The equivalence of some conjectures of Dade and Robinson

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Received 1 September 2001
Communicated by Michel Broué

Abstract

We demonstrate that Conjecture 4.1 of [G.R. Robinson, Proc. London Math. Soc. (3) 72 (1996) 312–330] and Dade’s projective conjecture are equivalent in a way which is compatible with the $p$-local rank. Further we consider refinements of these conjectures similar to those of Isaacs, Navarro and Uno, show their equivalence and demonstrate that in order to verify them it suffices to consider only those groups with no non-central normal $p$-subgroup.

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1. Introduction

We consider two variants of Alperin’s weight conjecture representing two different approaches to the problem. Both imply the weight conjecture and are refinements of the conjectures given in [5]. Conjecture 4.1 of Robinson [8] (which we shall refer to as “Robinson’s conjecture”) arises from the application of the results of [6] in order to make predictions about the relations between the local structure and the ordinary characters in a block, whilst keeping the notion of relative projectivity so fundamental to modular representation theory. Dade’s projective conjecture (see [2]) is a more direct refinement of the conjectures in [5], and is part of a series of conjectures formulated with the objective of giving a reduction to finite simple groups for the last (and strongest) of them. The object of this paper is twofold. Firstly to achieve a reduction to the case where all normal

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1 Partially supported by EPSRC award no. 95700575.
that $\chi$ with the $p$ where all normal $p$-subgroups are central. All equivalences and reductions are compatible with the $p$-local rank of a block as defined in [1].

Let $G$ be a finite group and $p$ a prime. Given a chain of $p$-subgroups $\sigma: Q_0 < Q_1 < \cdots < Q_n$ of $G$, define the length $|\sigma| = n$, the initial subgroup $V_\sigma = Q_0$, the $k$th initial subchain $\sigma_k: Q_0 < Q_1 < \cdots < Q_k$, and the normalizer $G_\sigma = N_G(\sigma) = N_G(Q_0) \cap N_G(Q_1) \cap \cdots \cap N_G(Q_n)$. Write $C(G|Q)$ for the set of those chains with initial subgroup $Q$ and $C(G)$ for the set of all chains in $G$.

A $p$-subgroup $Q$ of $G$ is radical if $O_p(N_G(Q)) = Q$, where $O_p(N_G(Q))$ is the maximal normal $p$-subgroup of the normalizer $N_G(Q)$. The $p$-chain $\sigma$ is said to be radical if $Q_i = O_p(N_G(\sigma_i))$ for each $i$. Denote by $R = R(G)$ the set of all radical $p$-chains of $G$.

Let $R$ be a complete local discrete valuation ring, with field of fractions $K$ of characteristic zero and residue field $k = R/J(R)$ algebraically closed of characteristic $p$.

If $X \subseteq C = C(G)$ and $B$ is a $p$-block of $G$, then write

$$X(G, B) = \{ \sigma \in X(G): \text{Blk}(G_\sigma|B) \neq \emptyset \},$$

where $\text{Blk}(G_\sigma|B)$ is the set of $p$-blocks of $G_\sigma$ whose Brauer correspondent in $G$ is $B$ (note that by [5, 3.2] every $p$-block of $G_\sigma$ does have a Brauer correspondent in $G$). Following [1] we define the $p$-local rank, $plr(B)$ of $B$ to be $\max(|\sigma|: \sigma \in R(G, B))$, and following [8] the $p$-local rank, $plr(G)$, of $G$ is $plr(G) = \max(|\sigma|: \sigma \in R(G))$.

For $H \leq G$ and $B$ a block of $G$, denote by $\text{Irr}(H, B)$ the set of irreducible characters of $H$ in Brauer correspondents of $B$. Write $k(H, B)$ for the cardinality of this set. Following Dade, define the defect of $\chi \in \text{Irr}(G)$ to be the integer $d$ such that $p^d \chi(1)_R = |G|_p$, and write $\text{Irr}_d(G)$ for the set of irreducible characters of $G$ with defect $d$. If $N = gG$, then we say that $\chi \in \text{Irr}(G)$ covers, or lies over $\mu \in \text{Irr}(N)$ if $(\chi|_N, \mu) \neq 0$, and write $\text{Irr}(G, \mu)$ for the set of such characters. $I_G(\mu)$ is the inertial subgroup. We combine all of these (and other) notations freely.

If $H$ is a section of $G$, then $f_0^H(B)(H)$ denotes the number of blocks of defect zero of $H$ corresponding to $B$ under the Brauer correspondence and the natural correspondence with a quotient group.

Let $H \leq G$. We say that $\chi \in \text{Irr}(G)$ is $H$-projective with respect to $R$ if there is some $H$-projective $RG$-module $M$ affording $\chi$. If $R$ is given then we just say that $\chi$ is $H$-projective.

Write $\nu(G, H)$ for the number of $H$-projective irreducible characters of $G$, and $\text{Irr}(G, H)$ for the set of such characters. As usual, we combine this with previous notation,
so that \( w_d(G, B, \mu, H) \) is the number of \( H \)-projective irreducible characters of defect \( d \) in the block \( B \) and covering \( \mu \).

Suppose that \( Q \triangleleft G \) is a \( p \)-group.

