A family of Prym–Tyurin varieties of exponent 3 ☆

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Abstract

We investigate a family of correspondences associated to étale coverings of degree 3 of hyperelliptic curves. They lead to Prym–Tyurin varieties of exponent 3. We identify these varieties and derive some consequences.
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1. Introduction

A *correspondence* on a smooth projective curve C is by definition a divisor D on the product $C \times C$. Any correspondence D on C induces an endomorphism γ_D of the Jacobian JC. Conversely, for every endomorphism $\gamma \in \operatorname{End} C$ there is a correspondence D on C such that $\gamma = \gamma_D$. However for most correspondences D which occur in the literature, γ_D is a multiple $d \cdot 1_{JC}$ of the identity of the Jacobian. In particular, this is the case for

any correspondence of a general curve. It were mainly these correspondences, called *of valency d*, which were studied by the classical Italian geometers (see, e.g., [8]).

At the beginning of the 1970s A. Tyurin suggested the investigation of another class of correspondences, namely effective symmetric correspondences D without fixed point on C such that γ_D satisfies an equation

$$\gamma_D^2 + (e-2)\gamma_D - (e-1) = 0.$$

For these correspondences $P = \mathrm{im}(\gamma_D - 1_{JC})$ is a *Prym–Tyurin variety* of exponent e, meaning that the restriction of the canonical polarization of JC to P is the e-fold of a principal polarization on P. Jacobians are Prym–Tyurin varieties of exponent 1, Prym varieties associated to étale double coverings are Prym–Tyurin varieties of exponent 2. On the other hand, it is not difficult to show (see [2, Corollary 12.2.4]) that every principally polarized abelian variety is a Prym–Tyurin of some high exponent. However, it seems not so easy to construct Prym–Tyurin varieties of low exponent $\geqslant 3$. First examples (associated to Fano threefolds) were investigated by Tyurin (see [9]). Other examples (associated to Weil groups of certain Lie algebras) were constructed by Kanev (see [5]).

It is the aim of this paper to study the following correspondence: let C be a hyperelliptic curve of genus $g \geqslant 3$ and $f : \widetilde{C} \to C$ an étale threefold covering. Consider the following curve in the symmetric product $\widetilde{C}^{(2)}$:

$$X = \left\{ p \in \widetilde{C}^{(2)} \mid f^{(2)}(p) \in g_2^1 \right\}.$$

Let ι denote the hyperelliptic involution of C and for $x \in C$ write $f^{-1}(x) = \{x_1, x_2, x_3\}$, $f^{-1}(\iota x) = \{y_1, y_2, y_3\}$ and moreover for abbreviation $P_{ij} = x_i + y_j \in \widetilde{C}^{(2)}$. Then the symmetric $(2, 2)$-correspondence D on X is defined by

$$D = \left\{ (P_{ij}, P_{kl}) \in X \times X \mid i = k \text{ and } j \neq l \text{ or } i \neq k \text{ and } j = l \right\}.$$

We show in Section 2 under the hypothesis that X is smooth and irreducible, that $P = \mathrm{im}(\gamma_D - 1_{JX})$ is a Prym–Tyurin variety of exponent 3. In Section 3 we realise a $(2g-1)$-dimensional family of pairs (C, f) such that X is smooth and irreducible. In fact, the Galois group G of the Galois extension $Y \to \mathbb{P}^1$ of $\widetilde{C} \to \mathbb{P}^1$ is necessarily isomorphic to $S_3 \times S_3 \subset S_6$. In Section 5 we compute the dimensions of the Jacobians and Prym varieties relevant to this situation. The main result of this section is Theorem 5.3 which says that there are two trigonal curves X_1 and X_2 associated to subgroups of G such that P is canonically isomorphic as a principally polarized abelian variety to the product of Jacobians $JX_1 \times JX_2$. The trigonal covers of X_1 and X_2 have disjoint ramification locus and X is their common fibre product over $\mathbb{P}^1$. As a consequence of this and the moduli considerations of Section 4 we obtain the following consequence which seems of interest to us and for which we could not find a different proof:

Corollary (of Theorems 4.1 and 5.2). *Let X_1 and X_2 be trigonal curves with simple ramification and disjoint branching and let X denote their fibre product over $\mathbb{P}^1$ with projections $f_i : X \to X_i$. Then X is a smooth projective curve and the map $f_1^* + f_2^* : JX_1 \times JX_2 \to JX$ is an embedding.*

Finally, in Section 6 we study the Abel–Prym map of the Prym–Tyurin variety P defined as the composition of the Abel map $X \to JX$ and the projection $JX \to P$. The main result is the following:

Theorem. *For $g \geqslant 6$ the Abel–Prym map $\beta_P : X \to P$ is an embedding.*

As a consequence we obtain that the cohomology class of three times the canonical product polarization of $JX_1 \times JX_2$ is represented by a smooth irreducible curve (see Corollary 6.5).

2. Construction of the Prym–Tyurin varieties

Let C be an hyperelliptic curve of genus $g \geqslant 3$, $i : C \to C$ the hyperelliptic involution and $h : C \to \mathbb{P}^1$ the map given by the g_2^1. Let $f : \widetilde{C} \to C$ be an étale morphism of degree n from a projective smooth irreducible curve $\widetilde{C}$.

Let us define a new curve X by the following cartesian diagram:

$$
\begin{array}{ccc}
X := \left(f^{(2)}\right)^{-1}\left(g_2^1\right) & \lhook\joinrel\longrightarrow & \widetilde{C}^{(2)} \\
{\scriptstyle \pi = f^{(2)}|_X} \downarrow & & \downarrow {\scriptstyle f^{(2)}} \\
\mathbb{P}^1 \cong g_2^1 & \lhook\joinrel\longrightarrow & C^{(2)},
\end{array}
\tag{2.1}
$$

where $f^{(2)} : C^{(2)} \to \widetilde{C}^{(2)}$ denotes the second symmetric product of f. Observe that π is of degree n^2.

For the rest of this section let us assume that the curve X is *smooth and irreducible*. (In the next section we will see that there exist étale coverings f such that this is the case.) Under this hypothesis we define a correspondence D on X. For this consider the canonical $2:1$-map $\lambda : \widetilde{C}^2 \to \widetilde{C}^{(2)}$ from the cartesian to the symmetric product of $\widetilde{C}$ and denote

$$\widetilde{X} := \lambda^{-1}(X).$$

Let $p_1 : \widetilde{X} \to \widetilde{C}$ denote the projection onto the first factor, where $\widetilde{X}$ is considered as a curve in $\widetilde{C}^2$. Then

$$\widetilde{D} := \left\{ (a, b) \in \widetilde{X} \times \widetilde{X} \mid p_1(a) = p_1(b) \right\}$$

with reduced subscheme structure is an effective divisor on $\widetilde{X}^2$ containing the diagonal $\widetilde{\Delta}$. Denote

$$Y := \widetilde{D} - \widetilde{\Delta}.$$

The divisor

$$D := (\lambda \times \lambda)_*(Y)$$

is an effective symmetric correspondence of X of bidegree $(2n - 2, 2n - 2)$.

In order to describe this correspondence set-theoretically, we fix some notation. Given $z \in \mathbb{P}^1$, let

$$h^{-1}(z) = x + ix$$

and

$$f^{-1}(x) = \{x_1, \ldots, x_n\}, \qquad f^{-1}(ix) = \{y_1, \ldots, y_n\}.$$

If we denote for $i, j = 1, \ldots, n$,

$$P_{ij} = x_i + y_j \in X \subset \widetilde{C}^{(2)},$$

then $\pi^{-1}(z) = \{P_{ij} \mid i, j = 1, \ldots, n\}$.

