ELSEVIER

Topology and its Applications 55 (1994) 289–305

TOPOLOGY AND ITS APPLICATIONS

Some conditions under which tri-quotient or compact-covering maps are inductively perfect *

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(Received 3 September 1992)
(Revised 10 December 1992)

Abstract

We solve two problems posed by E. Michael (1981) concerning whether certain compact-covering maps are inductively perfect, we generalize a theorem of A. Ostrovsky on a condition under which tri-quotient maps are inductively perfect and we present an example which answers a more recent question of Michael in the same general area.

Key words: Inductively perfect maps; Compact-covering maps; Tri-quotient maps; Sieve-completeness; Partition-completeness; W_δ-set

AMS (MOS) Subj. Class.: 54E40, 54C10, 54E99

1. Introduction and preliminaries

In order to state the problems we need a few definitions. All maps will be assumed to be continuous. Recall that a surjective map $f: X \to Y$ is *inductively perfect* if there is an $X' \subseteq X$ such that $f[X'] = Y$ and $f \mid X'$ is perfect. A surjective map $f: X \to Y$ is *(countable-)compact-covering* if every (countable and) compact $K \subseteq Y$ is the image of some compact $C \subseteq X$.

In [7] Michael asked the following questions (which we state verbatim):

* The authors would like to thank the referee for valuable remarks.
** Partially supported by NSF grant DMS 9016021.
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0166-8641/94/$07.00
SSDI 0166-8641(93)E0050-X

Question 1.1. Let $f: X \to Y$ be a map from a separable metrizable space X onto a metrizable space Y, with each $f^{-1}y$ compact.

(a) If f is countable-compact-covering, must f be compact-covering?

(b) If f is compact-covering, must f be inductively perfect?

Question 1.2. Let $f: X \to Y$ be a map from a separable metrizable space X onto a countable metrizable space Y. If f is compact-covering, must f be inductively perfect?

In [2] Question 1.1(a) is solved in the affirmative, actually without using separability of X. Here we present affirmative solutions to both Questions 1.1(b) and 1.2. These are given in Sections 2 and 3, respectively.

Questions 1.1 and 1.2 were originally asked in [6] as special cases, involving only elementary concepts, of two more general questions about tri-quotient maps. These maps, which were introduced in [5] and which will be defined in Section 4, include all open maps, all inductively perfect maps, and all countable-compact-covering maps with regular domain, first-countable Hausdorff range, and Lindelöf fibres. See [6] for details.

Ostrovsky proved in [9] that if $f: X \to Y$ is a tri-quotient map of a regular space X onto a paracompact space Y such that there exists a perfect extension $F: X^* \to Y$ of f such that X is a G_δ-set in X^*, then f is inductively perfect. In Section 4 we show that this remains true if we replace "G_δ-set" by the more general concept of "W_δ-set" (see Section 4 for the definition of a W_δ-set) and obtain the same conclusion. This generalization of Ostrovsky's theorem can be used to give an alternative proof of [5, Theorem 6.6].

In Section 5 we describe an example which provides a negative answer to a recent question of Michael [8] concerning partition-completeness.

Our topological terminology generally follows that of [3] and our set-theoretic terminology follows that of [4].

Cantor–Bendixson height 1.3. Let X be a topological space and α be an ordinal. The *αth derivative of* X, denoted by $D^{(\alpha)}X$, is defined inductively as follows:

$$D^{(0)}X = X,$$

$$D^{(\alpha+1)}X = D^{(\alpha)}X \setminus \{x\colon x \text{ is an isolated point in } D^{(\alpha)}X\},$$

$$D^{(\alpha)}X = \bigcap_{\beta<\alpha} D^{(\beta)}X \text{ for limit ordinals } \alpha.$$

The smallest α for which $D^{(\alpha)}X = D^{(\alpha+1)}X$ is called *the Cantor–Bendixson height of* X (abbreviated CB-height in the sequel). Note that the CB-heights of compact sets K are 0 or successor ordinals, and if X is a countable space, then the CB-height of X is a countable ordinal.

Claim 1.4. *Let X be a space, $K \subseteq X$ be compact, and let $(W_n)_{n\in\omega}$ be a decreasing base for K in X. If $C_i \subseteq W_i$ is compact for all $i \in \omega$, then $\bigcup_{n\in\omega} C_n \cup K$ is compact.*

Proof. Suppose $\mathscr{U}$ is an open cover of $C = \bigcup_{n \in \omega} C_n \cup K$. There is a finite $\mathscr{U}' \subseteq \mathscr{U}$ such that $K \subseteq \bigcup \mathscr{U}'$. Thus there is $j \in \omega$ such that $W_j \subseteq \bigcup \mathscr{U}'$. For each $i < j$ there is a finite $\mathscr{U}_i \subseteq \mathscr{U}$ such that $C_i \subseteq \bigcup \mathscr{U}_i$. Then $\bigcup\{\mathscr{U}_i\colon i < j\} \cup \mathscr{U}'$ is a finite subcollection of $\mathscr{U}$ which covers C. □

Remark 1.5. In Section 2 we shall need the following fact which can easily be deduced from Claim 1.4 by noting that the H appearing in the fact has a decreasing base in X:

Let H be a compact G_δ-subset of a compact space X and let $(W_i\colon i \in \omega)$ be a decreasing base for a compact subspace K of a space Y. If C_n is compact for all $n \in \omega$, and $C_n \subseteq H \times W_n$, then $\bigcup_{n \in \omega} C_n \cup (H \times K)$ is a compact subset of $X \times Y$.