**Proposition 1.1** ([6] and [8]). The \( Q \)-projective irreducible characters covering \( \mu \) are in 1–1 correspondence with the \( p \)-blocks of defect zero of \( I_G(\mu)/Q \). Further, the number of \( Q \)-projective irreducible characters of defect \( d \), covering a given \( \lambda \in \text{Irr}(O_p(Z(G))) \), and lying in a given \( p \)-block \( B \) of \( G \) is

\[
\sum_{\mu \in \text{Irr}_d(Q, \lambda)/G} f(B) \left( \frac{I_G(\mu)}{Q} \right).
\]

**Proof.** This is [8, 4.5] and the results of [6]. \( \square \)

**Proposition 1.2** [8]. Let \( \mu \in \text{Irr}(Q) \). Then \( \chi \in \text{Irr}(G, \mu) \) is \( Q \)-projective if and only if

\[
\chi(1) = [G : Q] \mu(1).
\]

**Proof.** This is [8, 4.4]. \( \square \)

A trivial but useful observation following from these results will be needed when dealing with chains of \( p \)-subgroups:

**Corollary 1.3.** Let \( G \) be a finite group and \( Q \) be a normal \( p \)-subgroup of \( G \). Let \( \mu \in \text{Irr}(Q) \). If \( w(G, \mu, Q) \neq 0 \), then \( Q = O_p(I_G(\mu)) \).

Let \( B \) be a \( p \)-block of \( G \) (of any defect), \( d \) an integer and \( \lambda \in \text{Irr}(O_p(Z(G))) \). Robinson’s conjecture [8, 4.1] predicts that

\[
k_d(G, B, \lambda) = \sum_{\sigma \in \mathcal{R}(G)/G} (-1)^{|\sigma|} \sum_{\mu \in \text{Irr}_d(V_\sigma, \lambda)/G_\sigma} f(B) \left( \frac{I_G(\mu)}{V_\sigma} \right).
\]

Using Proposition 1.1 this is equivalent to:

**Conjecture 1.4** (Robinson). Let \( B \) be a \( p \)-block of \( G \), \( d \) an integer and \( \lambda \in \text{Irr}(O_p(Z(G))) \). Then

\[
k_d(G, B, \lambda) = \sum_{\sigma \in \mathcal{R}(G)/G} (-1)^{|\sigma|} w_d(G_\sigma, B, \lambda, V_\sigma).
\]

Dade’s projective conjecture states that:
Conjecture 1.5 (Dade). If $B$ has defect groups not equal to $O_p(G)$ and $O_p(G)$ is cyclic and contained in $Z(G) \cap G'$, then

$$\sum_{\sigma \in \mathcal{R}(G|O_p(G))/G} (-1)^{|\sigma|} k_d(G_\sigma, B, \lambda) = 0.$$ 

Recently Isaacs and Navarro conjectured a refinement to the Alperin–McKay conjecture (see [4]). Uno generalised this to a refinement of Dade’s conjecture (see [12]). We give the natural refinements of Conjectures 1.4 and 1.5.

For any $\chi \in \text{Irr}(G)$, denote by $\kappa(\chi)$ the integer such that $1 \leq \kappa(\chi) \leq (p - 1)$ and

$$\kappa(\chi) \equiv |G| \chi(1) (\text{mod } p).$$

Write $k(G, [\kappa]) = |\text{Irr}(G, [\kappa])| = \{\chi \in \text{Irr}(G): \kappa(\chi) \equiv \kappa \text{ (mod } p)\}$. If $M \subseteq \{1, \ldots, p - 1\}$, write $k(G, [M]) = |\bigcup_{\kappa \in M} \text{Irr}(G, [\kappa])|$. Hence $k(G, [M]) = k(G)$ when $M = \{1, \ldots, p - 1\}$. We give $w(G, [M])$ the obvious meaning, and again combine this notation with that given previously.

We have

Conjecture 1.6. Let $B$ be a $p$-block of $G$, $d$ an integer, $\lambda \in \text{Irr}(O_p(Z(G)))$ and $M \subseteq \{1, \ldots, p - 1\}$. Then

$$k_d(G, B, \lambda, [M]) = \sum_{\sigma \in \mathcal{R}(G)/G} (-1)^{|\sigma|} w_d(G_\sigma, B, \lambda, V_\sigma, [M]).$$

Conjecture 1.7 (Uno). If $B$ has defect groups not equal to $O_p(G)$ and $O_p(G)$ is cyclic and contained in $Z(G) \cap G'$, and $M \subseteq \{1, \ldots, p - 1\}$, then

$$\sum_{\sigma \in \mathcal{R}(G|O_p(G))/G} (-1)^{|\sigma|} k_d(G_\sigma, B, \lambda, [M]) = 0.$$ 

Conjecture 1.4 is a special case of Conjecture 1.6 and Conjecture 1.5 is a special case of Conjecture 1.7. We thus prove everything in the setting of the refined conjectures, fixing a subset $M$ of $\{1, \ldots, p - 1\}$.

Our main results are the following:

Theorem 1.8. Fix $p$ and $M \subseteq \{1, \ldots, p - 1\}$ in the statement of the conjectures.

(i) If Conjecture 1.6 holds with respect to $M$ for all $p$-blocks of $p$-local rank at most $n$ of finite groups $G$ with $O_p(G) \subseteq Z(G) \cap G'$ and $O_p(G)$ cyclic, then the conjecture holds with respect to $M$ for all $p$-blocks of $p$-local rank $n$.