By construction, the image $D(P_{ij}) = (p_2)_*(D \cap (\{P_{ij}\} \times X))$ is given by

$$D(P_{ij}) = \sum_{l=1,\ l\neq j}^{n} P_{il} + \sum_{k=1,\ k\neq i}^{n} P_{kj}.$$

In particular, the correspondence D is fixed point free. Moreover, a straightforward computation shows that

$$D^2(P_{ij}) = (2n - 2)P_{ij} + (n - 2)D(P_{ij}) + 2\sum_{k\neq i,\ l\neq j} P_{kl},$$

thus we get:

$$D^2(P_{ij}) - (2n - 4)P_{ij} - (n - 4)D(P_{ij}) = 2\pi^*\big(\pi(P_{ij})\big).$$

This implies that the endomorphism γ_D of the Jacobian $J(X)$ induced by D satisfies the equation

$$\gamma_D^2 + (4 - n)\gamma_D - (2n - 4) = 0.$$

Recall that according to a theorem of Kanev (see [2, Theorem 12.9.1]) an effective fixed point free symmetric correspondence D on a smooth projective curve X defines a Prym–Tyurin variety of exponent e if and only if the endomorphism γ_D associated to D satisfies the equation

$$\gamma_D^2 + (e - 2)\gamma_D - (e - 1) = 0.$$

Together with the above reasoning this implies:

Proposition 2.1. $P = \mathrm{Im}(\gamma_D - 1)$ *is a Prym–Tyurin variety for the curve X if and only if $n = 3$. In this case the exponent of X is 3.*

Remark 2.2. Observe that on X there is another natural effective symmetric correspondence:

$$D'(P_{ij}) = \sum_{k \neq i,\, l \neq j} P_{kl},$$

which is symmetric and whose associated endomorphism $\gamma_D \in \mathrm{End}(JX)$ satisfies the equation $\gamma_{D'}^2 + (n-2)\gamma_{D'} - (n-1) = 0$. However D' has fixed points, so we do not know whether it induces a Prym–Tyurin variety.

3. Existence of the curve X

Let $h : C \to \mathbb{P}^1$ be a hyperelliptic covering of genus g as above. We want to determine those étale coverings $f : \tilde{C} \to C$ of degree 3 for which the curve X defined by diagram (2.1) is smooth and irreducible.

Recall that if $B_h = \{a_1, \ldots, a_{2g+2}\} \subset \mathbb{P}^1$ denotes the branch locus of h and σ_i denotes the class of the path from a fixed point $z_0 \in \mathbb{P}^1$ going around a_i once, then

$$\pi_1\left(\mathbb{P}^1 \setminus B_h, z_0\right) = \left\langle \sigma_1, \ldots, \sigma_{2g+2} \;\middle|\; \prod_{i=1}^{2g+2} \sigma_i = 1 \right\rangle. \tag{3.1}$$

Let

$$\mu : \pi_1\left(\mathbb{P}^1 \setminus B_h, z_0\right) \to S_6$$

be a classifying homomorphism for the composed map $f \circ h : \tilde{C} \to C \to \mathbb{P}^1$ and denote

$$G = \mathrm{Im}(\mu) \subset S_6.$$

By construction, $\mu(\sigma_i) = t_1 t_2 t_3$ where t_1, t_2 and t_3 are disjoint transpositions, but not all such products can occur.

In fact, if we denote as above $h^{-1}(z) = x + ix$, $f^{-1}(x) = \{x_1, x_2, x_3\}$ and $f^{-1}(ix) = \{y_1, y_2, y_3\}$ and if we identify (x_1, x_2, x_3) with $(1, 3, 5)$ and (y_1, y_2, y_3) with $(2, 4, 6)$, then exactly the following 6 permutations can occur:

$$\bigl\{(1\,2)(3\,4)(5\,6),\, (1\,4)(2\,5)(3\,6),\, (1\,6)(2\,3)(4\,5),$$
$$(1\,2)(3\,6)(4\,5),\, (1\,4)(2\,3)(5\,6),\, (1\,6)(2\,5)(3\,4)\bigr\}.$$

Hence the existence of an étale covering $f : \tilde{C} \to C$ is equivalent to the existence of a homomorphism $\mu : \pi_1(\mathbb{P}^1 \setminus B_h, z_0) \to S_6$ as above such that the image $G = \mathrm{Im}(\mu)$ is a transitive subgroup of S_6. By a direct computation, one checks that there are up to conjugation exactly 3 types of transitive subgroups of S_6 generated by a subset of the above set of permutations, namely:

I: $G = \langle (1\,2)(3\,4)(5\,6), (1\,4)(2\,5)(3\,6) \rangle \simeq S_3$,

II: $G = \langle (1\,2)(3\,4)(5\,6), (1\,4)(2\,5)(3\,6), (1\,2)(3\,6)(4\,5) \rangle \simeq S_2 \times S_3$,

III: G generated by all 6 permutations of above, i.e.,

$G = \langle (2\,4\,6), (1\,5)(2\,4), (1\,4)(2\,5)(3\,6) \rangle \simeq S_3 \times S_3$.

In order to see in which cases the associated curve X is smooth and irreducible, we describe the monodromy associated to the construction of X. For this we have to analyze the action of the group G on a fibre of $\pi : X \to \mathbb{P}^1$, i.e., the action of G on the set $\{x_1 + y_1, x_1 + y_2, \dots, x_3 + y_3\}$. One immediately checks that the in the cases I and II this action is not transitive. Hence the normalization of X is not connected in these cases. So for the rest of this section let G denote the group of case III. Then we have:

Lemma 3.1. *If $f : \widetilde{C} \to C$ is an étale covering of degree 3 of a hyperelliptic curve C such that the image of a classifying homomorphism $\mu : \pi_1(\mathbb{P}^1 \setminus B_h, z_0) \to S_6$ is a group of type* III, *then the curve X of diagram* (2.1) *is smooth and irreducible.*

Proof. The stabilizer of the element $P_{11} = x_1 + y_1$ of the fibre $\pi^{-1}(z)$ is the group $G_{P_{11}} = \langle (1\,2)(3\,4)(5\,6), (3\,4)(5\,6) \rangle$, which is Klein's group of 4 elements. Since G is of order 36, this means that G acts transitively on the set $\{x_1 + y_1, x_1 + y_2, \dots, x_3 + y_3\}$ implying that X is irreducible. The proof of the fact that X is smooth is a slight generalization of the proof of [2, Lemma 12.8.1]. $\quad\square$

Let

$$
\begin{array}{c}
\text{Gal}(h \circ f) \\[2em]
\end{array}
$$
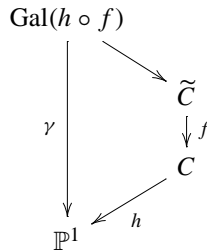
denote the Galois extension of $h \circ f : \widetilde{C} \to \mathbb{P}^1$. The next proposition identifies $\text{Gal}(h \circ f)$ in terms of the geometric construction of Section 2. For this consider the curve $Y = \widetilde{D} - \widetilde{\Delta} \subset \widetilde{X} \times \widetilde{X}$ of Section 1. It can be considered as a symmetric $(2, 2)$-correspondence on $\widetilde{X}$ without fixed points. So the projections q_1 and $q_2 : Y \to \widetilde{X}$ coincide and are étale of degree 2. Let us observe that in this case $(\deg f = 3)$ the map $p_1 : \widetilde{X} \to \widetilde{C}$ is étale (see proof of Lemma 5.1). Let δ denote the composed map $Y \xrightarrow{q_1} \widetilde{X} \to \mathbb{P}^1$.

