, n)$ is not chromatically unique if $n \geqslant 3$ (see [9] or Section 1), where $a_3 = (n-1)/\sqrt{3}$ and $n_1 + n_2 + n_3 = 2n + 1 < n^2 - 1 = 3a_3^2 + 2\sqrt{3}a_3$ if $n \geqslant 3$. But the condition of Theorem 2 for $t = 3$ is only a sufficient condition, since $K(2, 4, 6)$ is chromatically unique [12] while the condition is not satisfied.

3. Some preliminary lemmas

Let G be a graph and let $m_r(G)$ denote the number of distinct partitions of $V(G)$ into r color classes.

Lemma 1 (Zou [14]; Zou and Shi [17]). *For any two graphs G and Y, $Y \sim G$ if and only if $|V(Y)| = |V(G)|$ and $m_r(Y) = m_r(G)$ for $r = 1, 2, \ldots, |V(G)|$.*

Lemma 2. *Let $G = K(n_1, n_2, \ldots, n_t) \in \mathscr{L}_t$. Then $m_{t+1}(G) = 2^{n_1-1} + 2^{n_2-1} + \cdots + 2^{n_t-1} - t$.*

Proof. Let $(N_1, N_2, \ldots, N_t)$ denote the t-partition of $V(G)$, where $|N_i| = n_i$ for $i = 1, 2, \ldots, t$. Since any two vertices which lie in different partite sets of $(N_1, N_2, \ldots, N_t)$ are adjacent in G, a partition of $V(G)$ into $t+1$ color classes must be obtained from $(N_1, N_2, \ldots, N_t)$ by partitioning one of the N_i $(i = 1, 2, \ldots, t)$ into two color classes. For each $i = 1, 2, \ldots, t$, let D_i denote the number of ways of partitioning N_i into two color classes. Clearly

$$D_i = \sum_{j=1}^{n_i-1} \binom{n_i}{j} \Big/ 2 = 2^{n_i-1} - 1 \quad \text{for } i = 1, 2, \ldots, t.$$

Thus

$$m_{t+1}(G) = D_1 + D_2 + \cdots + D_t = 2^{n_1-1} + 2^{n_2-1} + \cdots + 2^{n_t-1} - t. \qquad \square$$

Lemma 3 (Zou [13,16]). *Let $G = K(n_1, n_2, \ldots, n_t) \in \mathscr{L}_t$ where $t \geqslant 3$, and let J be the set of integers, R the set of real numbers and R^t the t-dimensional cartesian product of R. Suppose that Y is a graph such that $Y \sim G$. Then*

$$Y = K(n_1 + \alpha_1, n_2 + \alpha_2, \ldots, n_t + \alpha_t) - A,$$

where $|A| = \sum_{1 \leqslant i < j \leqslant t} \alpha_i \alpha_j + \sum_{1 \leqslant i < j \leqslant t} (n_i \alpha_j + n_j \alpha_i) \geqslant 0$, $\sum_{i=1}^{t} \alpha_i = 0$, $\alpha_i \in J$ *and* $n_i + \alpha_i > 0$ for $i = 1, 2, \ldots, t$.

Moreover, let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_t) \in R^t$, $s(\alpha) = s = |A| \geqslant 0$, $a_t = \left(\sum_{1 \leqslant i < j \leqslant t} (n_i - n_j)^2 / (2t)\right)^{1/2}$, $d_{t-1} = \sqrt{2(t-1)/t}$ and $c = (c_1, c_2, \ldots, c_t)$ where $c_i = \left(\sum_{j=1}^{i-1} (n_j - n_i) + \sum_{j=i+1}^{t} (n_j - n_i)\right)/t$ for $i = 1, 2, \ldots, t$. Then

(i) $\alpha \in D = \left\{\alpha \mid c_i - d_{t-1} a_t \leqslant \alpha_i \leqslant c_i + d_{t-1} a_t,\ i = 1, 2, \ldots, t;\ \sum_{i=1}^{t} \alpha_i = 0\right\}$ *and* $s = s(\alpha) = 0$ *if each inequality is an equality*;
(ii) $\max_{\alpha \in D} \{s(\alpha)\} = s(c) = a_t^2$;
(iii) *Let* $w = \sum_{i=1}^{t} n_i/t - d_{t-1} a_t$. *If* $s = s(\alpha) > 0$, *then* $w \leqslant n_i$ *and* $w < n_i + \alpha_i$ *for* $i = 1, 2, \ldots, t$.