2. Solution of problem 1.1(b)

The aim of this section is to prove Theorem 2.4, which actually gives an affirmative answer to both parts of Question 1.1. Note, however, that our proof makes use of the affirmative answer to Question 1.1(a) in [2] (see the beginning of the proof of Theorem 2.0). We do not know whether one can obtain a positive answer to Question 1.1(b) without either quoting the positive answer to Question 1.1(a) or essentially repeating the argument from [2]. We believe that the proof of Theorem 2.0 given here is of independent interest.

First we concentrate on a special case.

Theorem 2.0. *Let L, M be compact metric spaces, let $A \subset L \times M$, and denote by p_2 the projection map on the second coordinate. Assume $p_2 | A : A \to p_2[A]$ is countable-compact-covering and, for all $y \in p_2[A]$, the set $(p_2 | A)^{-1}\{y\}$ is compact. Then $p_2 | A$ is inductively perfect.*

We will prove Theorem 2.0 using Lemma 2.1 below. Let A be as in Theorem 2.0. Define $A^+ = A \cup \{(t, z)\colon\ t \in L,\ z \in M \setminus p_2[A]\}$. We shall call $B \subseteq L \times M$ *countable-compact-covering* iff the map $p_2 | B : B \to p_2[B]$ is countable-compact-covering.

Now we are ready to formulate the key lemma. Since we believe that this lemma is of some independent interest, we shall formulate a slightly more general version than is needed in the proof of Theorem 2.0.

Lemma 2.1. *Suppose L is compact and perfectly normal, M is a first-countable T_3-space, and $A \subseteq L \times M$. If all fibres of $p_2 | A$ are compact, and A is countable-compact-covering, then A^+ is also countable-compact-covering.*

Proof. We shall prove the lemma only for the case that M is a Tychonoff space. One can prove the result for T_3-spaces M by an uninspiring modification of the present, more transparent argument, and we shall need the lemma only for metric M anyway.

In this proof, we use the following terminology: Let U, V be nonempty sets, $U \subseteq L$, $V \subseteq M$. Then, relative to A as in Theorem 2.0, the pair (U, V) is said to be *nice* iff for every compact, countable $K \subseteq V \cap p_2[A]$ there exists a compact $C \subseteq (U \times K) \cap A$ such that $p_2[C] = K$. (Such a set will be called a *compact lifting* of K.)

For $\alpha < \omega_1$, the pair (U, V) is called α-*nice* iff for every compact, countable $K \subseteq V$ of CB-height $\leqslant \alpha + 1$, there is a compact $C \subseteq (U \times K) \cap A^+$ such that $p_2[C] = K$. Note that the definition of "nice" involves A, whereas the definition of "α-nice" involves A^+. Also note that if $\beta < \alpha$, then α-niceness implies β-niceness.

Consider the following nice statements:

st(α, U, V): If the pair (U, V) is nice, then it is α-nice.

Claim 2.2. *If $\alpha < \omega_1$, then* st(α, U, V) *holds for every pair (U, V) such that U is a nonempty closed subset of L, and V is a nonempty open subset of M.*

Proof. Assume toward a contradiction that the claim is false. Let α be the smallest ordinal such that st(α, U, V) fails for some U, V as in the assumption, and let (U, V) be a witness. If K is countable and compact of CB-height 0, then $K = K^{(0)} = \emptyset$ is a compact lifting of K, so we may assume (since st(β, U, V) holds for every $\beta < \alpha$) that

$$\text{for every countable compact subset } K \subseteq V \text{ of CB-height} \leqslant \alpha \text{ there is a compact } C \subseteq (U \times V) \cap A^+ \text{ such that } p_2[C] = K. \tag{$*$}$$

Now let K be a compact, countable subset of V of CB-height $\alpha + 1$. Since $D^{(\alpha)}K$ is finite, we may restrict our attention to the case where $D^{(\alpha)}K = \{y\}$ for some y (if $K = K_0 \cup \cdots \cup K_l$ and C_i is a compact lifting of K_i for each $i \leqslant l$, then $C_0 \cup \cdots \cup C_l$ is a compact lifting of K).

Case 1: $y \notin p_2[A]$.

Let $(V_i\colon i \in \omega)$ be a decreasing open neighborhood base for y such that $K \subseteq V_0 \subseteq V$ and, for all $i \in \omega$, the boundary Fr(V_i) is disjoint from K (possible, since K is countable and M is Tychonoff and first-countable). For all $i \in \omega$, let $K_i = K \cap (V_i \setminus V_{i+1})$. Then K_i is countable and compact (K_i is a closed subspace of K since $\text{cl}_M(V_i) \cap K = (\text{Fr}(V_i) \cup V_i) \cap K = V_i \cap K$, thus $(V_i \cap K) \setminus V_{i+1}$ is closed in K), and of CB-height $\leqslant \alpha$, so by $(*)$ it has a compact lifting $C_i \subseteq (U \times V_i) \cap A^+$. Let $C = \bigcup_{i \in \omega} C_i \cup (U \times \{y\})$. Clearly, $p_2[C] = \bigcup_{i \in \omega} p_2[C_i] \cup \{y\} = K$. Moreover, by Remark 1.5, C is a compact subset of $(U \times V) \cap A^+$.