(ii) If Conjecture 1.6 holds with respect to $M$ for all finite groups $G$ with $\text{plr}(G) \leq n$ and $O_p(G) \subseteq Z(G) \cap G'$ cyclic, then the conjecture holds with respect to $M$ for all finite groups with $p$-local rank $n$. In particular in proving the conjecture it suffices to check it just for those groups with $O_p(G) \subseteq Z(G) \cap G'$ cyclic.
Theorem 1.9. Fix \( p \) and \( M \subseteq \{1, \ldots, p-1\} \). Conjecture 1.7 holds with respect to \( M \) for every finite group if and only if Conjecture 1.6 does.

2. Finite groups with normal \( p \)-subgroups

In this section we consider what happens when a finite group \( G \) has a non-central normal \( p \)-subgroup. We use methods developed by Robinson in [8] to ‘prepare’ the formula predicted by Robinson’s conjecture, so that we may apply Clifford theory to obtain a proof of Theorem 1.8.

Let \( \lambda \in \text{Irr}(O_p(Z(G))) \), let \( d \) be an integer and let \( M \subseteq \{1, \ldots, p-1\} \). Write \( U = O_p(G) \). By Clifford’s theorem

\[
kd(G, B, \lambda, [M]) = \sum_{\mu \in \text{Irr}(U, \lambda)/G} kd(G, B, \mu, [M]).
\]

We aim to prove the following result, which then allows us to apply our Clifford-theoretic techniques:

Proposition 2.1. Let \( B \) be a \( p \)-block of \( G \). Then

\[
\sum_{\sigma \in R(G)/G} (-1)^{|\sigma|} w_d(G_\sigma, B, \lambda, V_\sigma, [M]) = \sum_{\mu \in \text{Irr}(U, \lambda)/G} \left( \sum_{\sigma \in R(I_G(\mu))/I_G(\mu)} (-1)^{|\sigma|} w_d(I_G(\mu)_\sigma, B, \mu, V_\sigma, [M]) \right).
\]

Denote by \( C_U(G) \) the set of all \( p \)-chains in \( C(G) \) whose initial subgroup contains \( U \) (not necessarily properly). By [9, 1.1] we may replace \( R \) in the first alternating sum of Proposition 2.1 with \( C_U \), since the initial subgroup of every radical \( p \)-chain is radical and so contains \( U \). We obtain

\[
\sum_{\sigma \in R(G)/G} (-1)^{|\sigma|} w_d(G_\sigma, B, \lambda, V_\sigma, [M]) = \sum_{\sigma \in C_U(G)/G} (-1)^{|\sigma|} w_d(G_\sigma, B, \lambda, V_\sigma, [M]).
\]

For each \( \sigma \in C_U(G) \),

\[
w_d(G_\sigma, B, \lambda, V_\sigma, [M]) = \sum_{\mu \in \text{Irr}(U, \lambda)/G_\sigma} w_d(G_\sigma, B, \mu, V_\sigma, [M]),
\]

so

\[
\sum_{\sigma \in C_U(G)/G} (-1)^{|\sigma|} w_d(G_\sigma, B, \lambda, V_\sigma, [M])
\]
= \sum_{\sigma \in \mathcal{C}_U(G)/G} (-1)^{|\sigma|} \left( \sum_{\mu \in \text{Irr}(U, \lambda)/G} w_d(G_\sigma, B, \mu, V_\sigma, [M]) \right).

**Lemma 2.2.**

\[\sum_{\sigma \in \mathcal{C}_U(G)/G} (-1)^{|\sigma|} \left( \sum_{\mu \in \text{Irr}(U, \lambda)/G} w_d(G_\sigma, B, \mu, V_\sigma, [M]) \right) = \sum_{\mu \in \text{Irr}(U, \lambda)/G} \left( \sum_{\sigma \in \mathcal{C}_U(G)/G \mu} (-1)^{|\sigma|} w_d(G_\sigma, B, \mu, V_\sigma, [M]) \right)\]

**Proof.** Observe that the pair \((\sigma, \mu)\) lies in an orbit of length \([G : G_\sigma] [I_{G_\sigma}(\mu)] = [G : I_{G_\sigma}(\mu)]\) in the first instance and \([G : I_G(\mu)] [I_{G_\sigma}(\mu)] = [G : I_{G_\sigma}(\mu)]\) in the second, and that \(w_d(G_\sigma, B, \mu, V_\sigma, [M])\) is constant under conjugation of \((\sigma, \mu)\) in \(G\).

We conclude that if \(k_d(G, B, \mu, [M]) = \sum_{\sigma \in \mathcal{C}_U(G)/G} (-1)^{|\sigma|} \left( \sum_{\mu \in \text{Irr}(U, \lambda)/G_\sigma} w_d(G_\sigma, B, \mu, V_\sigma, [M]) \right)\) for each \(\mu \in \text{Irr}(U, \lambda)\), then Conjecture 1.6 holds for that choice of \(B, d\) and \(\lambda\). We fix \(\mu \in \text{Irr}(U, \lambda)\) and write \(H = I_G(\mu)\). Clifford theory then allows us to move from counting characters of \(G\) and \(G_\sigma\) to counting characters of \(H\) and \(H_\sigma\):

**Lemma 2.3.** \(k_d(G, B, \mu, [M]) = k_d(H, B, \mu, [M])\) and \(w_d(G_\sigma, B, \mu, V_\sigma, [M]) = w_d(H_\sigma, B, \mu, V_\sigma \cap H, [M])\) for each \(\sigma \in \mathcal{C}_U(G)\).

