Note

The chromatic uniqueness of certain complete \( t \)-partite graphs

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Abstract

Let \( G \) be a simple graph and \( P(G, \lambda) \) denote the chromatic polynomial of \( G \). Then \( G \) is said to be chromatically unique if for any simple graph \( H \), \( P(H, \lambda) = P(G, \lambda) \) implies that \( H \) is isomorphic to \( G \). A class \( \mathcal{L} \) of graphs is called a class of chromatically normal graphs if, for any \( Y, G \in \mathcal{L} \), \( P(Y, \lambda) = P(G, \lambda) \) implies that \( Y \) is isomorphic to \( G \). Let \( K(n_1, n_2, \ldots, n_t) \) denote a complete \( t \)-partite graph and \( \mathcal{L}_t = \{ K(n_1, n_2, \ldots, n_t) \mid 0 < n_1 \leq n_2 \leq \cdots \leq n_t \} \). The main results of the paper are as follows.

Let \( G = K(n_1, n_2, \ldots, n_t) \in \mathcal{L}_t \), \( t \geq 3 \), \( \mathcal{L}(G) = \{ Y \mid P(Y, \lambda) = P(G, \lambda) \} \) and \( a_t = \left( \sum_{1 \leq i < j \leq t} (n_i - n_j)^2 / (2t) \right)^{1/2} \). If

\[
\sum_{i=1}^{t} n_i > ta_t^2 + \sqrt{2t(t-1)a_t},
\]

then \( \mathcal{L}(G) \subseteq \mathcal{L}_t \). Furthermore, if \( \mathcal{L}_t \) is also a class of chromatically normal graphs, then \( G \) is chromatically unique. In particular, if \( G \) satisfies the condition \( (*) \) and one of the following conditions:

\[
(i) \ n_1 = n_2 = \cdots = n_t, \quad (ii) \ n_1 < n_2 < \cdots < n_t, \quad (iii) \ t = 3, \quad (iv) \ t = 4,
\]

then \( G \) is chromatically unique.

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Keywords: Complete \( t \)-partite graph; Chromatically unique graph; Chromatically normal graphs; Partition into color classes

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1. Introduction

We consider only finite, undirected and simple graphs. Notation and terminology that are not defined here may be found in [1,2,7,8].

Let \( P(G, \lambda) \) denote the chromatic polynomial of a graph \( G \). Two graphs \( H \) and \( G \) are said to be chromatically equivalent (in notation: \( H \sim G \)) if \( P(H, \lambda) = P(G, \lambda) \). A graph \( G \) is said to be chromatically unique if, for any graph \( H \), \( H \sim G \) implies that \( H \cong G \).

The notion of chromatic uniqueness was first introduced and studied by Chao and Whitehead in 1978 [4]. Koh and Teo, in their expository paper [7,8], gave a survey of most of the work done before 1997.

In this paper, let \( K(n_1, n_2, \ldots, n_t) \) denote a complete \( t \)-partite graph with partite sets \( N_i \) such that \( |N_i| = n_i \) for \( i = 1, 2, \ldots, t \), and let \( K(n_1, n_2, \ldots, n_t) - A \) denote the \( t \)-partite graph obtained by deleting a set \( A \) of edges from the graph \( K(n_1, n_2, \ldots, n_t) \).

When \( t = 2 \), the beautiful results are that \( K(m, n) \) (for \( 2 \leq m \leq n \)) and \( K(m, n) - \{e\} \) (for \( 3 \leq m \leq n \)) are chromatically unique [10]. On the chromatic uniqueness of \( K(n_1, n_2, \ldots, n_t) \) for \( t \geq 3 \), the authors pointed out in [3,5–9] that the following graphs (under certain conditions) are chromatically unique:

- \( K(n, n, n + k) \) (for \( n \geq 2 \) and \( 0 \leq k \leq 3 \)), \( K(n - k, n, n) \) (for \( n \geq k + 2 \) and \( 0 \leq k \leq 3 \)), \( K(n - k, n, n + k) \) (for \( n \geq 5 \) and \( 0 \leq k \leq 2 \)) [5];
- \( K(n_1, n_2, \ldots, n_t) \) (for \( |n_i - n_j| \leq 1 \) where \( 1 \leq i \leq t \) and \( 1 \leq j \leq t \)) [3];
- \( K(n - 1, n, \ldots, n, n + 1) \) and \( K(n, n, \ldots, n) - \{e\} \) (for \( n \geq 3 \)) [6];
- \( K(1, n_2, \ldots, n_t) \) (if and only if \( \max\{n_2, \ldots, n_t\} \leq 2 \)) [9].