Case 2: $y \in p_2[A]$.

Let $S = \{x\colon (x, y) \in A\} = (p_2 \mid A)^{-1}\{y\}$. By assumption, S is compact, so $S \cap U$ is a compact subspace of U (recall that U is a closed subspace of the compact space L). Since $S \cap U$ is G_δ in U, there exists an external base (relative to U) $\{W_i\colon i \in \omega\}$ of $S \cap U$ such that $\text{cl}_U W_{i+1} \subseteq W_i$ for every i.

Subclaim 2.3. *For every $i \in \omega$ there exists an open $V_i \subseteq V$ such that $y \in V_i$ and the pair (W_i, V_i) is nice.*

Proof. If not, there exist an $i \in \omega$ and a decreasing neighborhood base $\{V^j: j \in \omega\}$ of y such that $V^0 = V$ and for all $j \in \omega$, there is a compact, countable $K_j \subseteq V^j \cap p_2[A]$ such that every compact lifting $C_j \subseteq A$ of K_j contains a point not in $W_i \times V^j$. The set $K = \bigcup_{j \in \omega} K_j \cup \{y\}$ is countable and compact by Claim 1.4. Since (U, V) is nice and $K \subseteq V \cap p_2[A]$, there is a compact lifting $C \subseteq (U \times V) \cap A$ of K. Since $p_2 | C$ is perfect, $C_j = C \cap p_2^{-1} K_j$ is a compact lifting of K_j. Thus for all $j \in \omega$, there exists $(x_j, y_j) \in C_j \setminus (W_i \times V^j)$, so that $x_j \notin W_i$. By the compactness of C, the sequence $((x_j, y_j): j \in \omega)$ has a cluster point $(u, v) \in C$. Since $y_j \in V^j$ for all j, we have $\lim_{j \to \infty} y_j = y$, thus $v = y$. But $x_j \in U \setminus W_i$ implies that $u \notin W_i$. On the other hand, $u \in S \cap U \subseteq W_i$, which is a contradiction. □

Proof of Claim 2.2 (continued). We may now choose for every $i \in \omega$ a V_i as in Subclaim 2.3 in such a way that $V_{i+1} \subseteq V_i$ and $\{V_i: i \in \omega\}$ is a base at y such that $\mathrm{Fr}(V_i) \cap K = \emptyset$ (note that if (W_i, V_i) is nice and $G_i \subseteq V_i$, then (W_i, G_i) is also nice).

For all $i \in \omega$, let $K_i = K \setminus V_i$. Then K_i is of CB-height $\leqslant \alpha$. Also $K_{i+1} \setminus K_i = (V_i \setminus V_{i+1}) \cap K$ is closed in K (for $\mathrm{cl}_M(V_i) \cap K \subseteq (\mathrm{Fr}(V_i) \cup V_i) \cap K = V_i \cap K$). Thus $K_{i+1} \setminus K_i$ is a compact, countable subspace of V_i of CB-height $\leqslant \alpha$, so by the choice of α it has a compact lifting $C_i \subseteq (\mathrm{cl}_L(W_i) \times V_i) \cap A^+$. (We use one more monotonicity property of niceness: If (W_i, V_i) is nice, then so is $(\mathrm{cl}_L(W_i), V_i)$.) Let $C = \bigcup_{i \in \omega} C_i \cup ((S \cap U) \times \{y\})$. Note that $\mathrm{cl}_L(W_{i+1}) \subseteq W_i$ and thus $\bigcap_{i \in \omega} \mathrm{cl}_L(W_i) = S \cap U$.) Now Remark 1.5 implies that $C \subseteq (U \times V) \cap A^+$ is compact. Since $p_2[C] = K$, this is the lifting required in Case 2. □

Proof of Lemma 2.1 (continued). Assume A is countable-compact-covering. Then the pair (L, M) is nice. Let $K \subseteq M$ be countable and compact. Suppose K has CB-height $\leqslant \alpha + 1$. By Claim 2.2, the pair (L, M) is α-nice, hence K has a compact lifting $C \subseteq A^+$. □

Proof of Theorem 2.0. Let A be as in the assumptions of Theorem 2.0. Then in particular, A satisfies the assumptions of Lemma 2.1, and thus A^+ is also countable-compact-covering. By [2], the set A^+ is also compact-covering. Hence there exists a compact set $C \subseteq A^+$ such that $(p_2 | A^+)[C] = M$. Let $\tilde{C} = C \cap A$, and $f = p_2 | \tilde{C}$. Then f is a perfect mapping of $\tilde{C}$ onto $p_2[A]$: Note first that if $y \in p_2[A]$, there exists x such that $(x, y) \in C$. Since C is contained in the union of the disjoint sets $\tilde{C}$ and $L \times (M \setminus p_2[A])$, it follows that $(x, y) \in \tilde{C}$. Hence f is onto. Also, for $y \in p_2[A]$, we have $f^{-1}\{y\} = p_2^{-1}\{y\} \cap \tilde{C} = (p_2^{-1}\{y\} \cap A) \cap C$, which is compact.