**Proof.** For each \(\sigma \in \mathcal{C}_U(G)\), \(I_{G_\sigma}(\mu) = H_\sigma\), and so Clifford’s theorem gives a 1–1 correspondence \(\text{Irr}(H_\sigma, \mu) \leftrightarrow \text{Irr}(G_\sigma, \mu)\) given by induction of characters, and note that \(G = G_\sigma\) for the chain \(\sigma = U\) of length 0. This correspondence is clearly defect-preserving. Suppose that \(\chi \in \text{Irr}(G_\sigma, \mu)\) corresponds to \(\eta \in \text{Irr}(H_\sigma, \mu)\). Note that \([H : H_\sigma]/\eta(1)_{H_\sigma} = [G : G_\sigma]/\eta(1)_{G_\sigma}\). Now \(\chi\) is \(V_\sigma\)-projective if and only if \(\eta\) is \(V_\sigma \cap H\)-projective (for let \(\varphi \in \text{Irr}(V_\sigma \cap H, \mu)\) be a character covered by \(\eta\). Now \(I_{V_\sigma}(\mu) = H \cap V_\sigma\), so by Clifford’s theorem \(\varphi |_{V_\sigma} = \theta\) for some \(\theta \in \text{Irr}(V_\sigma, \mu)\). But

\[(\chi |_{V_\sigma}, \theta) = (\chi |_{V_\sigma}, \varphi |_{V_\sigma}) = (\eta |_{V_\sigma \cap H}, \varphi) = (\eta |_{H_\sigma}, \varphi |_{H_\sigma})\]

But \((\eta, \varphi |_{H_\sigma}) \neq 0\) and \((\eta, \varphi |_{H_\sigma}) \neq 0\), so \(\chi\) covers \(\theta\). We have \(\chi(1)_p = [G_\sigma : H_\sigma] \eta(1)_p\) and

\([G_\sigma : V_\sigma] \eta(1)_p = [G_\sigma : V_\sigma] [V_\sigma : V_\sigma \cap H] \varphi(1)_p = [G_\sigma : H_\sigma] [H_\sigma : V_\sigma \cap H] \varphi(1)_p\).
So \( \chi(1)_p = [G_\sigma : V_\sigma]_p \theta(1)_p \) if and only if \( \eta(1)_p = [H_\sigma : V_\sigma \cap H]_p \psi(1)_p \), and we may apply Proposition 1.2.

Every \( p \)-block of \( H_\sigma \) containing an irreducible character lying over \( \mu \) has a Brauer correspondent in \( G_\sigma \) and the character correspondence given by induction respects the Brauer correspondence, i.e., if \( \eta \in \text{Irr}(H_\sigma, \mu) \), then \( \eta^{G_\sigma} \) lies in the \( p \)-block of \( G_\sigma \) which is a Brauer correspondent of the \( p \)-block of \( H_\sigma \) containing \( \eta \). Hence \( w_d(G_\sigma, B, \mu, V_\sigma, [M]) = w_d(H_\sigma, B, \mu, V_\sigma \cap H, [M]) \) since the Brauer correspondence is transitive, and we’re done.

\[ \square \]

It follows that
\[
\sum_{\sigma \in \mathcal{C}(G)/H} (-1)^{||\sigma||} w_d(G_\sigma, B, \mu, V_\sigma, [M]) = \sum_{\sigma \in \mathcal{C}(G)/H} (-1)^{||\sigma||} w_d(H_\sigma, B, \mu, V_\sigma \cap H, [M]).
\]

We use the theory of deficient \( p \)-chains introduced in [8] in order to overcome the problem of summing over \( IG(\mu) \)-orbits of \( p \)-chains of \( G \) (note in particular that a given \( \sigma \in \mathcal{C}(G) \) need not lie in the stabilizer of \( \mu \)). Many of the arguments that follow have their origins in [8]. Given a subgroup \( T \) of \( G \), we say that a \( p \)-chain \( \sigma : Q_0 < \cdots < Q_n \in \mathcal{C}(G) \) is \( T \)-deficient if \( Q_n \cap T \leqslant O_p(G) \). Given a \( p \)-chain \( \sigma \in \mathcal{C}(G) \) we call the longest deficient initial subchain the \( T \)-deficient part, and denote it by \( d_T(\sigma) \). Note that \( d_T(\sigma) \) may be empty. Denote the empty chain by \( \emptyset \).

Returning to our original hypotheses, for brevity we write \( \mathcal{D}(G) = \mathcal{D}_H(G) \) for the set of non-empty \( H \)-deficient chains in \( \mathcal{C}_U(G) \). Write \( \mathcal{D}_T(G) = \mathcal{D}(G) - \{U\} \) and \( d(\sigma) = d_H(\sigma) \).

Observe that we may write
\[
\sum_{\sigma \in \mathcal{C}(G)/H} (-1)^{||\sigma||} w_d(H_\sigma, B, \mu, V_\sigma \cap H, [M])
\]
as
\[
\sum_{\tau \in \mathcal{D}(G)/H} \left( \sum_{\sigma \in \mathcal{C}(G^{V_\tau})/H_\tau} (-1)^{||\tau||} w_d((H_\tau)_\sigma, B, \mu, U, [M]) \right)
\]
\[
+ \sum_{\sigma \in \mathcal{C}(G)/H} (-1)^{||\sigma||} w_d(H_\sigma, B, \mu, V_\sigma \cap H, [M]),
\]
since \( V_\tau \cap H = U \) when \( \tau \in \mathcal{D}(G) \).