Proposition 3.2. *The map $\delta : Y \to \mathbb{P}^1$ coincides with the Galois extension $\gamma : \text{Gal}(h \circ f) \to \mathbb{P}^1$.*

Proof. As in Section 2, for $z \in \mathbb{P}^1$, denote $h^{-1}(z) = x + ix$ and $f^{-1}(x) = \{x_1, x_2, x_3\}$, $f^{-1}(ix) = \{y_1, y_2, y_3\}$. Then the fibre $\delta^{-1}(z)$ for a general $z \in \mathbb{P}^1$ consists of the 36 ele-

ments $\{((x_1, y_1), (x_1, y_2)), ((x_1, y_1), (x_1, y_3)), ((y_1, x_1), (y_1, x_2)), ((y_1, x_1), (y_1, x_3)), \ldots,$
$((y_3, x_3), (y_3, x_2))\}$. It is immediate to check that the group G acts transitively on these
fibres or equivalently that the stabilizer of a point, say $((x_1, y_1), (x_1, y_2))$, is trivial. This
implies that the Galois covering $\text{Gal}(h \circ f)$ is a normalization of Y. The smoothness of Y
follows from the fact that Y is a symmetric fixed point free correspondence on the smooth
curve $\widetilde{X}$. $\quad \square$

In order to study the Prym–Tyurin variety P of Proposition 2.1, we have to take into
account also the subgroups of G, since to every such subgroup there corresponds an in-
termediate covering of $\delta : Y \to \mathbb{P}_1$. For this note that the triple products of transpositions
generating the group G form two conjugation classes in G, namely

$$C_1 = \{(1\,2)(3\,4)(5\,6), (1\,4)(2\,5)(3\,6), (1\,6)(2\,3)(4\,5)\},$$
$$C_2 = \{(1\,2)(3\,6)(4\,5), (1\,4)(2\,3)(5\,6), (1\,6)(2\,5)(3\,4)\}.$$

Moreover, we have

$$G = \langle C_1, C_2 \rangle \simeq \langle C_1 \rangle \times \langle C_2 \rangle \simeq S_3 \times S_3. \tag{3.2}$$

Let $\tau : G \to G$ denote the outer automorphism interchanging the direct factors $\langle C_1 \rangle$ and
$\langle C_2 \rangle$ of G. The subgroup diagram of G consists of 8 conjugacy classes of subgroups in-
variant under τ and of 14 pairs of different conjugacy classes of subgroups (G_1, G_2) with
$\tau(G_1) = G_2$. We need only the following part of it:
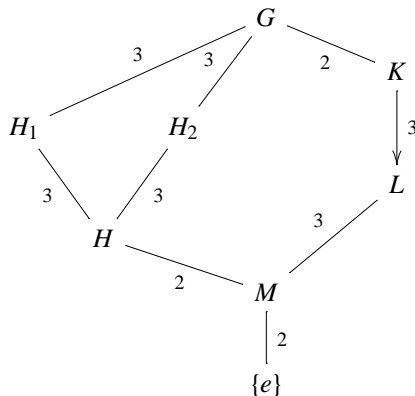
where

$$H = \langle (1\,2)(3\,4)(5\,6), (3\,5)(4\,6) \rangle,$$
$$H_1 = \langle (1\,2)(3\,4)(5\,6), (3\,5)(4\,6), (1\,3\,5)(2\,4\,6) \rangle = \langle (1\,2)(3\,4)(5\,6), C_2 \rangle,$$
$$H_2 = \langle (1\,2)(3\,4)(5\,6), (3\,5)(4\,6), (1\,3\,5)(2\,6\,4) \rangle = \langle (1\,2)(3\,6)(4\,5), C_1 \rangle,$$

$$K = \langle (2\,4\,6), (1\,3\,5), (1\,5)(2\,4) \rangle,$$
$$L = \langle (2\,4\,6), (3\,5)(4\,6) \rangle \quad \text{and}$$
$$M = \langle (3\,5)(4\,6) \rangle.$$

Lemma 3.3. *Up to conjugation in G, we have*

(a) $Y/L = \widetilde{C}$ *and* $Y/K = C$,
(b) $Y/H = X$ *and* $Y/M = \widetilde{X}$.

Proof. Observe first that $L = G \cap S_5$, where S_5 denotes the stabilizer of the symbol 1 in S_6. Hence $\mu^{-1}(L) \subset \pi_1(\mathbb{P}^1 \setminus B_h, z_0)$ is isomorphic to the fundamental group of $\widetilde{C} \setminus (h \circ f)^{-1} B_h$. Hence $Y/L = \widetilde{C}$ and by Galois theory it follows that $Y/K = C$, since there is no other subgroup between L and G. This completes the proof of (a).

In order to see (b), note first that the subgroup H is the stabilizer in G of the set $\{1, 2\}$. This implies that the action of G on the set of classes G/M gives a homomorphism $\lambda : G \to S_9$, which is injective and with image a transitive subgroup of S_9. Hence, by construction of the curve X the composition $\lambda \circ \mu : \pi_1(\mathbb{P}^1 \setminus B_h, z_0) \to S_9$ is a classifying morphism for the covering $\pi : X \to \mathbb{P}^1$. Denoting by S_8 the stabilizer of a symbol in S_9, this implies that $(\lambda \circ \mu)^{-1}(S_8)$ is isomorphic to the fundamental group of $X \setminus \pi^{-1}(B_h)$. But a short computation shows that $\lambda^{-1}(S_8) = H$. Hence the fundamental group of $X \setminus \pi^{-1}(B_h)$ is isomorphic to $\mu^{-1}(H)$ implying $Y/H = X$. Since M is the stabilizer of the ordered set $(1, 2)$ in G, one shows in a similar way that $Y/M = \widetilde{X}$. $\square$

Combining everything and denoting $X_1 = Y/H_1$ and $X_2 = Y/H_2$, we obtain the following diagram of morphisms of smooth projective curves:

$$(3.3)$$
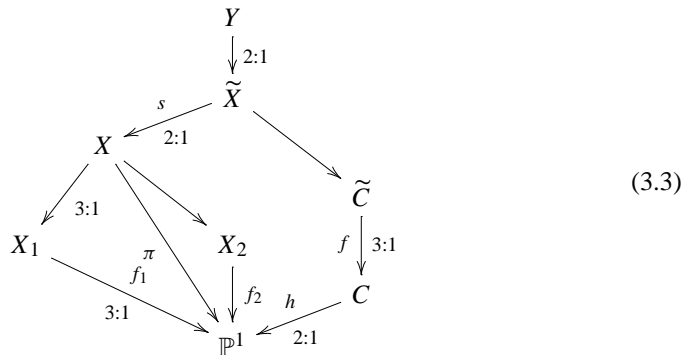
Recall from Section 2 the correspondence $D = (\lambda \times \lambda)_*(Y) \subset X \times X$.