Thus, \( K(1, n_2, \ldots, n_t) \) is not chromatically unique if \( \max\{n_2, \ldots, n_t\} \geq 3 \).

The authors in [5,7] also put forward the following problem and conjecture:

**Problem A** (Koh and Teo [7]). For each \( t \geq 2 \), is the graph \( K(n_1, n_2, \ldots, n_t) \) chromatically unique if \( |n_i - n_j| \leq 2 \) where \( 1 \leq i \leq t \) and \( 1 \leq j \leq t \), and if \( \min\{n_1, n_2, \ldots, n_t\} \) is sufficiently large?

**Conjecture B** (Chia et al. [5]; Koh and Teo [7]). The graph \( K(n - k, n, n) \) is chromatically unique for all \( n, k \) with \( n \geq k + 2 \).

In this paper, we discuss the chromatic uniqueness of more general graphs \( K(n_1, n_2, \ldots, n_t) \) (for \( t \geq 3 \)) and give a partial solution to the above problem and conjecture.

2. The main results

First, we define a class of graphs as follows.

A class \( \mathcal{L} \) of graphs is called a class of *chromatically normal graphs* if, for any \( Y, G \in \mathcal{L} \), \( Y \sim G \) implies that \( Y \cong G \).

Clearly, if a graph \( G \) is chromatically unique, then \( \mathcal{A}(G) = \{Y \mid Y \sim G\} \) is a class of chromatically normal graphs. Thus the following property holds.

**Property 1.** A graph \( G \) is chromatically unique if and only if there exists a class \( \mathcal{L} \) of chromatically normal graphs such that \( \mathcal{A}(G) \subseteq \mathcal{L} \).
Let $\mathcal{L}_t = \{K(n_1, n_2, \ldots, n_t) \mid 0 < n_1 \leq n_2 \leq \cdots \leq n_t\}$.

**Property 2** (Zou [15]). $\mathcal{L}_3$ and $\mathcal{L}_4$ are classes of chromatically normal graphs.

Our main results are as follows.

**Theorem 1.** Let $G = K(n_1, n_2, \ldots, n_t) \in \mathcal{L}_t$, $t \geq 3$, $a_i = \left(\sum_{1 \leq i < j \leq t} (n_i - n_j)^2/(2t)\right)^{1/2}$ and $\mathcal{A}(G) = \{Y \mid Y \sim G\}$. If

$$\sum_{i=1}^t n_i > ta_i^2 + \sqrt{2t(t-1)a_i},$$

then $\mathcal{A}(G) \subseteq \mathcal{L}_t$.

Furthermore, if $\mathcal{L}_t$ is also a class of chromatically normal graphs, then $G$ is chromatically unique.

**Corollary 1.** Let $G = K(n_1, n_2, \ldots, n_t) \in \mathcal{L}_t$ where $t \geq 3$. If $G$ satisfies condition (*) of Theorem 1 and also satisfies one of the following conditions:

(i) $n_1 = n_2 = \cdots = n_t$, (ii) $n_1 < n_2 < \cdots < n_t$, (iii) $t = 3$, (iv) $t = 4$,

then $G$ is chromatically unique.

**Theorem 2.** Let $G = K(m, n, r)$ where $m \leq n \leq r$ and $r - m = u \geq 0$. If $m > (u + u^2)/3$, then $G$ is chromatically unique.

**Theorem 3.** If $n > k + k^2/3$ and $k \geq 0$, then $K(n-k, n, n)$ is chromatically unique.

**Theorem 4.** Let $G = K(h, m, n, r)$ where $h \leq m \leq n \leq r$ and $r - h = u \geq 0$. If $h > (2\sqrt{3} - 1)u/4 + u^2/2$, then $G$ is chromatically unique.