It remains to show that the map f is closed. If $B \subseteq \tilde{C}$ is closed, then $B = \mathrm{cl}_{L \times M}(B) \cap \tilde{C}$. If $y \in p_2[\mathrm{cl}_{L \times M}(B)] \cap p_2[A]$, then there is an x such that $(x, y) \in \mathrm{cl}_{L \times M}(B) \subseteq C$. Since $y \in p_2[A]$, the point $(x, y) \in \mathrm{cl}_{L \times M}(B) \cap \tilde{C} = B$. Thus $f[B] = p_2[B] = p_2[\mathrm{cl}_{L \times M}(B)] \cap p_2[A]$. Since the closed subset $\mathrm{cl}_{L \times M}(B)$ of the compact metric space $L \times M$ is compact, its continuous image $p_2[\mathrm{cl}_{L \times M}(B)]$ is also compact. Hence $f[B]$ is closed in $p_2[A]$, and we have shown that f is a closed and thus a perfect mapping. □

The next theorem covers the general case and thus yields a positive answer to Question 1.1(b).

Theorem 2.4. *Let $f: X \to Y$ be a countable-compact-covering map from a separable metrizable space X onto a metrizable space Y such that each fibre $f^{-1}\{y\}$ is compact. Then f is inductively perfect.*

Proof. We apply Theorem 2.0. We assume X and Y are embedded in I^{ω} and, in the notation of Theorem 2.0, let $L = M = I^{\omega}$. Let $A = \Gamma(f)$, the graph of f, and let $p_X = p_1 \mid \Gamma(f)$, $p_Y = p_2 \mid \Gamma(f)$. Clearly, p_Y is countable-compact-covering. Also $p_Y^{-1}\{y\} = f^{-1}\{y\} \times \{y\}$ is compact for all $y \in Y$. Thus Theorem 2.0 implies that p_Y is inductively perfect. Thus there exists $B \subseteq \Gamma(f)$ such that $p_Y \mid B$ is a perfect mapping of B onto Y. Let $C = \{x \in X\colon (x, f(x)) \in B\} = p_X[B]$, and let $\tilde{f} = f \mid C$. Then $\tilde{f}: C \to Y$ is continuous, and if $y \in Y$, then there is $b \in B$ such that $p_Y(b) = y$. Since $y = f(x)$ for some $x \in X$ with $(x, y) \in B$ (so $x \in C$), the function $\tilde{f}$ is onto.

If $y \in Y$, the inverse image $\tilde{f}^{-1}\{y\} = p_X[p_Y^{-1}\{y\} \cap B] = p_X[(p_Y \mid B)^{-1}\{y\}]$. Since $p_Y \mid B$ is perfect, $(p_Y \mid B)^{-1}\{y\}$ is compact, so $\tilde{f}^{-1}\{y\}$ is also compact. Suppose E is closed in C. Then $\tilde{f}[E] = (p_Y \mid B)[p_X^{-1}(E)]$. Since $p_X^{-1}E$ is closed in B (for $p_X^{-1}E \subseteq B$ and p_X is continuous on B), and since the map $p_Y \mid B$ is closed, $\tilde{f}$ is also a closed map. $\square$

3. Solution of problem 1.2

Let (X, m_X) be a metric space, and let $f: X \to Y$ be a continuous surjection of X onto a topological space Y. By $\mathscr{K}$ we denote the family of all nonempty compact subsets of X. On $\mathscr{K}^2$, we define two functions:

$$d(K, L) = \inf\{\varepsilon > 0\colon (\forall q \in L)(\exists p \in K)[m(p, q) < \varepsilon]\}$$

(where m is the metric on X),

$$\rho(K, L) = \max\{d(K, L), d(L, K)\}.$$

The function ρ is a metric, sometimes called the *Hausdorff metric,* which induces a compact T_2-topology on $\mathscr{K}$, which we shall refer to as the *hyperspace topology.*

The function d is not a metric; however, the family $\{B_\varepsilon(K)\colon \varepsilon > 0,\ K \in \mathscr{K}\}$ (where $B_\varepsilon(K) = \{L \in \mathscr{K}\colon d(K, L) < \varepsilon\}$) is still the base for a topology. We shall refer to the latter as the *d-topology*. Whenever topological properties of $\mathscr{K}$ are considered, and no topology is specified, we have in mind the d-topology.

The hyperspace topology is richer than the d-topology. Therefore, the d-topology is still compact, but not T_2. In fact, the d-topology is not even T_1, but only T_0. It also follows immediately that the d-topology is hereditarily Lindelöf.

Also, suppose $K, L, R, S \in \mathscr{K}$ are such that $K \subseteq L$. Then

(a) $d(K, R) \geqslant d(L, R)$,

(b) $d(R, K) \leqslant d(R, L)$,
(c) $d(R, S) = 0$ iff $S \subseteq R$,
(d) $d(L, S) \leqslant d(L, R) + d(R, S)$.