Proposition 3.4. $\lambda \times \lambda|_Y : Y \to D$ *is an isomorphism.*

Proof. The map is given by $((x_i, y_j), (x_i, y_k)) \to (x_i + y_j, x_i + y_k)$. The morphism $D \to Y$ defined by $(a, b) \to ((a \cap b, a - a \cap b), (a \cap b, b - a \cap b))$ is inverse to it. Here

a and b are considered as divisors on the curve $\widetilde{C}$ and $a \cap b$ denotes the greatest common divisor of a and b. $\quad \square$

4. The moduli spaces

Let the notation be as at the end of the last section. So $f : \widetilde{C} \to C$ is an étale covering of degree 3 of the hyperelliptic curve $h : C \to \mathbb{P}^1$ with branch locus $B_h = \{a_1, \dots, a_{2g+2}\}$, such that the composition $h \circ f$ is given by a classifying homomorphism $\mu : \pi_1(\mathbb{P}^1 \setminus B_h, z_0) \to G \subset S_6$ as in (3.1) and $G \simeq S_3 \times S_3$, of type III. In this section we want to study the moduli of this situation.

Consider again the direct product decomposition (3.2). Since $\mu(\sigma_i) \in C_1 \cup C_2$, we can enumerate the σ_i in such a way that

$$\mu(\sigma_i) = (g_i, 1) \in \langle C_1 \rangle \times \langle C_2 \rangle \quad \text{for } i = 1, \dots, \alpha,$$

$$\mu(\sigma_i) = (1, g_{\alpha+i}) \in \langle C_1 \rangle \times \langle C_2 \rangle \quad \text{for } i = 1, \dots, \beta,$$

for some α and β with $\alpha + \beta = 2g + 2$. In particular, $g_i \in C_1$ for $i = 1, \dots, \alpha$ and $g_i \in C_2$ for $i = \alpha + 1, \dots, \alpha + \beta$. Then the condition $\prod_{i=1}^{2g+2} \sigma_i = 1$ is equivalent to the two conditions

$$\prod_{i=1}^{\alpha} g_i = 1 \quad \text{and} \quad \prod_{i=1}^{\beta} g_{\alpha+i} = 1. \tag{4.1}$$

So we must have α and β even with

$$\alpha \geqslant 4 \quad \text{and} \quad \beta \geqslant 4,$$

since under the isomorphisms $\langle C_1 \rangle \simeq S_3$ and $\langle C_2 \rangle \simeq S_3$ the elements of C_1 and C_2 correspond to transpositions, that is are of order 2.

Using the notation of above it makes sense to call an étale covering $f : \widetilde{C} \to C$ of degree 3 *of type* (α, β) if $\mu(\sigma_i) \in C_1$ for $i = 1, \dots, \alpha$ and $\mu(\sigma_i) \in C_2$ for $i = \alpha + 1, \dots, \alpha + \beta = 2g + 2$. The following theorem is the main result for studying the moduli space of étale degree 3 coverings of hyperelliptic curves of type (α, β).

Theorem 4.1. *Suppose* α, β *are even integers* $\geqslant 4$ *with* $\alpha + \beta = 2g + 2$ *and* $a_1, \dots, a_{2g+2} \in \mathbb{P}^1$ *pairwise different. There is a canonical* $(1 : 1)$-*correspondence between the sets of*

(1) *coverings* $\widetilde{C} \xrightarrow{f} C \xrightarrow{h} \mathbb{P}^1$ $\deg h = 2$ *ramified exactly over* $a_1, \dots, a_{2g+2}$ *and* f *unramified of type* (α, β) *of degree 3, and*
(2) *pairs of trigonal curves* $f_1 : X_1 \to \mathbb{P}^1$ *simply ramified exactly over* $a_1, \dots, a_\alpha$ *and* $f_2 : X_2 \to \mathbb{P}^1$ *simply ramified exactly over* $a_{\alpha+1}, \dots, a_{2g+2}$.

Proof. Given (f, h) as in (1) it follows from (4.1) that the homomorphism μ induces homomorphisms μ_1 and μ_2 such that the following diagram is commutative:

$$
\begin{array}{ccc}
\pi_1\big(\mathbb{P}^1 \setminus \{a_1, \ldots, a_\alpha\}, z_0\big) & \xrightarrow{\ \mu_1\ } & S_3 \\
{\scriptstyle l_1} \big\uparrow & & \big\uparrow {\scriptstyle p_1} \\
\pi_1\big(\mathbb{P}^1 \setminus \{a_1, \ldots, a_{2g+2}\}, z_0\big) & \xrightarrow{\ \mu\ } & G = \langle C_1 \rangle \times \langle C_2 \rangle \\
{\scriptstyle l_2} \big\downarrow & & \big\downarrow {\scriptstyle p_2} \\
\pi_1\big(\mathbb{P}^1 \setminus \{a_{\alpha+1}, \ldots, a_{2g+2}\}, z_0\big) & \xrightarrow{\ \mu_2\ } & S_3,
\end{array}
\qquad (4.2)
$$

where l_1 is induced by the inclusion $\mathbb{P}^1 \setminus B_h \to \mathbb{P}^1 \setminus \{a_1, \ldots, a_\alpha\}$ and similarly l_2 by $\mathbb{P}^1 \setminus B_h \to \mathbb{P}^1 \setminus \{a_{\alpha+1}, \ldots, a_{2g+2}\}$ and $p_i \colon G \to \langle C_i \rangle \simeq S_3$ for $i = 1$ and 2 is the projection followed by a fixed isomorphism $\langle C_i \rangle \simeq S_3$.

To μ_1 corresponds a triple covering $f_1 \colon X_1 \to \mathbb{P}^1$ simply ramified exactly over $a_1, \ldots, a_\alpha$. Similarly to μ_2 corresponds a triple covering $f_2 \colon X_2 \to \mathbb{P}^1$ simply ramified exactly over $a_{\alpha+1}, \ldots, a_{2g+2}$. The coverings f_1 and f_2 are uniquely determined up to an automorphism, since every automorphism of S_3 is inner.

Conversely, given (f_1, f_2) as in (2), to f_1 corresponds a homomorphism μ_1 and to f_2 a homomorphism μ_2 as in the diagram. Define $i_1 \colon S_3 \to G$ as the composition of a fixed isomorphism $S_3 \simeq \langle C_1 \rangle$ with the embedding of $\langle C_1 \rangle$ as the first factor and similarly $i_2 \colon S_3 \to G$ as the composition of a fixed isomorphism $S_3 \simeq \langle C_2 \rangle$ with the embedding of $\langle C_2 \rangle$ as the second factor. Then we can define $\mu \colon \pi_1(\mathbb{P}^1 \setminus \{a_1, \ldots, a_{2g+2}\}, z_0) \to G$ such that $\mu = i_1 \circ \mu_1 \circ l_1 = i_2 \circ \mu_2 \circ l_2$ by setting

$$
\mu(\sigma_i) = i_1 \circ \mu_1 \circ l_1(\sigma_i) \quad \text{for } i = 1, \ldots, \alpha \text{ and}
$$

$$
\mu(\sigma_i) = i_2 \circ \mu_2 \circ l_2(\sigma_i) \quad \text{for } i = \alpha + 1, \ldots, 2g + 2.
$$

Since i_1 and i_2 map the transpositions of S_3 to the products of transpositions in $\langle C_1 \rangle$ and $\langle C_2 \rangle$, i.e., to C_1 and C_2, the homomorphism μ defines a composition of coverings $\widetilde{C} \xrightarrow{f} C \xrightarrow{h} \mathbb{P}^1$ as in (1). Certainly these maps are inverse to each other which completes the proof. $\quad\square$