Lemma 4. *Let $H = K(r_1, r_2, \ldots, r_t) \in \mathscr{L}_t$, $Y = H - A$, where A is a nonempty set of s edges of H and $\eta = m_{t+1}(Y) - m_{t+1}(H)$. If $\min\{r_1, r_2, \ldots, r_t\} > s$, then $s \leqslant \eta \leqslant 2^s - 1$.*

Proof. It is clear that any partition of $V(H)$ into $t+1$ color classes is a partition of $V(Y)$ into $t+1$ color classes. Therefore η is the number of partitions of $V(Y)$ into $t+1$ color classes which are not partitions of $V(H)$.

Let $(R_1, R_2, \ldots, R_t)$ denote the t-partition of $V(H)$, where $|R_i| = r_i$ for $i = 1, 2, \ldots, t$, and let V' denote the set of end vertices of the edges in A.

We next prove the following:

Claim. *A necessary and sufficient condition for P to be a partition of $V(Y)$ into $t+1$ color classes but not a partition of $V(H)$ is that P be a partition of $V(Y)$ into $t+1$ color classes of which one is the set V_0 of end vertices of the edges in some fixed nonempty subset of A and the remaining t are the sets $R_i - V_0$ where $i = 1, 2, \ldots, t$.*

(*Necessity*). If P is a partition of $V(Y)$ into $t+1$ color classes but not a partition of $V(H)$, then there exists at least one color class (say V_0) of the $t+1$ color classes of P which contains some vertices of different partite sets of $(R_1, R_2, \ldots, R_t)$. Since any two vertices of V_0 are not adjacent in Y, V_0 contains the end vertices of some i $(0 < i \leqslant s)$ edges of A.

Because $\min\{r_1, r_2, \ldots, r_t\} > s$, we conclude that none of $R_i - V'(\subset R_i - V_0)$, $i = 1, 2, \ldots, t$, is null. Clearly, $R_i - V'$, $i = 1, 2, \ldots, t$, must be contained, respectively, in t different color classes of P of which none contains vertices of different partite sets of $(R_1, R_2, \ldots, R_t)$. But there are only $t+1$ color classes. Therefore, the $t+1$ color classes must be V_0 and $R_i - V_0$, $i = 1, 2, \ldots, t$.

(*Sufficiency*). If one of the $t+1$ color classes of P contains vertices of different partite sets of $(R_1, R_2, \ldots, R_t)$, it is clear that P is not a partition of $V(H)$ into $t+1$ color classes.

Now we complete the proof of the lemma.

Since η is equal to the number of partitions P described in the claim and P is determined by V_0, we can easily see that

$$s = \binom{s}{1} \leqslant \eta \leqslant \sum_{j=1}^{s} \binom{s}{j} = 2^s - 1. \qquad \square$$

4. Proofs of the theorems and corollary

4.1. Proof of Theorem 1

Let J be the set of integers and let J^t be the t-dimensional cartesian product of J. Suppose that $Y \in \mathcal{Q}(G)$. Then, by Lemmas 1 and 3, we have

$$m_{t+1}(Y) = m_{t+1}(G), \tag{1}$$

$$Y = K(n_1 + \alpha_1, n_2 + \alpha_2, \ldots, n_t + \alpha_t) - A, \tag{2}$$

where $|A| = s = s(\alpha) = \sum_{1 \leqslant i < j \leqslant t} \alpha_i \alpha_j + \sum_{1 \leqslant i < j \leqslant t} (n_i \alpha_j + n_j \alpha_i) \geqslant 0$, $\alpha \in D \cap J^t$ and $n_i + \alpha_i > 0$ for $i = 1, 2, \ldots, t$. (See Lemma 3(i) for the definition of D.)