**Remark 1.** 1. Theorems 2 and 4 give a partial solution to Problem A for $t = 3$ and 4, respectively. Theorem 3 gives a partial solution to Conjecture B. When $k = 4$, Conjecture B is true [11].

2. Let $K(n_1, n_2, n_3) = K(n-k, n, n+i)$, where $k$ and $i$ are non-negative integers. By Theorem 2 if $3n > k - i + (k^2 + i^2 + ki) + 2(k^2 + i^2 + ki)^{1/2}$, then $K(n-k, n, n+i)$ is chromatically unique. In [5,7,8], it was shown that $K(n-k, n, n+i)$ is chromatically unique for some particular cases such as $k = 0$, $0 \leq i \leq 3$ and $n \geq 2$; or $i = 0$, $0 \leq k \leq 3$ and $n \geq k + 2$; or $0 \leq k \leq 2$ and $n \geq 5$ (see also Section 1).

3. If $n_1 + n_2 + n_3 \leq 3a_3^2 + 2\sqrt{3}a_3$, then $K(n_1, n_2, n_3)$ might not be chromatically unique. For example, $K(1, n, n)$ is not chromatically unique if $n \geq 3$ (see [9] or Section 1), where $a_3 = (n-1)/\sqrt{3}$ and $n_1 + n_2 + n_3 = 2n + 1 < n^2 - 1 = 3a_3^2 + 2\sqrt{3}a_3$ if $n \geq 3$. But the condition of Theorem 2 for $t = 3$ is only a sufficient condition, since $K(2, 4, 6)$ is chromatically unique [12] while the condition is not satisfied.
3. Some preliminary lemmas

Let $G$ be a graph and let $m_r(G)$ denote the number of distinct partitions of $V(G)$ into $r$ color classes.

**Lemma 1** (Zou [14]; Zou and Shi [17]). For any two graphs $G$ and $Y$, $Y \sim G$ if and only if $|V(Y)| = |V(G)|$ and $m_r(Y) = m_r(G)$ for $r = 1, 2, \ldots, |V(G)|$.

**Lemma 2.** Let $G = K(n_1, n_2, \ldots, n_t) \in \mathcal{L}_t$. Then $m_{t+1}(G) = 2^{n-1} + 2^{n-1} + \cdots + 2^{n-1} - t$.

**Proof.** Let $(N_1, N_2, \ldots, N_t)$ denote the $t$-partition of $V(G)$, where $|N_i| = n_i$ for $i = 1, 2, \ldots, t$. Since any two vertices which lie in different partite sets of $(N_1, N_2, \ldots, N_t)$ are adjacent in $G$, a partition of $V(G)$ into $t + 1$ color classes must be obtained from $(N_1, N_2, \ldots, N_t)$ by partitioning one of the $N_i$ ($i = 1, 2, \ldots, t$) into two color classes. For each $i = 1, 2, \ldots, t$, let $D_i$ denote the number of ways of partitioning $N_i$ into two color classes. Clearly

$$D_i = \sum_{j=1}^{n_i-1} \binom{n_i}{j} / 2 = 2^{n_i-1} - 1 \quad \text{for} \quad i = 1, 2, \ldots, t.$$  

Thus

$$m_{t+1}(G) = D_1 + D_2 + \cdots + D_t = 2^{n_1-1} + 2^{n_2-1} + \cdots + 2^{n_t-1} - t. \quad \Box$$

**Lemma 3** (Zou [13,16]). Let $G = K(n_1, n_2, \ldots, n_t) \in \mathcal{L}_t$ where $t \geq 3$, and let $J$ be the set of integers, $R$ the set of real numbers and $R^t$ the $t$-dimensional cartesian product of $R$. Suppose that $Y$ is a graph such that $Y \sim G$. Then

$$Y = K(n_1 + \alpha_1, n_2 + \alpha_2, \ldots, n_t + \alpha_t) - A,$$

where $|A| = \sum_{1 \leq i < j \leq t} \alpha_i \alpha_j + \sum_{1 \leq i < j \leq t} (n_i \alpha_j + n_j \alpha_i) \geq 0$, $\sum_{i=1}^t \alpha_i = 0$, $\alpha_i \in J$ and $n_i + \alpha_i > 0$ for $i = 1, 2, \ldots, t$.