By an argument given in the proof of [8, 1.2] we may cancel the of contributions chains with non-radical deficient part in the above alternating sum.

Now fix \( \tau \in \mathcal{D}(G) \cap \mathcal{R}(G) \). By an argument similar to that given for [8, 1.2] we have further
\[
\sum_{\sigma \in \mathcal{C}(G^{V_\tau})/H_\tau} (-1)^{||\tau||+||\sigma||} w_d((H_\tau)_\sigma, B, \mu, U, [M])
\]
= ∑_{σ \in \mathcal{C}(G_1 | V^1) / H_i \atop d(σ) = σ_0} (-1)^{|τ| + |σ|} w_d((H_τ)_σ, B, U, [M]).

We next modify the alternating sum in a way which will later allow us to reduce to the consideration of chains of $H$ rather than $G$, the key to proving Proposition 2.1.

Denote by $\mathcal{C}(G_1 | V^1)$ the set of those chains $σ$: $Q_0 < \cdots < Q_n$ in $\mathcal{C}(G_1 | V^1)$ satisfying $Q_i = V^i (H \cap Q_i)$ for each $i = 0, \ldots, n$. We also allow $τ$ to be the empty chain, in which case we set $G_τ = G$ and set $\mathcal{C}_τ(G) = \mathcal{C}_U(H)$.

By a cancellation argument similar to that given in the proof of [8, 2.1], using the map of posets $X \to V^i (H \cap X)$, we see that

$$
\sum_{σ \in \mathcal{C}_U(G) / H \atop d(σ) = 0} (-1)^{|σ|} w_d((H_σ)_σ, B, U, [M])
= \sum_{σ \in \mathcal{C}_U(G) / H \atop d(σ) = σ_0} (-1)^{|σ|} w_d((H_σ)_σ, B, U, [M])
$$

(representing the case $τ$ is empty), and

$$
\sum_{σ \in \mathcal{C}(G_1 | V^1) / H_i \atop d(σ) = σ_0} (-1)^{|τ| + |σ|} w_d((H_τ)_σ, B, U, [M])
= \sum_{σ \in \mathcal{C}_U(H) / H \atop d(σ) = σ_0} (-1)^{|σ|} w_d((H_σ)_σ, B, U, [M]).
$$

Finally, by another argument given in the proof of [8, 2.1] we obtain

$$
\sum_{σ \in \mathcal{C}_U(G_1 | V^1) / H_i \atop d(σ) = 0} (-1)^{|σ|} w_d((H_σ)_σ, B, U, [M])
= \sum_{σ \in \mathcal{C}_U(H_1 | V^1) / H_i \atop d(σ) = σ_0} (-1)^{|σ|} w_d((H_σ)_σ, B, U, [M]),
$$

and

$$
\sum_{σ \in \mathcal{C}(G_1 | V^1) / H_i \atop d(σ) = σ_0} (-1)^{|τ| + |σ|} w_d((H_τ)_σ, B, U, [M])
= \sum_{σ \in \mathcal{C}(H_1 | V^1) / H_i \atop d(σ) = σ_0} (-1)^{|τ| + |σ|} w_d((H_τ)_σ, B, U, [M]).$$
(The first equality is immediate since $C^*_U(G)$ is by definition $C_U(H)$.)

**Proof of Proposition 2.1.** We have seen that

$$
\sum_{\sigma \in R(G)/G} (-1)^{|\sigma|} w_d\left(G_\sigma, B, \lambda, V_\sigma, [M]\right)
= \sum_{\mu \in \text{Irr}(U, \lambda)/G} \left( \sum_{\sigma \in C_U(G)/IG(\mu)} (-1)^{|\sigma|} w_d\left(G_\sigma, B, \mu, V_\sigma, [M]\right) \right)
$$

(see Lemma 2.2 and the discussion preceding it). We fix $\mu \in \text{Irr}(U, \lambda)$ and write $H = I_G(\mu)$. Further we have seen that we may write $\sum_{\sigma \in C_U(G)/H} (-1)^{|\sigma|} w_d\left(G_\sigma, B, \mu, V_\sigma, [M]\right)$ as

$$
\sum_{\tau \in D(G)/H} \left( \sum_{\sigma \in C(H_{\tau}|U)/H_{\tau}} (-1)^{|\tau| + |\sigma|} w_d\left((H_{\tau})_\sigma, B, \mu, U, [M]\right) \right) + \sum_{\sigma \in C_U(H)/H, d(\sigma) = \emptyset} (-1)^{|\sigma|} w_d\left(H_\sigma, B, \mu, V_\sigma\right),
$$

where the final alternating sum represents those chains in $C_U(G)$ with empty deficient part.

Now if we set $\alpha \in D(G)$ to be the chain $\alpha = U$ (of length zero), then

$$
\sum_{\sigma \in C(H_{\alpha}|U)/H_{\alpha}} (-1)^{|\alpha| + |\sigma|} w_d\left((H_\alpha)_\sigma, B, \mu, U, [M]\right)
= \sum_{\sigma \in C_U(H)/H} (-1)^{|\sigma|} w_d\left(H_\sigma, B, \mu, V_\sigma, [M]\right),
$$

as we are considering on the one hand chains whose initial subgroup is $U$ and on the other chains whose initial subgroup strictly contains $U$. It suffices to show that

$$
\sum_{\tau \in D(G)/H} \left( \sum_{\sigma \in C(H_{\tau}|U)/H_{\tau}} (-1)^{|\tau| + |\sigma|} w_d\left((H_{\tau})_\sigma, B, \mu, U, [M]\right) \right) = 0,
$$

since then we have

$$
\sum_{\sigma \in C_U(G)/H} (-1)^{|\sigma|} w_d\left(G_\sigma, B, \mu, V_\sigma, [M]\right)
= \sum_{\sigma \in C_U(H)/H} (-1)^{|\sigma|} w_d\left(H_\sigma, B, \mu, V_\sigma, [M]\right)$$
as required.