Proposition 4.2. *The maps $f_i \colon X_i \to \mathbb{P}^1$ of Theorem* 4.1 *and of diagram* (3.3) *coincide. In particular,*

$$
X_i = Y/H_i \quad \text{for } i = 1, 2.
$$

Proof. Observe that if we fix the isomorphism $S_3 \xrightarrow{\sim} \langle C_1 \rangle$ by

$$
(1\,2) \mapsto (1\,2)(3\,4)(5\,6), \quad (1\,3) \mapsto (1\,4)(2\,5)(3\,6),
$$

and $S_3 \xrightarrow{\sim} \langle C_2 \rangle$ by

$$
(1\,2) \mapsto (1\,2)(3\,6)(4\,5), \quad (1\,3) \mapsto (1\,4)(2\,3)(5\,6),
$$

then with the notation of diagram (4.2) we have that the group $\mu^{-1}(\{1, (1\,2)\})$ is isomorphic to the fundamental group of $X_1 \setminus f_1^{-1}(\{a_1, \ldots, a_\alpha\})$. Hence the group $l_1^{-1}\mu_1^{-1}(\{1, (1\,2)\})$ is isomorphic to the fundamental group of $X_1 \setminus f_1^{-1}(\{a_1, \ldots, a_{2g+2}\})$. On the other hand,

$$l_1^{-1}\big(\mu_1^{-1}(\{1, (1\,2)\})\big) = \mu^{-1}\big(\langle (1\,2)(3\,4)(5\,6)\rangle \times \langle C_2\rangle\big) = \mu^{-1}(H_1),$$

which gives $X_1 \simeq Y/H_1$. Similarly

$$l_2^{-1}\big(\mu_2^{-1}(\{1, (1\,2)\})\big) = \mu^{-1}\big(\langle C_1\rangle \times \langle (1\,2)(3\,6)(4\,5)\rangle\big) = \mu^{-1}(H_2),$$

which gives $X_2 \simeq Y/H_2$. □

Corollary 4.3. *Let α, β be even integers $\geqslant 4$ with $\alpha + \beta = 2g + 2$ and $g \geqslant 3$. The moduli space $\mathcal{M}(\alpha, \beta)$ of étale degree 3 coverings of hyperelliptic curves of genus g of type (α, β) is isomorphic to the moduli space of pairs of trigonal coverings of genus $\frac{\alpha}{2} - 2$ and $\frac{\beta}{2} - 2$ with simple ramification and disjoint branching. In particular,* $\dim \mathcal{M}(\alpha, \beta) = \alpha + \beta - 3 = 2g - 1$.

Proof. Theorem 4.1 or to be more precise a slight generalization of it concerning families of the corresponding coverings implies that the moduli functors in question are isomorphic. This implies the statement about the moduli spaces. It remains to compute the dimension. Consider

$$A := \big(\times_{i=1}^{\alpha} \mathbb{P}^1\big) \setminus \Delta_\alpha \quad \text{and} \quad B := \big(\times_{i=1}^{\beta} \mathbb{P}^1\big) \setminus \Delta_\beta,$$

where Δ_α and Δ_β denote the corresponding discriminants and let

$$\pi_\alpha : H^{3,\alpha} \to A \quad \text{and} \quad \pi_\beta : H^{3,\beta} \to B$$

denote the Hurwitz spaces of triple covers of $\mathbb{P}^1$ simply ramified in α respectively β points. Let $R \subset A \times B$ be the open set

$$R = \big\{(a_1, \ldots, a_\alpha, a_{\alpha+1}, \ldots, a_{\alpha+\beta}) \mid a_i \neq a_{\alpha+j} \text{ for } 1 \leqslant i \leqslant \alpha, \ 1 \leqslant j \leqslant \beta\big\}.$$

Then $(\pi_\alpha \times \pi_\beta)^{-1}(R) \to R$ parametrizes pairs of simple triple covers with disjoint branching. The action of $\mathrm{PGL}(1)$ on R lifts to an action on $(\pi_\alpha \times \pi_\beta)^{-1}(R)$ and the quotient $(\pi_\alpha \times \pi_\beta)^{-1}(R)/\mathrm{PGL}(1)$ is the moduli space of pairs of trigonal coverings with simple ramification and disjoint branching. So $\dim \mathcal{M}(\alpha, \beta) = \alpha + \beta - 3$. □

5. Comparison with other ppav's

It is the aim of this section to relate the Prym–Tyurin varieties P introduced in Section 2 to the other principally polarized abelian varieties occurring in this situation. For

this we first determine the ramification of the maps and the genera of the curves occurring in diagram (3.3).

Recall that C is a hyperelliptic curve of genus g ramified over the points $a_1, \ldots, a_{2g+2} \in \mathbb{P}^1$ and $\widetilde{C}$ an étale covering of degree 3 of C and thus of genus $3g - 2$. Moreover, from the construction we get that all maps in the diagram (3.3) are unramified over $\mathbb{P}^1 \setminus \{a_1, \ldots, a_{2g+2}\}$. Suppose $f : \widetilde{C} \to C$ is of type (α, β). Then we have

Lemma 5.1.

(a) $g(\widetilde{C}) = 3g - 2$, $g(\widetilde{X}) = 9g - 8$, $g(Y) = 18g - 17$;
(b) $g(X) = 3g - 5$;
(c) $g(X_1) = \frac{\alpha}{2} - 2$ and $g(X_2) = \frac{\beta}{2} - 2$;
(d) $\dim P = g - 3$.

Proof. The covering $\widetilde{X} \to C$ is defined by restricting the étale map $f \times f$ in the following diagram:

$$
\begin{array}{ccc}
\widetilde{X} = (f \times f)^{-1}(C) & \hookrightarrow & \widetilde{C}^2 \\
\downarrow & & \downarrow {\scriptstyle f \times f} \\
C = \{(x, ix) \mid x \in C\} & \hookrightarrow & C^2.
\end{array}
$$

It follows that $\widetilde{X} \to \widetilde{C}$ is étale of degree 3. Recall from Section 2 that Y is a fixed point free symmetric $(2, 2)$-correspondence on the curve $\widetilde{X}$. Hence $Y \to \widetilde{X}$ is an étale double covering and Hurwitz formula gives (a). From the description of the fibre $(fh)^{-1}(z)$ for any $z \in \mathbb{P}^1$ in Section 2 we see that π is of ramification type $(2, 2, 2, 1, 1, 1)$. Again Hurwitz formula gives (b). (c) was proven already in Corollary 4.3.

Proof of (d): Let N_P denote the norm endomorphism associated to the endomorphism γ_D of the correspondence D. According to [2, 5.3.10] $\dim P$ is related to the analytic trace of N_P by

$$
\dim P = \frac{1}{3} \mathrm{Tr}_a(N_P).
$$

Since $\gamma_D = 1_X - N_P$, we have

$$
\mathrm{Tr}_r(\gamma_D) = 2g(X) - \mathrm{Tr}_r(N_P),
$$

where Tr_r denotes the rational trace, which is related to the analytic trace by $\mathrm{Tr}_r = 2 \,\mathrm{Re}\,\mathrm{Tr}_a$. On the other hand, according to a theorem of Weil (see [2, 11.5.2 and 3.1.3]) we have

$$
\mathrm{Tr}_r(\gamma_D) = 8,
$$

since the correspondence D is without fixed points. Putting everything together gives the assertion. $\square$

From Lemma 5.1 we deduce

$$\dim f_1^* J X_1 + \dim f_2^* J X_2 = \dim P.$$

This suggests that there should be a relation between $J X_1 \times J X_2$ and P. The following theorem is the main result of this section.

Theorem 5.2. *The canonical map $f_1^* + f_2^* : J X_1 \times J X_2 \to J X$ induces an isomorphism*

$$J X_1 \times J X_2 \simeq P.$$

The proof consists of a careful analysis of the action of the group G. Recall that S_3 admits 3 (absolutely) irreducible $\mathbb{Q}$-representations, the trivial and the alternating representations 1_{S_3} and U of dimension 1 and the standard 2-dimensional representation V. The tensor products of these are all irreducible representations of the group $G = S_3 \times S_3$. For any subgroup H of G let $\rho_H(1_H)$ denote the representation of G induced by the trivial representation 1_H of H.