We first show that $s = s(\alpha) = 0$.

Let $H = K(n_1 + \alpha_1, n_2 + \alpha_2, \ldots, n_t + \alpha_t)$ and $\eta = m_{t+1}(Y) - m_{t+1}(H)$. By Lemma 2, we obtain $m_{t+1}(G) = \sum_{i=1}^{t} 2^{n_i-1} - t$ and $m_{t+1}(H) = \sum_{i=1}^{t} 2^{n_i+\alpha_i-1} - t$. It follows that

$$m_{t+1}(G) - m_{t+1}(Y) = \sum_{i=1}^{t} 2^{n_i-1} - \sum_{i=1}^{t} 2^{n_i+\alpha_i-1} - \eta. \tag{3}$$

Suppose that $s = s(\alpha) > 0$. We shall deduce that $m_{t+1}(G) - m_{t+1}(Y) \neq 0$. By Lemma 3, we have

$$s \leqslant \max_{\alpha \in D}\{s(\alpha)\} = a_t^2 = \sum_{1 \leqslant i < j \leqslant t} (n_i - n_j)^2/(2t). \tag{4}$$

Let $w = \sum_{i=1}^{t} n_i/t - \sqrt{2t(t-1)}a_t/t$. Also by Lemma 3, we find that

$$w \leqslant n_i \quad \text{and} \quad w < n_i + \alpha_i \ \text{ for } i = 1, 2, \ldots, t. \tag{5}$$

From the hypothesis of Theorem 1, we see that

$$w > a_t^2 \geqslant s. \tag{6}$$

From (5) and (6), we have

$$\max_{\alpha \in D \cap J^t}\{n_1 + \alpha_1, n_2 + \alpha_2, \ldots, n_t + \alpha_t\} > w > s. \tag{7}$$

Thus, by Lemma 4, we arrive at

$$0 < s \leqslant \eta \leqslant 2^s - 1. \tag{8}$$

Now, from (5) and (6), we have $n_i - 1 \geqslant s$ and $n_i + \alpha_i - 1 \geqslant s$, $i = 1, 2, \ldots, t$. Thus the following expression is divisible by 2^s:

$$\sum_{i=1}^{t} 2^{n_i-1} - \sum_{i=1}^{t} 2^{n_i+\alpha_i-1}.$$

But $0 < \eta < 2^s$. Hence $m_{t+1}(G) - m_{t+1}(Y)$ is not divisible by 2^s and, of course, not equal to 0, a contradiction.

Thus $s = s(\alpha) = 0$ and $Y = H = K(n_1 + \alpha_1, n_2 + \alpha_2, \ldots, n_t + \alpha_t)$.

Let $r_i = n_i + \alpha_i$ for $i = 1, 2, \ldots, t$. We might as well assume that $r_1 \leqslant r_2 \leqslant \cdots \leqslant r_t$. Then $Y = H = K(r_1, r_2, \ldots, r_t) \in \mathscr{L}_t$. Hence $\mathscr{Q}(G) \subseteq \mathscr{L}_t$.

If $\mathscr{L}_t$ is a class of chromatically normal graphs, then, from Property 1, G is chromatically unique.

4.2. Proof of Corollary 1

Follow the proof of Theorem 1.

(i) $n_1 = n_2 = \cdots = n_t$.

By Lemma 3, we have $a_t = 0$, $\alpha = c = 0$ and $s = s(\alpha) = 0$. Then, from (2), we obtain $Y = K(n_1, n_2, \ldots, n_t) \cong G$, i.e., G is chromatically unique.

(ii) $n_1 < n_2 < \cdots < n_t$.