Definition 3.0. We say that a function $f: X \to Y$ has *Property* A if there exists an indexed family $\{\mathscr{K}_y\colon y \in Y\}$ of nonempty subspaces of $\mathscr{K}$ such that for all $y \in Y$ and $K \in \mathscr{K}_y$:
(i) $K \subseteq f^{-1}\{y\}$,
(ii) for every $\varepsilon > 0$ there exists a neighborhood U of y such that for all $y' \in U$ there exists $K' \in \mathscr{K}_{y'}$ such that $d(K, K') < \varepsilon$.

The solution of Michael's Problem 1.2 is an immediate consequence of the following two theorems:

Theorem 3.1. *Suppose X is a metric space, Y is a first-countable zero-dimensional space, and $f: X \to Y$ is a countable-compact-covering surjection such that every fibre of f is separable. Then f has property* A.

Martin's axiom for families of size less than κ, MA_κ, is the statement: "Whenever $\langle P, \leqslant \rangle$ is a nonempty partial order satisfying the ccc, and $\mathscr{D}$ is a family of $\leqslant \kappa$ dense subsets of P, then there is a filter $\mathscr{G}$ in P such that $G \cap D \neq \emptyset$ for each $D \in \mathscr{D}$." [4, p. 54].

Theorem 3.2. *Suppose* MA_κ *holds, Y is zero-dimensional and second-countable, $|Y| \leqslant \kappa$, and $f: X \to Y$ is a surjection of a metric space X onto Y that has Property* A. *Then f is inductively perfect.*

Remark 3.3. To see how the answer to Michael's question follows from Theorems 3.1 and 3.2, note that $\mathrm{MA}_{\aleph_0}$ is a theorem of ZFC, and that every metric space of cardinality less than that of the continuum is zero-dimensional.

Proof of Theorem 3.1. For $y \in Y$, let $\mathscr{K}_y$ be the family of all $K \in \mathscr{K}$ such that
(I) $K \subseteq f^{-1}\{y\}$,
(II) for every compact, countable $E \subseteq Y$ there exists a compact $C \subseteq X$ such that $C \cap f^{-1}\{y\} \subseteq K$ and $f[C] = E$.
Since (I) is the same as (i) of Definition 3.0, it suffices to prove (ii) of that definition.

Definition 3.4. Let E be a compact, countable subset of Y, let $y \in Y$, and let $K \in \mathscr{K}$ be such that $K \subseteq f^{-1}\{y\}$. We say that E *eliminates* K if there is no compact $C \subseteq X$ with $f[C] = E$ and $C \cap f^{-1}\{y\} \subseteq K$.

Claim 3.5. (a) *If E eliminates $K \subseteq f^{-1}\{y\}$, then $y \in E$.*
(b) *Let $y \in Y$. Then $\mathscr{K}_y$ is the family of all $K \subseteq f^{-1}\{y\}$ such that $K \in \mathscr{K}$ and no compact, countable $E \subseteq Y$ eliminates K.*

(c) *If E eliminates $K \subseteq f^{-1}(y\}$, and U is a clopen neighborhood of y in Y, then $E \cap U$ also eliminates K.*

(d) *If $E \subseteq F$ are compact, countable subsets of Y, and if E eliminates K, then so does F.*

(e) *If E eliminates some $K \subseteq f^{-1}\{y\}$, then there is some neighborhood V of K such that E eliminates every $L \in V$ which is a subset of $f^{-1}\{y\}$.*

Proof. (a) and (b) are obvious. For the proof of (c), note that both $E \cap U$ and $E \setminus U$ are compact, countable, and, if C, D are compact subsets of X such that $f[C] = E \cap U$, $f[D] = E \setminus U$, then $C \cup D$ is a compact subset of X with $f[C \cup D] = E$ such that $(C \cup D) \cap f^{-1}\{y\} = C \cap f^{-1}\{y\}$. For (d), note that if C is compact with $f[C] = F$, then $D = C \cap f^{-1}E$ is also compact with $f[D] = E$.

Now suppose (e) is false, and let E and $K \subseteq f^{-1}\{y\}$ witness this fact. Then for every $\varepsilon > 0$ there is a compact lifting $D(\varepsilon) \subseteq X$ of E such that $(D(\varepsilon)) \cap f^{-1}\{y\} \in B_\varepsilon(K)$. Fix such $D(\varepsilon)$ for every $\varepsilon > 0$.

Subclaim 3.6. *Let $\varepsilon > 0$. There is a clopen neighborhood U of y such that $D(\varepsilon) \cap f^{-1}U \in B_\varepsilon(K)$.*

Proof. Suppose not. Then, since Y is zero-dimensional and has countable pseudocharacter, we find a sequence $(U_n)_{n \in \omega}$ of clopen neighborhoods of y with

$$\bigcap_{n \in \omega} U_n = \{y\}, \tag{$*$}$$

and a sequence $(w_n)_{n \in \omega}$ of points in $D(\varepsilon)$ such that $f(w_n) \in U_n$ and

$$d\big(D(\varepsilon) \cap f^{-1}\{y\}, \{w_n\}\big) \geqslant \varepsilon \quad \text{for all } n \in \omega. \tag{$**$}$$

By compactness of $D(\varepsilon)$, the sequence $(w_n)_{n \in \omega}$ must cluster at some point $w \in D(\varepsilon)$. It follows from $(*)$ that $w \in f^{-1}\{y\}$, which by the monotonicity property of the d-function mentioned at the beginning of this section and the choice of $D(\varepsilon)$ contradicts $(**)$. □

Proof of Claim 3.5 (continued). Now let $(\varepsilon_n)_{n \in \omega}$ be a sequence of positive reals such that $(\varepsilon_n) \searrow 0$.