Lemma 5.3. $\rho_{H_1}(1_{H_1}) - 1_G \simeq V \otimes 1_{S_3}$, $\rho_{H_2}(1_{H_2}) - 1_G \simeq 1_{S_3} \otimes V$.

Proof. Fix isomorphisms $\langle C_1 \rangle \simeq S_3$ and $\langle C_2 \rangle \simeq S_3$ and thus $G \simeq S_3 \times S_3$. Then $H_1 \simeq \langle (1\,2)(3\,4)(5\,6) \rangle \times \langle C_2 \rangle \simeq S_2 \times S_3$.

The representation $\rho_{H_1}(1_{H_1})$ is then given by the action of the group $G = S_3 \times S_3$ on the quotient $(S_3 \times S_3)/(S_2 \times S_3)$ or equivalently by the action of the first factor S_3 on the quotient S_3/S_2. But it is easy to see that this is just the representation $V + 1_{S_3}$. This implies the first equation. The second equation is proved in the same way. $\quad\square$

Proof of Theorem 5.2. The action of the group G on the curve Y induces a homomorphism $\mathbb{Q}[G] \to \mathrm{End}_0(JY)$ of the rational group ring $\mathbb{Q}[G]$. Using this, one can associate an abelian subvariety of JX to every projector $p \in \mathbb{Q}[G]$ and thus to every subrepresentation of $\mathbb{Q}[G]$ in a natural way. In particular, if W is an irreducible $\mathbb{Q}$-representation of G, and $(\,,\,)$ a G-invariant scalar product of W, then for any nonzero $w \in W$ a projector for W is given as follows (see [2, p. 434]):

$$p_w = \frac{\dim W}{|G| \cdot \|w\|^2} \sum_{g \in G} (w, gw) g.$$

It is well known (see, e.g., [7, Corollary 3.2]), that the pull-back of the Prym variety $P(f_i)$ of the morphism $f_i : X_i = Y/H_i \to Y/G = \mathbb{P}^1$ in J_Y corresponds to the representation $\rho_{H_i}(1_{H_i}) - 1_G$. But according to Lemma 5.3 both representations are irreducible. So choosing nonzero vectors $w_1 \in V \otimes 1_{S_3}$ and $w_2 \in 1_{S_3} \otimes V$ and denoting φ the composed map $Y \to \widetilde{X} \xrightarrow{s} X$ of diagram (3.3), this implies

$$p_{w_i}(JY) \sim \varphi^* f_i^* J X_i \tag{5.1}$$

for $i = 1$ and 2, where $\sim$ denotes isogeny.

Now an easy computation shows that the vector spaces $(V \otimes 1_{S_3})^H$ and $(1_{S_3} \otimes V)^H$ of H-invariants are one-dimensional. Choosing $w_1 \in (V \otimes 1_{S_3})^H \setminus 0$ and $w_2 \in (1_{S_3} \otimes V)^H \setminus 0$, we have

$$(w_i, hgw_i) = (w_i, ghw_i) = (w_i, gw_i)$$

for any $h \in H$ and $i = 1, 2$. Hence applying [6, Proposition 3.5], we get that the projector p_{w_i} descends to an endomorphism $\tilde{p}_{w_i} : JX \to JX$ such that

$$p_{w_i}(JY) = \varphi^* \big(\tilde{p}_{w_i}(JX) \big).$$

In fact,

$$\tilde{p}_{w_i}(x) = \frac{\dim V}{|G| \cdot \|w\|^2} \sum_{g \in G} (w, gw) \bar{g}(x),$$

where

$$\bar{g}(x) = \sum_{h \in H} \varphi(ghy) \tag{5.2}$$

with $y \in Y$ such that $\varphi(y) = x$.

So (5.1) implies

$$\tilde{p}_{w_i}(JX) \sim f_i^* JX_i. \tag{5.3}$$

Now the projector $\tilde{p}_{w_i}$ does not depend on the choice of the vector w_i the vector spaces $(V \otimes 1_{S_3})^H$ and $(1_{S_3} \otimes V)^H$ being one-dimensional. This implies that in (5.3) we actually have equality instead of only isogeny (see [2, Proposition 13.6.4]). So we conclude

$$\big(f_1^* + f_2^* \big)(JX_1 \times JX_2) = (\tilde{p}_{w_1} + \tilde{p}_{w_2})(JX). \tag{5.4}$$

In order to complete the proof of the theorem, we have to show that the image of $\tilde{p}_{w_1} + \tilde{p}_{w_2}$ is just P. For this we have to compute the projectors p_{w_1} and p_{w_2} explicitly.

Consider the decomposition

$$G = H_1 \cup (2\,4\,6)H_1 \cup (1\,3\,5)H_1.$$

If we define $v_1 = H_1$, $v_2 = (2\,4\,6)H_1$, $v_3 = (1\,3\,5)H_1$ and $e_1 = v_1 - v_2$, $e_2 = v_2 - v_3$, then the action of G on G/H_1 gives us the representation $V \otimes 1_{S_3} = e_1 \mathbb{Q} \oplus e_2 \mathbb{Q} \simeq \mathbb{Q}^2$.

Let us for example consider the action of $(3\,5)(4\,6) \in H$. We have $(3\,5)(4\,6)(H_1) = H_1$, $(3\,5)(4\,6)((2\,4\,6)H_1) = (1\,3\,5)H_1$ and hence $(3\,5)(4\,6)(e_1) = e_1 + e_2$ and $(3\,5)(4\,6)(e_2) = -e_2$. This means in matrix form $(3\,5)(2\,4) = \left(\begin{smallmatrix} 1 & 0 \\ 1 & -1 \end{smallmatrix} \right) : \mathbb{Q}^2 \to \mathbb{Q}^2$.

The next step is to introduce a G-invariant scalar product on $\mathbb{Q}^2$. One checks directly that

$$\langle (u_1, u_2), (v_1, v_2) \rangle = 2u_1v_1 - u_1v_2 - u_2v_1 + 2u_2v_2$$

is a G-invariant scalar product on $V \otimes 1_{S_3} = \mathbb{Q}^2$. We noted already above that $\dim(V \otimes 1_{S_3})^H = 1$. In fact, one observes that the matrix representation for the elements of H is either the above example or the identity, implying that $(2, 1)^t \in \mathbb{Q}^2$ is a generator of $(V \otimes 1_{S_3})^H$.