From (1), Lemma 2 and $Y = K(r_1, r_2, \ldots, r_t) \in \mathscr{L}_t$, we deduce that

$$\sum_{i=1}^{t} 2^{n_i-1} = \sum_{i=1}^{t} 2^{r_i-1}. \tag{9}$$

Let N and M denote respectively the set consisting of the exponents of the non-zero terms in the binary expansion of $\sum_{i=1}^{t} 2^{n_i-1}$ and of $\sum_{i=1}^{t} 2^{r_i-1}$. (For example, for $2 + 2^2 + 2^2 + 2^3 = 2 + 2^4$ we see that $N = \{1, 4\}$.) Then, from (9), we have $N = M$.

Since $n_1 < n_2 < \cdots < n_t$, $N = \{n_1 - 1, n_2 - 1, \ldots, n_t - 1\}$ and $|M| = |N| = t$. Therefore $M = \{r_1 - 1, r_2 - 1, \ldots, r_t - 1\}$ and hence $\{n_1, n_2, \ldots, n_t\} = \{r_1, r_2, \ldots, r_t\}$.

Therefore, $Y \cong G$, i.e., G is chromatically unique.

(iii) $t = 3$ and (iv) $t = 4$.

Since $\mathscr{Q}(G) \subseteq \mathscr{L}_t$, from Properties 1 and 2, G is chromatically unique.

4.3. Proof of Theorem 2

From Theorem 1 and Corollary 1, we need only prove that

$$m + n + r > 3a_3^2 + 2\sqrt{3}a_3.$$

Since $m \leqslant n \leqslant r$ and $r - m = u$, $m + n + r \geqslant 3m + u$.

Let $n - m = i$. Then $0 \leqslant i \leqslant u$. Therefore $i^2 + (u - i)^2 = 2i(i - u) + u^2 \leqslant u^2$. Hence

$$a_3^2 = ((n - m)^2 + (r - m)^2 + (r - n)^2)/6 = (i^2 + u^2 + (u - i)^2)/6 \leqslant u^2/3,$$

$$\text{i.e.,}\quad u^2 \geqslant 3a_3^2 \quad \text{and} \quad u \geqslant \sqrt{3}a_3.$$

From the assumption that $m > (u + u^2)/3$, we deduce that

$$m + n + r \geqslant 3m + u > 2u + u^2 \geqslant 3a_3^2 + 2\sqrt{3}a_3.$$

4.4. Proof of Theorem 3

From Theorem 1 and Corollary 1, we need only prove that

$$3n - k > 3a_3^2 + 2\sqrt{3}a_3.$$

Clearly, $a_3^2 = k^2/3$. From the assumption that $n > k + k^2/3$, we deduce that

$$3n - k > 2k + k^2 = 3a_3^2 + 2\sqrt{3}a_3.$$

4.5. Proof of Theorem 4

From Theorem 1 and Corollary 1, we need only prove that

$$h + m + n + r > 4a_4^2 + 2\sqrt{6}a_4.$$

Since $h \leqslant m \leqslant n \leqslant r$ and $r - h = u$, $h + m + n + r \geqslant 4h + u$.

Let $m - h = i$ and $n - h = j$. Then $0 \leqslant i \leqslant u$ and $0 \leqslant j \leqslant u$. Thus

$$i^2 + (u-i)^2 \leqslant u^2, \quad j^2 + (u-j)^2 \leqslant u^2, \quad (j-i)^2 \leqslant u^2.$$

Hence

$$a_4^2 = ((m-h)^2 + (n-h)^2 + (r-h)^2 + (n-m)^2 + (r-m)^2 + (r-n)^2)/8$$

$$= (i^2 + j^2 + u^2 + (j-i)^2 + (u-i)^2 + (u-j)^2)/8 \leqslant u^2/2,$$

$$\text{i.e.,} \ u^2 \geqslant 2a_4^2 \quad \text{and} \quad u \geqslant \sqrt{2}a_4.$$

From the assumption that $h > (2\sqrt{3} - 1)u/4 + u^2/2$, we deduce that

$$h + m + n + r \geqslant 4h + u > 2\sqrt{3}u + 2u^2 \geqslant 4a_4^2 + 2\sqrt{6}a_4.$$

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