Using Subclaim 3.6, find a decreasing sequence $(U_n)_{n \in \omega}$ of clopen neighborhoods of y such that

$$D(\varepsilon_n) \cap f^{-1}U_n \in B_{\varepsilon_n}(K) \quad \text{for all } n \in \omega. \tag{!}$$

Let $D = \bigcup_{n \in \omega}(D(\varepsilon_n) \cap f^{-1}(U_n \setminus U_{n+1})) \cup K$.

Then $f[D] = E \cup \{y\} = E$. The set D is compact by Claim 1.4, since (!) implies that there exists a decreasing base $(W_n)_{n \in \omega}$ of K in X such that each $D(\varepsilon_n) \cap f^{-1}(U_n \setminus U_{n+1}) \subseteq W_n$. If $x \in D$ and $x \notin K$, there exists $n \in \omega$ such that $x \in D(\varepsilon_n) \cap f^{-1}(U_n \setminus U_{n+1})$. Thus $f(x) \neq y$. Hence $D \cap f^{-1}y \subseteq K$. This contradicts the assumption that E eliminates K. □

Corollary 3.7. *For every $y \in Y$, the set $\mathscr{H}_y$ is a closed, nonempty subspace of $\mathscr{H}$.*

Proof. Closedness follows immediately from Claim 3.5(b) and (e). To show that $\mathscr{H}_y$ is nonempty, assume otherwise. That is, consider some $y \in Y$ such that every nonempty compact $K \subseteq f^{-1}\{y\}$ is eliminated by some compact, countable $E_K \subseteq Y$. By Claim 3.5(e) again, for each K we find an open neighborhood V_K of K in the d-topology on $\mathscr{H}$ such that E_K eliminates all compact $L \in V$ such that $L \subseteq f^{-1}\{y\}$. Since the d-topology on $\mathscr{H}$ is hereditarily Lindelöf, there is a sequence $(K_n)_{n \in \omega}$ of compact subsets of $f^{-1}\{y\}$ such that every compact $L \subseteq f^{-1}\{y\}$ is eliminated by some E_{K_n}. Choose a decreasing sequence $(U_n)_{n \in \omega}$ of clopen subsets of Y such that $\bigcap_{n \in \omega} U_n = \{y\}$. Let $E = \bigcup_{n \in \omega}(E_{K_n} \cap U_n)$. A standard argument shows that E is a countable, compact subset of Y. Moreover, by what was said above and Claim 3.5(c), every compact $L \subseteq f^{-1}\{y\}$ is eliminated by some $E_{K_n} \cap U_n$. It follows now from Claim 3.5(d) that E eliminates all compact $L \subseteq f^{-1}\{y\}$, which is impossible, since there is a compact $C \subseteq X$ with $f[C] = E$. Then $C \cap f^{-1}\{y\}$ is obviously compact, is nonempty by Claim 3.5(a), and is certainly not eliminated by E. So we have reached a contradiction. □

Let us remark that the proof of Corollary 3.7 gives in fact something stronger, namely:

Claim 3.8. *Let $y \in Y$ and $G \subseteq \mathscr{H} \setminus \mathscr{H}_y$. Then there is a single compact, countable $E \subseteq Y$ such that E eliminates all compact $L \in G$ such that $L \subseteq f^{-1}\{y\}$.*

Proof of Theorem 3.1 (continued). Now we are ready to show that (ii) of Definition 3.0 holds. Let $y \in Y$ and $K \in \mathscr{H}$ be such that $K \subseteq f^{-1}\{y\}$. Let $(U_n)_{n \in \omega}$ be a decreasing sequence of clopen sets such that $\{U_n\colon n \in \omega\}$ is a neighborhood base of y. Suppose that for some fixed $\varepsilon > 0$ we can find a sequence $(y_n)_{n \in \omega}$ such that $y_n \in U_n$ and $\mathscr{H}_{y_n} \cap B_\varepsilon(K) = \emptyset$ for every $n \in \omega$. Passing to a subsequence if necessary, we may for simplicity assume that $y_n \in U_n \setminus U_{n+1}$ for every $n \in \omega$. We show that there is a compact, countable $E \subseteq Y$ that eliminates K. This will prove (ii).

Note that by Claim 3.8, for every $n \in \omega$ we may choose a compact, countable subset E_n of Y that eliminates all sets in $B_\varepsilon(K) \cap \mathscr{H} \cap \mathscr{P}(f^{-1}\{y_n\})$ (where $\mathscr{P}(X)$ denotes the family of all subsets of a set X). By Claim 3.5(c), we may assume that $E_n \subseteq U_n \setminus U_{n+1}$ for all $n \in \omega$. Now let $E = \bigcup_{n \in \omega} E_n \cup \{y\}$. By Claim 1.4, E is compact.