Choosing $w_1 = (2, 1)^t$, we are ready to compute the projector p_{w_1}. For example, the coefficient of $g = (24)(35)$ in p_{w_1} is:

$$\frac{\dim(V \otimes 1_{S_3})}{|G| \cdot \|w\|^2}(w, gw) = \frac{2}{36 \cdot 6} \left\langle (2, 1), \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle$$

$$= \frac{2}{36 \cdot 6}(-4 - 2 + 1 + 2) = \frac{2}{36 \cdot 6}(-3).$$

Proceeding in this way with all elements of G and the representation $V \otimes 1_{S_3}$ and similarly the representation $1_{S_3} \otimes V$, we obtain:

$$
\begin{aligned}
p_{w_1} + p_{w_2} = \frac{2}{36 \cdot 6} \cdot \Big\{ & 12 \sum h \\
& - 6 \sum (135)h - 6 \sum (153)h \\
& - 6 \sum (246)h - 6 \sum (264)h \\
& + 3 \sum (135)(246)h + 3 \sum (135)(264)h \\
& + 3 \sum (153)(246)h + 3 \sum (153)(264)h \Big\},
\end{aligned}
$$

where the sum is always to be taken over all $h \in H$. (Notice that even though H is not a normal subgroup of G, the same expression for $p_{w_1} + p_{w_2}$ is valid if we use right cosets instead of left cosets.) So we get

$$
\begin{aligned}
36(\tilde{p}_{w_1} + \tilde{p}_{w_2}) = {}& 4 \cdot 1_{JX} - 2\overline{(135)} - 2\overline{(153)} - 2\overline{(246)} - 2\overline{(264)} \\
& + \overline{(135)(246)} + \overline{(135)(264)} \\
& + \overline{(153)(246)} + \overline{(153)(264)},
\end{aligned}
$$

where $\overline{(135)}$ is the endomorphism of JX induced by (135) using (5.2).

Now with the notation of identifications of Sections 2 and 3 we have at the level of the curve X:

$$\overline{(1\,3\,5)}(P_{11}) = P_{21}, \qquad \overline{(1\,5\,3)}(P_{11}) = P_{31},$$

$$\overline{(2\,4\,6)}(P_{11}) = P_{12}, \qquad \overline{(2\,6\,4)}(P_{11}) = P_{13},$$

$$\overline{(1\,3\,5)(2\,4\,6)}(P_{11}) = P_{22}, \qquad \overline{(1\,3\,5)(2\,6\,4)}(P_{11}) = P_{23},$$

$$\overline{(1\,5\,3)(2\,4\,6)}(P_{11}) = P_{32}, \qquad \overline{(1\,5\,3)(2\,6\,4)}(P_{11}) = P_{33}.$$

So

$$36(\tilde{p}_{w_1} + \tilde{p}_{w_2})(P_{11}) = 4P_{11} - 2P_{21} - 2P_{31} - 2P_{12} - 2P_{13} + P_{22} + P_{23} + P_{32} + P_{33}.$$

On the other hand, recalling that $D(P_{11}) = P_{12} + P_{13} + P_{21} + P_{31}$, we get

$$3(1_X - D)(P_{11}) + \pi^*\big(\pi(P_{11})\big) = 36(\tilde{p}_{V \otimes 1_{S_3}} + \tilde{p}_{1_{S_3} \otimes V})(P_{11}).$$

This implies

$$(\tilde{p}_{w_1} + \tilde{p}_{w_2})(JX) = (1_{JX} - \gamma_D)(JX) = P,$$

which with (5.4) implies

$$\big(f_1^* + f_2^*\big)(JX_1 \times JX_2) = P.$$

But on the one hand, the restriction of the canonical principal polarization Θ of JX to P is of type $(3, \dots, 3)$ and on the other hand, the pull-back of Θ via $f_1^* + f_2^*: JX_1 \times JX_2 \to JX$ is also of type $(3, \dots, 3)$. This implies that $f_1^* + f_2^*: JX_1 \times JX_2 \to JX$ is a closed embedding, thus completing the proof of the theorem. □

Remark 5.4. We computed the dimension of all Jacobians and Prym varieties arising from the subgroup graph of the group G. It turns out that they are different from $\dim P$. So none of them is isogenous to the Prym–Tyurin variety P. The details will not be included here.

Now we are in a position to prove the corollary stated in the introduction.

Proof of the corollary of Theorems 4.1 and 5.2. Let X_1, X_2 and X be as stated in the corollary. According to Theorem 4.1 the pair of trigonal covers determines an étale degree 3 covering $f : \widetilde{C} \to C$ of a hyperelliptic curve C. So we are in the situation of Theorem 5.2 in the proof of which we saw that $f_1^* + f_2^* : JX_1 \times JX_2 \to JX$ is an embedding. It only remains to be noted that the curve X of Theorem 5.2 coincides with the curve X of the corollary, i.e., is the fibre product of the trigonal covers. But this comes from the fact that

$$\langle H_1, H_2 \rangle = G \quad \text{and} \quad H_1 \cap H_2 = H,$$
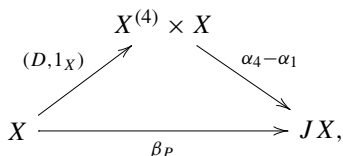
since the branchings of the trigonal covers are disjoint. □

6. The Abel–Prym map

Let the notation be as above. In particular, X is a smooth projective curve of genus $3g - 5$ and $D \subset X \times X$ the correspondence given by $D(P_{ij}) = \sum_{k \neq j} P_{ik} + \sum_{l \neq i} P_{lj}$ for $i, j = 1, \ldots, 3$. For any positive integer d let $X^{(d)}$ denote the d-fold symmetric product of X and $\alpha_d : X^{(d)} \to JX$ be the Abel map with respect to a base point $p_0 \in X$. In this section we analyse the Abel–Prym map of P which is by definition the composition

$$\beta_P : X \xrightarrow{\alpha_1} JX \xrightarrow{\gamma_D - 1_{JX}} P \subset JX.$$

By definition, β_P is given by the following diagram:

$$
\begin{array}{ccc}
 & X^{(4)} \times X & \\
{\scriptstyle (D, 1_X)} \nearrow & & \searrow {\scriptstyle \alpha_4 - \alpha_1} \\
X & \xrightarrow{\hspace{2cm}} & JX, \\
 & {\scriptstyle \beta_P} &
\end{array}
$$

where we consider the correspondence as a morphism $D : X \to X^{(4)}$.

Proposition 6.1. *If $g \geqslant 6$, the Abel–Prym map $\beta_P : X \to P$ is injective.*

Proof. Suppose β_P is not injective. Then according to the diagram there are two points $p_1, p_2 \in X$ such that $D(p_1) - p_1 \sim D(p_2) - p_2$, where $\sim$ means linear equivalence. This implies

$$D(p_1) + p_2 \sim D(p_2) + p_1,$$

which means that X admits a g_5^1, i.e., covering $X \to \mathbb{P}^1$ of degree $\leqslant 5$. On the other hand, X is a three to one covering of a curve of genus $g_1 \leqslant \frac{g-3}{2}$, namely $X \to X_1$. But then Castelnuovo's inequality (see [3]) implies

$$3g - 5 = g(X) \leqslant 2 \cdot 4 + 3 \cdot \frac{g-3}{2} \leqslant \frac{1}{2}(3g + 7).$$

So $g \leqslant 5$, a contradiction. $\quad\square$

Let us now analyse the local behaviour of the Abel–Prym map β_P. The tangent space $T_0 JX$ of JX at 0 can and will be identified with $H^0(X, \omega_X)^*$. The differential $(d\gamma_D)_0 : T_0 JX \to T_0 JX$ has just 2 eigenvalues, namely 1 with multiplicity $g(X) - \dim P = 2g - 2$ and -2 with multiplicity $\dim P = g - 3$. Let V_+ and V_- denote the corresponding eigenspaces. Clearly

$$T_0 P = V_-.$$

Define the *Prym–Tyurin canonical map* φ_P of P to be the composition of the canonical map $\varphi_X : X \to \mathbb{P}(T_0 J X)$ and the linear projection $r : \mathbb{P}(T_0 J X) \to \mathbb{P}(T_0 P)$ with center $\mathbb{P}(V_+)$:

$$\varphi_P : X \xrightarrow{\varphi_X} \mathbb{P}(T_0 J X) \setminus \mathbb{P}(V_+) \xrightarrow{r} \mathbb{P}(T_0 P).$$