Suppose that \( \tau \in \mathcal{D}^G(G) \), and consider a chain \( \sigma \in \mathcal{C}(H_\tau(U)) \), \( \sigma : Q_0 \prec \cdots \prec Q_n \), of length \( |\sigma| > 0 \). Then \( U < Q_1 \triangleleft N_{H_\tau}(\sigma) = I_{N_{H_\tau}(\sigma)}(\mu) \), i.e., \( Q_1 \) is a normal \( p \)-subgroup of \( (H_\tau)^{\sigma} \) strictly containing \( U \) and stabilizing \( \mu \), and so \( w_{d^\tau}(\mu(U), \mu, U, [M]) = 0 \) by Corollary 1.3. Hence

\[
\sum_{\tau \in \mathcal{D}^G(G)/H} \left( \sum_{\sigma \in \mathcal{C}(H_\tau(U))/H} (-1)^{|\tau|+|\sigma|} w_d((H_\tau)^{\sigma}, \mu, U, [M]) \right) = \sum_{\tau \in \mathcal{D}^G(G)/H} (-1)^{|\tau|} w_d(H_\tau, \mu, U, [M]).
\]

But notice that we may pair each deficient chain \( \tau \in \mathcal{D}^G(G) \) satisfying \( V_\tau = U \) with another chain in \( \mathcal{D}^G(G) \) with initial term strictly containing \( U \), since we have defined \( \mathcal{D}^G(G) \) to exclude the chain \( \tau = U \). Clearly paired chains lie in \( H \)-orbits of the same size, and the lengths of the chains in each pair differ in parity, so we may cancel their contributions to this last alternating sum. This gives \( \sum_{\tau \in \mathcal{D}^G(G)/H} (-1)^{|\tau|} w_d(H_\tau, \mu, U, [M]) = 0 \), and Proposition 2.1. \( \square \)

Before continuing with the proof of Theorem 1.8 we give a brief summary of the Clifford-theoretic tools we use.

Suppose that \( N \lhd G \) and \( \mu \in \text{Irr}(N) \) is \( G \)-stable. Then following [3], we may construct a central extension \( \tilde{G} \) of \( G \) by a central cyclic subgroup \( \tilde{W} \) (whose order has prime divisors which are also divisors of \( |N| \)—see [6, 2.1]), so that \( \mu \) extends to an irreducible character, say \( \tilde{\mu} \), of \( \tilde{G} \). Let \( \tilde{G} = \tilde{G}/\tilde{N} \), where \( \tilde{N} \) is naturally isomorphic to \( N \) and \( \tilde{N} \cap \tilde{W} = 1 \). Let \( \tilde{W} \) be the image of \( \tilde{W} \) under the natural homomorphism, so that \( \tilde{G}/\tilde{W} \cong G/N \). Let \( \tilde{\theta} \) be the linear character of \( \tilde{W} \) covered by \( \tilde{\mu} \) and \( \tilde{\theta} \in \text{Irr}(\tilde{W}) \) be identified with the complex conjugate \( \tilde{\theta} \). For each \( d \) there is a 1–1 correspondence \( \text{Irr}_d(G, \mu) \leftrightarrow \text{Irr}_{d^{\tilde{G}}}((\tilde{G}, \tilde{\theta})) \) given by \( \chi = \tilde{\chi} \tilde{\mu} \) (regarding \( \chi \) as a character of \( \tilde{G} \)), where \( d = d + \log_p(\mu(1)_p |\tilde{W}|_p/|N|_p) \). If two characters \( \chi_1, \chi_2 \in \text{Irr}(G, \mu) \) lie in the same block of \( G \), then the corresponding \( \tilde{\chi}_1, \tilde{\chi}_2 \) lie in the same block of \( \tilde{G} \). We call the blocks of \( \tilde{G} \) thus associated to a given block \( B \) of \( G \) the Dade correspondents of \( B \) with respect to \( \mu \). The Dade correspondence commutes with the Brauer correspondence. Full details of the above are given in [3], and it should be noted that those results are purely an elementary version of the results of [2]. Note that \( \tilde{W} \leq Z(\tilde{G}) \cap [\tilde{G}, \tilde{G}] \). In general we denote objects associated to \( \tilde{G} \) with a ‘‘\( \tilde{\cdot} \)’’ and objects associated to \( G \) with a ‘‘\( \cdot \)’’.
An elementary computation using Proposition 1.2 tells us that when $N$ is a $p$-group, $\chi$ is $N$-projective if and only if the corresponding $\tilde{\chi}$ is $W$-projective.

**Remark.** Note that given a block $B$ of $G$, there is a unique block $\tilde{B}$ of $\tilde{G}$ corresponding to $B$ (since $\tilde{W} \subseteq Z(\tilde{G})$). Then $\tilde{B}$ dominates the Dade correspondents of $B$ through an $R\tilde{G}$-module affording $\mu$ in the sense of [7]. This will be important when we come to use [1, 4.1].