Let us show that E eliminates K. If C is a compact subset of X such that $f[C] = E$, then by the choice of E_n and Claim 3.5(d), we find a sequence $(w_n)_{n \in \omega}$ of elements of C such that $f(w_n) = y_n$ and

$$d(K, \{w_n\}) \geqslant \varepsilon \quad \text{for all } n \in \omega. \tag{@}$$

Since C is compact, this sequence clusters at some $w \in C$. Now $w \in f^{-1}\{y\} \cap C$, since $\{U_n\colon n \in \omega\}$ is a base at y. But by (@), $d(K, \{w\}) \geqslant \varepsilon$, hence $w \notin K$. Thus $C \cap f^{-1}\{y\}$ is not a subset of K, and we have shown that E eliminates K. □

Proof of Theorem 3.2. In this proof we shall use a more set-theoretical language than in the rest of the paper. In particular, partial orders will be referred to as "forcing notions". The following theorem will be referred to as the *Δ-system lemma*: "If $\mathscr{A}$ is any uncountable family of finite sets, there is an uncountable $\mathscr{B} \subseteq \mathscr{A}$ which forms a Δ-system, i.e., there is a fixed set r such that $a \cap b = r$ whenever a and b are distinct members of $\mathscr{B}$." For a proof see [4, p. 49]. The set r will be called the *kernel* of the Δ-system.

We shall construct an appropriate forcing notion $\boldsymbol{P}$. First, we choose a countable base $\mathscr{B}$ for Y that consists of clopen subsets of Y such that $\mathscr{B} \cup \{\emptyset\}$ is a field of subsets of Y. Next, we choose a countable dense (in the topology induced by ρ) subset $\mathscr{K}$ of $\mathscr{H}$, and, for every $y \in Y$, a countable dense (in the topology induced by ρ) subset $\mathscr{K}_y$ of $\mathscr{H}_y$. Now our forcing notion $\boldsymbol{P}$ will consist of pairs of the form $p = \langle f_p, s_p \rangle$, where:

(1) f_p is a function such that
  (a) $\operatorname{dom}(f_p)$ is a partition of Y into finitely many elements of $\mathscr{B}$. We shall denote $\operatorname{dom}(f_p)$ by $\mathscr{U}_p$ in the sequel.
  (b) For every $U \in \mathscr{U}_p$, $f_p(U)$ is a set of the form $B_\varepsilon(K)$ for some $K \in \mathscr{K}$ and $\varepsilon \in \mathbb{Q}^+ \cup \{\infty\}$ such that $\mathscr{H}_y \cap f_p(U) \neq \emptyset$ for all $y \in U$.

(2) s_p is a function whose domain is a finite subset Y_p of Y. Instead of $s_p(y)$ we shall write K_y.
  (a) If $U \in \mathscr{U}_p$ and $y \in U \cap \operatorname{dom}(s_p)$, then $K_y \in f_p(U)$.
  (b) $K_y \in \mathscr{K}_y$ for all $y \in \operatorname{dom}(s_p)$.

We define a partial order $\preccurlyeq$ on $\boldsymbol{P}$ as follows: $p \preccurlyeq q$ iff

(3) $\mathscr{U}_p$ is a refinement of $\mathscr{U}_q$,
(4) if $U' \subseteq U$, where $U' \in \mathscr{U}_p$, $U \in \mathscr{U}_q$, then $f_p(U') \subseteq f_q(U)$,
(5) $s_p \supseteq s_q$.

Claim 3.9. *$(\boldsymbol{P}, \preccurlyeq)$ satisfies the ccc.*

Proof. We actually show that $\boldsymbol{P}$ has $\aleph_1$ as a precaliber. Of course, MA_κ implies that every ccc forcing notion of size $\leqslant \kappa$ has $\aleph_1$ as a precaliber. However, the present proof does not use Martin's axiom. Let A''' be an uncountable subset of $\boldsymbol{P}$. There are only countably many objects that can appear as first coordinates of elements of $\boldsymbol{P}$. Thus, there is f such that for an uncountable $A'' \subseteq A'''$ we have $f_p = f$ for all $p \in A''$. By the Δ-system lemma, there exists an uncountable $A' \subseteq A''$ such that the family $\{Y_p\colon p \in A'\}$ forms a Δ-system. Let us denote its kernel by v. Since v is finite, the set $\prod_{y \in v} \mathscr{K}_y$ is countable. Therefore, there exist a function $g \in \prod_{y \in v} \mathscr{K}_y$ and an uncountable $A \subseteq A'$ such that $\langle y, g(y) \rangle \in s_p$ for all $y \in v$ and $p \in A$. Now it follows immediately from the definition of $\boldsymbol{P}$ that if $p, q \in A$, then $r = \langle f, s_p \cup s_q \rangle \in \boldsymbol{P}$ and $r \preccurlyeq p, q$. $\square$