The following lemma is an immediate consequence of the fact that the canonical map φ_X is everywhere defined:

Lemma 6.2. *A point $x \in X$ is a base point of the linear system $|T_0 P| = |V_-|$ if and only if $\varphi_X(x) \in |V_+|$.*

For any point $p \in P$ there is a canonical isomorphism of tangent spaces $T_p P \simeq T_0 P$ via which we identify the two vector spaces. In particular, we consider the differential of β_P at x as a map $(d\beta_P)_x : T_x X \to T_0 P$. Varying x in X, we obtain a homomorphism of the corresponding tangent bundles $d\beta_P : T_X \to T_P = P \times T_0 P$. It is easy to see that the projectivization of $d\beta_P$ coincides with the Prym–Tyurin canonical map

$$\varphi_P = P(d\beta_P) : X \to \mathbb{P}(T_0 P).$$

Using these facts, we are in a position to show:

Proposition 6.3. *If $g \geqslant 5$, the differential $(d\beta_P)_x : T_x X \to T_0 P$ of the Abel–Prym map is injective for every $x \in X$.*

Proof. Assume that for some $x \in X$ the differential of the Abel–Prym map at x is not injective, i.e., $(d\beta_P)_x = 0$. According to Lemma 6.2 and by what we have said above this means that x is a base point of the linear system $|T_0 P|$ which is the case if and only if the image of the canonical map satisfies $\varphi_X(x) \in \mathbb{P}(V_+)$. We will show that this leads to a contradiction. For this consider the commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ \ D\ \ } & X^{(4)} \\
{\scriptstyle \alpha_1}\downarrow & & \downarrow{\scriptstyle \alpha_4} \\
J X & \xrightarrow{\ \ \gamma_D\ \ } & J X.
\end{array}
$$

On the level of tangent spaces this gives

$$
\begin{array}{ccc}
T_x X & \xrightarrow{\ (dD)_x\ } & T_{D(x)} X^{(4)} \\
{\scriptstyle (d\alpha_1)_x}\downarrow & & \downarrow{\scriptstyle (d\alpha_4)_{D(x)}} \\
T_0 J X & \xrightarrow{\ (d\gamma_D)_0\ } & T_0 J X.
\end{array}
$$

In the same way as in the proof of Proposition 6.1 we conclude from Castelnuovo's inequality that for $g(X) \geqslant 10$ or equivalently $g \geqslant 5$ the curve X does not admit a g_4^1. This implies that the map $(d\alpha_4)_{D(x)} : T_{D(x)} X^{(4)} \to T_0 J X$ is an isomorphism onto its image.

Now let $t \in T_x X$ be a nonzero vector. Since the projectivized differential of the Abel map α_1 is the canonical map φ_X, we get from the assumption on x that $d\alpha_1(t) \in V_+$. But $(d\gamma_D)_0$ is the identity on V_+, implying

$$(d\gamma_D)_0 (d\alpha_1)_x(t) = (d\alpha_1)_x(t).$$

The commutativity of the above diagram implies

$$(d\alpha_1)_x(t) \in (d\alpha_4)_{D(x)} T_{D(x)} X^{(4)} \subset T_0 J X.$$

Projectivizing and using the fact that the projectivization of $(d\alpha_1)_x$ is the canonical map, we get

$$\varphi_X(x) \in \overline{D(x)} \subset |\omega_X|^*. \tag{6.1}$$

Here $\overline{D(x)}$ means the linear span of the divisor $D(x)$ in the projective space $|\omega_X|^* = \mathbb{P}^{3g-6}$. Now $h^0(D) = 1$ since X does not admit a g_4^1. Hence the geometric version of Riemann–Roch (see [1, p. 12]) implies

$$\dim \overline{D(x)} = 3.$$

But then (6.1) implies $\dim \overline{D(x) + x} = 3$ which again by the geometric version of Riemann–Roch implies $h^0(D(x) + x) = 2$. So the linear system $|D(x) + x|$ is a g_5^1. But $D(x) + x$ is part if a fibre of $\pi : X \to \mathbb{P}^1$ and the corresponding linear system does not admit fixed points. So this cannot occur completing the proof of the proposition. $\quad\square$

Combining Propositions 6.1 and 6.3, we proved the theorem stated in the introduction.

Let (A, Θ) denote a principally polarized abelian variety of dimension g. The cohomology class $\frac{1}{(g-1)!}[\Theta]^{g-1}$ is not divisible in $H^{2g-2}(A, \mathbb{Z})$ therefore called the dimension-one *minimal cohomology class* of (A, Θ). If $g \geqslant 3$ it is not at all clear whether a multiple of it contains a smooth irreducible curve. We obtain as a consequence of Theorem 5.3 and the result of this section

Corollary 6.4. *Let X_1 and X_2 be trigonal curves of positive genus with simple ramification and disjoint branching and let Θ denote the canonical product principal polarization of $JX_1 \times JX_2$. If $g := g(X_1) + g(X_2) \geqslant 3$, then the class $\frac{3}{(g-1)!}[\Theta]^{g-1}$ is represented by a smooth irreducible curve.*

For the proof we need the following well-known lemma, for which we include a proof in lack of a reference:

Lemma 6.5. *For $i = 1, 2$ let (A_i, Θ_i) be principally polarized abelian varieties with* $\text{End}_{\mathbb{Q}}(A_i) = \mathbb{Q}$ *and* $\text{Hom}(A_1, A_2) = 0$. *Then $A_1 \times A_2$ admits no principal polarization apart from the canonical product polarization.*

Proof. Let Θ denote the canonical product polarization of $A_1 \times A_2$. For any polarization L let $\varphi_L : A_1 \times A_2 \to (A_1 \times A_2)^*$ denote the associated isogeny onto the dual abelian variety. According to [2, Theorem 5.2.8] the map $NS(A_1 \times A_2) \to \text{End}(A_1 \times A_2)$, $L \mapsto \varphi_L \circ \varphi_{\Theta}^{-1}$, induces a bijection between the sets of principal polarizations and totally positive automorphisms of $A_1 \times A_2$ symmetric with respect to the Rosati involution. But $\text{End}(A_1 \times A_2) = \mathbb{Z} \times \mathbb{Z}$ and the only such automorphism is $1_{A_1} \times 1_{A_2}$, the Rosati involution being the identity. Since $1_{A_1} \times 1_{A_2}$ corresponds to the polarization Θ, this implies the assertion. $\quad\square$

Proof of Corollary 6.4. By Theorem 5.3, it suffices to show that the principal polarization Ξ of P coincides with the canonical product principal polarization of $JX_1 \times JX_2$. According to Lemma 6.5 it is enough to show that for general trigonal curves as in the corollary we have $\text{End}_{\mathbb{Q}}(JX_i) = \mathbb{Q}$ for $i = 1, 2$ and $\text{Hom}(JX_1, JX_2) = 0$, since if the polarizations coincide for general trigonal curves they do so for all such curves.

The second condition being obvious it suffices to show that a general trigonal curve X satisfies $\text{End}_{\mathbb{Q}}(JX) = \mathbb{Q}$. But for $1 \leqslant g(X) \leqslant 4$ any curve is trigonal and for $g(X) \geqslant 5$ the subspace of trigonal curves in the moduli space $\mathcal{M}_{g(X)}$ is of dimension $2g(X) + 1$. Hence the assertion follows from the main result of [4] which says that the Jacobian of a general member of a family of curves of genus g and dimension $> 2g - 2$ has endomorphism algebra $\mathbb{Q}$. $\quad\square$

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