Moreover, let $x = (x_1, x_2, \ldots, x_t) \in R^t$, $s(x) = s = |A| \geq 0$, $a_i = \left(\sum_{1 \leq i < j \leq t} (n_i - n_j)^2 / (2t)\right)^{1/2}$, $d_{i-1} = \sqrt{2(t - 1)/t}$ and $c = (c_1, c_2, \ldots, c_t)$ where $c_i = \left(\sum_{j=i+1}^(t-1) (n_j - n_i) + \sum_{j=i+1}^t (n_j - n_i)\right) / t$ for $i = 1, 2, \ldots, t$. Then

(i) $x \in D = \{x | c_i - d_{i-1}a_i \leq \alpha_i \leq c_i + d_{i-1}a_i, \ i = 1, 2, \ldots, t; \ \sum_{i=1}^t \alpha_i = 0\}$ and $s = s(x) = 0$ if each inequality is an equality;

(ii) $\max_{x \in D} \{s(x)\} = s(c) = a_i^2$,

(iii) Let $w = \sum_{i=1}^t n_i / t - d_{t-1}a_t$. If $s = s(x) > 0$, then $w \leq n_i$ and $w < n_i + \alpha_i$ for $i = 1, 2, \ldots, t$.

**Lemma 4.** Let $H = K(r_1, r_2, \ldots, r_t) \in \mathcal{L}_t$, $Y = H - A$, where $A$ is a nonempty set of $s$ edges of $H$ and $\eta = m_{t+1}(Y) - m_{t+1}(H)$. If $\min\{r_1, r_2, \ldots, r_t\} > s$, then $s \leq \eta \leq 2^s - 1$. 

Proof. It is clear that any partition of $V(H)$ into $t + 1$ color classes is a partition of $V(Y)$ into $t + 1$ color classes. Therefore $\eta$ is the number of partitions of $V(Y)$ into $t + 1$ color classes which are not partitions of $V(H)$.

Let $(R_1, R_2, \ldots, R_t)$ denote the $t$-partition of $V(H)$, where $|R_i| = r_i$ for $i = 1, 2, \ldots, t$, and let $V'$ denote the set of end vertices of the edges in $A$.

We next prove the following:

Claim. A necessary and sufficient condition for $P$ to be a partition of $V(Y)$ into $t + 1$ color classes but not a partition of $V(H)$ is that $P$ be a partition of $V(Y)$ into $t + 1$ color classes of which one is the set $V_0$ of end vertices of the edges in some fixed nonempty subset of $A$ and the remaining $t$ are the sets $R_i - V'$ where $i = 1, 2, \ldots, t$.

(Necessity). If $P$ is a partition of $V(Y)$ into $t + 1$ color classes but not a partition of $V(H)$, then there exists at least one color class (say $V_0$) of the $t + 1$ color classes of $P$ which contains some vertices of different partite sets of $(R_1, R_2, \ldots, R_t)$. Since any two vertices of $V_0$ are not adjacent in $Y$, $V_0$ contains the end vertices of some $\ell$ ($1 \leq \ell \leq t$) edges of $A$.

Because $\min\{r_1, r_2, \ldots, r_t\} > s$, we conclude that none of $R_i - V' (\subseteq R_i - V_0)$, $i = 1, 2, \ldots, t$, is null. Clearly, $R_i - V'$, $i = 1, 2, \ldots, t$, must be contained, respectively, in $t$ different color classes of $P$ of which none contains vertices of different partite sets of $(R_1, R_2, \ldots, R_t)$. But there are only $t + 1$ color classes. Therefore, the $t + 1$ color classes must be $V_0$ and $R_i - V_0$, $i = 1, 2, \ldots, t$.

(Sufficiency). If one of the $t + 1$ color classes of $P$ contains vertices of different partite sets of $(R_1, R_2, \ldots, R_t)$, it is clear that $P$ is not a partition of $V(H)$ into $t + 1$ color classes.

Now we complete the proof of the lemma.

Since $\eta$ is equal to the number of partitions $P$ described in the claim and $P$ is determined by $V_0$, we can easily see that

$$s = \left( \begin{array}{c} s \\ 1 \end{array} \right) \leq \eta \leq \sum_{j=1}^{s} \left( \begin{array}{c} s \\ j \end{array} \right) = 2^s - 1.$$ 

4. Proofs of the theorems and corollary

4.1. Proof of Theorem 1

Let $J$ be the set of integers and let $J'$ be the $t$-dimensional cartesian product of $J$.