**Proof of Theorem 1.8.** (i) Let $B$ be a $p$-block of $G$ with $plr(B) = n$. Write $U = O_p(G)$. We have seen (by Proposition 2.1, its proof and the accompanying discussion) that in order to verify Robinson’s conjecture for $B$ (for a given $\lambda \in \text{Irr}(O_p(Z(G)))$ and integer $d$) it suffices to show that

$$k_d(I_G(\mu), B', \mu, [M]) = \sum_{\sigma \in \mathcal{R}(I_G(\mu), B')/I_G(\mu)} (-1)^{|\sigma|} w_d(I_G(\mu)\sigma, B', \mu, V_\sigma, [M])$$

for each $\mu \in \text{Irr}(U, \lambda)$ and each block $B'$ of $I_G(\mu)$ which is a Brauer correspondent of $B$. Fix $\mu \in \text{Irr}(U, \lambda)$ and write $H = I_G(\mu)$. We now use the results described above and the notation introduced there freely, with $H$ and $U$ in place of $G$ and $N$ respectively.

Let $\tilde{B}$ be the sum of Dade correspondents for $B'$ with respect to $\mu$. It is clear that $\mathcal{R}(\tilde{H}/\tilde{W})/\tilde{H}$ and $\mathcal{R}(H/U)/H$ may be identified, and since every radical $p$-subgroup contains $O_p(H)$ and $U \trianglelefteq O_p(H)$, we may further identify $\mathcal{R}(H)/H$ and $\mathcal{R}(\tilde{H})/\tilde{H}$. For each $\sigma \in \mathcal{R}(H)$, define $\tilde{\sigma} \in \mathcal{R}(\tilde{H})$ in the obvious way. We may make the identification

$$\text{Irr}_d(H_\sigma, B', \mu, V_\sigma, [M]) \leftrightarrow \text{Irr}_d(\tilde{H}_\sigma, \tilde{B}, \tilde{\sigma}, V_\sigma, [M]),$$

where $\tilde{d} = d + \log_p(\mu(1)\tilde{W}/|U|)$.

since the Dade correspondence commutes with the Brauer correspondence. Note that since $U$ and $W$ are $p$-groups we need make no modification to the set $M$ used.

Hence (1) if and only if

$$k_{\tilde{d}}(\tilde{H}, \tilde{B}, \tilde{\sigma}, [M]) = \sum_{\tilde{\sigma} \in \mathcal{R}(\tilde{H})/\tilde{H}} (-1)^{|\tilde{\sigma}|} w_{\tilde{d}}(\tilde{H}_\sigma, \tilde{B}, \tilde{\sigma}, V_\sigma, [M]).$$

Now $\tilde{W} \trianglelefteq O_p(\tilde{H})$, and this is precisely the statement of Conjecture 1.6 for the sum of blocks $\tilde{B}$. Observe that each component of $\tilde{B}$ is related to $B$ by the Brauer correspondence and by domination, and so by [1, 3.2, 4.1] each has $p$-local rank at most $n$. If $O_p(\tilde{H}) \leq \tilde{W}$, then we are done. Otherwise, observe that $[\tilde{H} : O_p(\tilde{H})] < [G : O_p(G)]$. For each component block of $\tilde{B}$, we may repeat the whole argument with $\tilde{H}$ and $O_p(\tilde{H})$ in place of $G$ and $U$ respectively until $[\tilde{H} : O_p(\tilde{H})] = [G : O_p(G)]$ (with a slight abuse of notation), in which case $O_p(\tilde{H}) \leq \tilde{W}$ and we are done.

The proof of (ii) is almost identical to that of part (i), using the $p$-local rank of a group rather than that of a block. □
3. The equivalence of Robinson’s conjecture and Dade’s projective conjecture

Lemma 3.1. Let \( B \) be a \( p \)-block of \( G \) and fix \( M \subseteq \{1, \ldots, p−1\} \). Suppose that \( plr(B) = n \) and that Conjecture 1.6 holds with respect to \( M \) for blocks of \( p \)-local rank strictly less than \( n \). Write \( U = O_p(G) \). Then

\[
\sum_{\sigma \in R(G)/G} (-1)^{|\sigma|} k_d(G_\sigma, B, \lambda, \,[M]) = \sum_{\sigma \in R(G)/G} (-1)^{|\sigma|} w_d(G_\sigma, B, \lambda, \,[M])
\]

if and only if

\[
k_d(G, B, \lambda, \,[M]) = \sum_{\sigma \in R(G)/G} (-1)^{|\sigma|} w_d(G_\sigma, B, \lambda, \, V_\sigma, \,[M]).
\]

The analogous result holds considering radical \( p \)-chains for \( G \) rather than \( B \), and the \( p \)-local rank of a group instead of that for a block.