Suppose that $Y \in \mathcal{Z}(G)$. Then, by Lemmas 1 and 3, we have

$$m_{t+1}(Y) = m_{t+1}(G),$$

$$Y = K(n_1 + x_1, n_2 + x_2, \ldots, n_t + x_t) - A,$$

where $|A| = s = s(x) = \sum_{1 \leq i < j \leq t} x_i x_j + \sum_{1 \leq i < j \leq t} (n_i x_j + n_j x_i) \geq 0$, $x \in D \cap J'$ and $n_i + x_i > 0$ for $i = 1, 2, \ldots, t$. (See Lemma 3(i) for the definition of $D$.)

We first show that $s = s(x) = 0$. 


Let \( H = K(n_1 + x_1, n_2 + x_2, \ldots, n_t + x_t) \) and \( \eta = m_{t+1}(Y) - m_{t+1}(H) \). By Lemma 2, we obtain \( m_{t+1}(G) = \sum_{i=1}^{t} 2^{n_i-1} - t \) and \( m_{t+1}(H) = \sum_{i=1}^{t} 2^{n_i+x_i-1} - t \). It follows that

\[
m_{t+1}(G) - m_{t+1}(Y) = \sum_{i=1}^{t} 2^{n_i-1} - \sum_{i=1}^{t} 2^{n_i+x_i-1} - \eta.
\] (3)

Suppose that \( s = s(x) > 0 \). We shall deduce that \( m_{t+1}(G) - m_{t+1}(Y) \neq 0 \). By Lemma 3, we have

\[
s \leq \max_{s \in \mathcal{D}} \{ s(x) \} = a_t^2 = \sum_{1 \leq i < j \leq t} (n_i - n_j)^2/(2t).
\] (4)

Let \( w = \sum_{i=1}^{t} n_i/t - \sqrt{2t(t-1)a_i/t} \). Also by Lemma 3, we find that

\[
w \leq n_i \quad \text{and} \quad w < n_i + x_i \quad \text{for} \; i = 1, 2, \ldots, t.
\] (5)

From the hypothesis of Theorem 1, we see that

\[
w \geq a_t^2 \geq s.
\] (6)

From (5) and (6), we have

\[
\max_{s \in \mathcal{D} \cup \mathcal{L}} \{ n_1 + x_1, n_2 + x_2, \ldots, n_t + x_t \} > w > s.
\] (7)

Thus, by Lemma 4, we arrive at

\[
0 < s \leq \eta \leq 2^t - 1.
\] (8)

Now, from (5) and (6), we have \( n_i - 1 \geq s \) and \( n_i + x_i - 1 \geq s \), \( i = 1, 2, \ldots, t \). Thus the following expression is divisible by \( 2^t \):

\[
\sum_{i=1}^{t} 2^{n_i-1} - \sum_{i=1}^{t} 2^{n_i+x_i-1}.
\]

But \( 0 < \eta < 2^t \). Hence \( m_{t+1}(G) - m_{t+1}(Y) \) is not divisible by \( 2^t \) and, of course, not equal to 0, a contradiction.

Thus \( s = s(x) = 0 \) and \( Y = H = K(n_1 + x_1, n_2 + x_2, \ldots, n_t + x_t) \).

Let \( r_i = n_i + x_i \) for \( i = 1, 2, \ldots, t \). We might as well assume that \( r_1 \leq r_2 \leq \cdots \leq r_t \).

Then \( Y = H = K(r_1, r_2, \ldots, r_t) \in \mathcal{L}' \). Hence \( \mathcal{L}(G) \subseteq \mathcal{L}' \).

If \( \mathcal{L}' \) is a class of chromatically normal graphs, then, from Property 1, \( G \) is chromatically unique.

### 4.2. Proof of Corollary 1

Follow the proof of Theorem 1.

(i) \( n_1 = n_2 = \cdots = n_t \).

By Lemma 3, we have \( a_t = 0, \; x = c = 0 \) and \( s = s(x) = 0 \). Then, from (2), we obtain \( Y = K(n_1, n_2, \ldots, n_t) \cong G \), i.e., \( G \) is chromatically unique.

(ii) \( n_1 < n_2 < \cdots < n_t \).
From (1), Lemma 2 and \( Y = K(r_1, r_2, \ldots, r_t) \in L_t \), we deduce that
\[
\sum_{i=1}^{t} 2^{n_i - 1} = \sum_{i=1}^{t} 2^{r_i - 1}.
\]

Let \( N \) and \( M \) denote respectively the set consisting of the exponents of the non-zero terms in the binary expansion of \( \sum_{i=1}^{t} 2^{n_i - 1} \) and of \( \sum_{i=1}^{t} 2^{r_i - 1} \). (For example, for \( 2 + 2^2 + 2^3 = 2 + 2^4 \) we see that \( N = \{1, 4\} \).) Then, from (9), we have \( N = M \).

Since \( n_1 < n_2 < \cdots < n_t \), \( N = \{n_1 - 1, n_2 - 1, \ldots, n_t - 1\} \) and \( |M| = |N| = t \). Therefore \( M = \{r_1 - 1, r_2 - 1, \ldots, r_t - 1\} \) and hence \( \{n_1, n_2, \ldots, n_t\} = \{r_1, r_2, \ldots, r_t\} \).

Therefore, \( Y \sim G \), i.e., \( G \) is chromatically unique.

### 4.3. Proof of Theorem 2

From Theorem 1 and Corollary 1, we need only prove that
\[
m + n + r > 3a_3^2 + 2\sqrt{3}a_3.
\]

Since \( m \leq n \leq r \) and \( r - m = u \), \( m + n + r \geq 3m + u \).

Let \( n - m = i \). Then \( 0 \leq i \leq u \). Therefore \( i^2 + (u - i)^2 = 2i(i - u) + u^2 \leq u^2 \). Hence
\[
a_3^2 = ((n - m)^2 + (r - m)^2 + (r - n)^2)/6 = (i^2 + u^2 + (u - i)^2)/6 \leq u^2/3,
\]
i.e., \( u^2 \geq 3a_3^2 \) and \( u \geq \sqrt{3}a_3 \).

From the assumption that \( m > (u + u^2)/3 \), we deduce that
\[
m + n + r \geq 3m + u > 2u + u^2 \geq 3a_3^2 + 2\sqrt{3}a_3.
\]

### 4.4. Proof of Theorem 3

From Theorem 1 and Corollary 1, we need only prove that
\[
3n - k > 3a_3^2 + 2\sqrt{3}a_3.
\]

Clearly, \( a_3^2 = k^2/3 \). From the assumption that \( n > k + k^2/3 \), we deduce that
\[
3n - k > 2k + k^2 = 3a_3^2 + 2\sqrt{3}a_3.
\]

### 4.5. Proof of Theorem 4

From Theorem 1 and Corollary 1, we need only prove that
\[
h + m + n + r > 4a_4^2 + 2\sqrt{6}a_4.
\]
Since \( h \leq m \leq n \leq r \) and \( r - h = u \), \( h + m + n + r \geq 4h + u \).

Let \( m - h = i \) and \( n - h = j \). Then \( 0 \leq i \leq u \) and \( 0 \leq j \leq u \). Thus
\[
i^2 + (u - i)^2 \leq u^2, \quad j^2 + (u - j)^2 \leq u^2, \quad (j - i)^2 \leq u^2.
\]

Hence
\[
a_4^2 = \frac{((m - h)^2 + (n - h)^2 + (r - h)^2 + (n - m)^2 + (r - m)^2 + (r - n)^2)}{8}
\]
\[
= \frac{(i^2 + j^2 + u^2 + (j - i)^2 + (u - i)^2 + (u - j)^2)}{8} \leq \frac{u^2}{2},
\]
i.e., \( u^2 \geq 2a_4^2 \) and \( u \geq \sqrt{2}a_4 \).

From the assumption that \( h > (2\sqrt{3} - 1)u/4 + u^2/2 \), we deduce that
\( h + m + n + r \geq 4h + u > 2\sqrt{3}u + 2u^2 \geq 4a_4^2 + 2\sqrt{6}a_4 \).

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