Proof. Note that by the inductive hypotheses both equalities hold for all blocks of \( p \)-local rank less than \( n \). Suppose that the former holds. Let \( R_0(G, B) \) be the set of all radical \( p \)-subgroups of \( B \) (i.e., terms of radical \( p \)-chains of length zero). We have

\[
\sum_{\sigma \in R(G)/G} (-1)^{|\sigma|} w_d(G_\sigma, B, \lambda, \,[M])
= \sum_{Q \in R_0(G)/G} \left( \sum_{\sigma \in R(N_G(Q))/N_G(Q)} (-1)^{|\sigma|} w_d(N_G(Q)_\sigma, B, \lambda, \, Q, \,[M]) \right)
= \sum_{Q \in R_0(G)/G} \left( \sum_{\sigma \in R(N_G(Q))/N_G(Q)} (-1)^{|\sigma|} k_d(N_G(Q)_\sigma, B, \lambda, \,[M]) \right)
= \sum_{\sigma \in R(G)/G} (-1)^{|\sigma|} k_d(G_\sigma, B, \lambda, \,[M]).
\]

We may pair every chain \( Q_0 < \cdots < Q_r \) in \( R(G) \) in which \( Q_0 \neq U \) with the chain \( U < Q_0 < \cdots < Q_r \) in \( R(G) \) of length \( r+1 \). These two chains have the same stabilizer, so their contributions cancel. We may cancel all chains in \( R(G) \) in this way except for the chain \( U \). Hence

\[
\sum_{\sigma \in R(G)/G} (-1)^{|\sigma|} w_d(G_\sigma, B, \lambda, \, V_\sigma, \,[M]) = k_d(G, B, \lambda, \,[M])
\]

and Conjecture 1.6 is satisfied in that case.

Now suppose that \( B \) satisfies the second equality. Then
\[
kd(G, B, \lambda, [M]) = \sum_{\sigma \in \mathcal{R}(G)/G} (-1)^{|\sigma|}w_d(G_\sigma, B, \lambda, V_\sigma, [M])
\]
\[
= \sum_{Q \in \mathcal{R}_0(G)/G} \left( \sum_{\sigma \in \mathcal{R}(N_G(Q))/N_G(Q)} (-1)^{|\sigma|}w_d(N_G(Q)_\sigma, B, \lambda, Q, [M]) \right)
\]
\[
= \sum_{\sigma \in \mathcal{R}(G)/G} (-1)^{|\sigma|}w_d(G_\sigma, B, \lambda, U, [M])
\]
\[
+ \sum_{U \neq Q \in \mathcal{R}_0(G)/G} \left( \sum_{\sigma \in \mathcal{R}(N_G(Q)/Q)/N_G(Q)} (-1)^{|\sigma|}kd(N_G(Q)_\sigma, B, \lambda, [M]) \right).
\]

Replacing each chain \(Q_0 < \cdots < Q_r\) in \(\mathcal{R}(G)\) in which \(U \neq Q_0\) by \(U<Q_0<\cdots<Q_r\), we have
\[
\sum_{Q \neq Q_0 \in \mathcal{R}_0(G)/G} \left( \sum_{\sigma \in \mathcal{R}(N_G(Q)/Q)/N_G(Q)} (-1)^{|\sigma|}kd(N_G(Q)_\sigma, B, \lambda, [M]) \right)
\]
\[
= \sum_{\sigma \in \mathcal{R}(G(U)/G)} (-1)^{|\sigma|+1}kd(G_\sigma, B, \lambda, [M]) + kd(G, B, \lambda, [M]),
\]
and so
\[
\sum_{\sigma \in \mathcal{R}(G(U)/G)} (-1)^{|\sigma|}kd(G_\sigma, B, \lambda, [M]) = \sum_{\sigma \in \mathcal{R}(G(U)/G)} (-1)^{|\sigma|}w_d(G_\sigma, B, \lambda, U, [M])
\]
as required. \(\square\)

**Proof of Theorem 1.9.** Suppose that Conjecture 1.6 holds with respect to \(M\). Then it follows immediately from Lemma 3.1 that Conjecture 1.7 also holds with respect to \(M\).

Suppose that Conjecture 1.7 holds with respect to \(M\). Let \(plr(B) = n\) and suppose that Conjecture 1.6 holds with respect to \(M\) for every block with \(p\)-local rank strictly less than \(n\). Then by Lemma 3.1 Conjecture 1.6 holds with respect to \(M\) for every \(p\)-block with \(p\)-local rank \(n\) of every finite group \(H\) with \(Op(H) \leq Z(H) \cap H'\) cyclic. Then by Theorem 1.8, Conjecture 1.6 holds with respect to \(M\) for \(B\). Since Conjecture 1.6 is identical to Conjecture 1.7 for blocks of \(p\)-local rank one of finite groups with \(Op(H) \leq Z(H) \cap H'\), the result follows by induction on the \(p\)-local rank. \(\square\)

**Corollary 3.2.** Fix \(M \subseteq \{1, \ldots, p - 1\}\).

(i) If
\[
\sum_{\sigma \in \mathcal{R}(G(U)/G)} (-1)^{|\sigma|}kd(G_\sigma, B, \lambda, [M]) = \sum_{\sigma \in \mathcal{R}(G(U)/G)} (-1)^{|\sigma|}w_d(G_\sigma, B, \lambda, U, [M])
\]
for all $p$-blocks of $p$-local rank at most $n$ of finite groups $G$ with $O_p(G) \leq Z(G) \cap G'$ and $O_p(G)$ cyclic, then Conjecture 1.7 holds with respect to $M$ for all blocks of $p$-local rank $n$.

(ii) The analogous result to (i) holds for the $p$-local rank of a group rather than the $p$-local rank of a block.

**Proof.** This follows by induction from Theorem 1.8 and Lemma 3.1. □

**Acknowledgments**

The material for this paper consists of part of my doctoral research, and I thank my supervisor Geoffrey Robinson for all his help. I also thank Burkhard Külshammer and Wayne Wheeler for their useful comments and suggestions. I also thank the referee for their thorough reading and for their helpful comments.

**References**