Claim 3.10. *For every $\varepsilon > 0$ and $y \in Y$ the following set is dense in $\boldsymbol{P}$:*

$$D_y^\varepsilon = \{p\colon y \in Y_p \text{ and there exists a } U \in \mathscr{U}_p \text{ such that } y \in U \text{ and } f_p(U) \subseteq B_\varepsilon(K_y)\}.$$

Proof. Let $y \in Y$. As a first step, we show that the set $D_y = \{p\colon y \in Y_p\}$ is dense in $\boldsymbol{P}$. If $p \in \boldsymbol{P}$ and $y \notin Y_p$, then by (1a) there exists $U \in \mathscr{U}_p$ such that $y \in U$, and by (1b), $\mathscr{K}_y \cap f_p(U) \neq \emptyset$. Since $\mathscr{K}_y$ is dense in $\mathscr{K}_y$, we may choose $K_y \in \mathscr{K}_y \cap f_p(U)$. Then $q = \langle f_p, s_p \cup \{\langle y, K_y\rangle\}\rangle \in D_y$, and clearly, $q \preccurlyeq p$.

Next we show that for every neighborhood V of y the set $D_y^V = \{p \in D_y\colon$ there exists a $U \in \mathscr{U}_p$ such that $y \in U \subseteq V\}$ is dense in $\boldsymbol{P}$. Let $p \in D_y$ and let V be a neighborhood of y. Let U be the element of $\mathscr{U}_p$ that contains y. Since $\mathscr{B}$ is a base for Y, we find a $U' \in \mathscr{B}$ such that $y \in U' \subseteq U \cap V$. Since $\mathscr{B} \cup \{\emptyset\}$ is a field of sets, the family $\mathscr{U}_q = (\mathscr{U}_p \setminus \{U\}) \cup \{U', U \setminus U'\}$ is a legitimate partition of Y (in the nontrivial case where $U' \neq U$ at least). Thus we can put $f_q(W) = f_p(W)$ for $W \in \mathscr{U}_p \setminus \{U\}$, $f_q(U') = f_q(U \setminus U') = f_p(U)$, and $s_q = s_p$. Then $q \in D_y^V$, and $q \preccurlyeq p$.

Now we are ready to tackle our task proper. Let $\varepsilon > 0$, and let $p \in \boldsymbol{P}$. We want to construct $q \preccurlyeq p$ such that $q \in D_y^\varepsilon$. We may assume $y \in Y_p$, i.e., $\langle y, K_y\rangle \in s_p$ for some K_y. Choose ε' such that $0 < \varepsilon' \leqslant \varepsilon$ and $B_{\varepsilon'}(K_y) \subseteq f_p(U)$, where $y \in U$. By Property A, there is a neighborhood V of y such that for each $y' \in V$ there is some $K' \in \mathscr{K}_{y'}$ with $d(K_y, K') < \varepsilon'$. Fix such a V. Replacing V by a smaller neighborhood if necessary, we may assume that $V \cap Y_p = \{y\}$. As was shown above, we may replace p by a condition $p' \preccurlyeq p$ such that $p' \in D_y^V$ and $f_{p'}(U') = f_p(U)$, where $U' \in \mathscr{U}_{p'}$ is such that $y \in U' \subseteq V$. Then $U' \cap Y_p = \{y\}$. Now put $s_q = s_{p'}$, $\mathscr{U}_q = \mathscr{U}_{p'}$, $f_q(U') = B_{\varepsilon'}(K_y)$, and let $f_q(W) = f_{p'}(W)$ for $W \neq U'$. It is not hard to see that $q \in D_y^\varepsilon$ and $q \preccurlyeq p' \preccurlyeq p$. $\square$

Proof of Theorem 3.2 (continued). By MA_κ, if $|Y| \leqslant \kappa$, there exists a filter $F \subseteq \boldsymbol{P}$ that meets each of the sets D_y^ε for $y \in Y$ and $\varepsilon \in \mathbb{Q}^+ \cap (0, \infty)$.

Let $X' = \bigcup\{K_y\colon$ there exist $p \in F$ and $y \in Y$ such that $\langle y, K_y\rangle \in s_p\}$. We show that the restriction of f to the subspace X' of X is a perfect map onto Y. Since $F \cap D_y^1 \neq \emptyset$ for every $y \in Y$, the restriction $f \mid X'$ is onto. Moreover, since every two elements of F are compatible, it follows that for every $y \in Y$ there is exactly one K_y such that $\langle y, K_y\rangle \in s_p$ for all $p \in F \cap D_y$. Therefore, $X' \cap f^{-1}\{y\} = K_y$, and thus every fibre of $f \mid X'$ is compact.

Now let $M \subseteq X'$ be closed in X' and let $y \in \mathrm{cl}_Y(f[M])$. Let $K_y = X' \cap f^{-1}\{y\}$. For every $n \in \omega$ choose some $p_n \in D_y^{1/(n+1)} \cap F$. Let $U_n \in \mathscr{U}_{p_n}$ be such that $y \in U_n$. Choose $y_n \in U_n \cap f[M]$ and x_n such that $f(x_n) = y_n$. Pick $q_n \preccurlyeq p_n$ such that $q_n \in D_{y_n}^1$. It follows that $\langle y_n, K_{y_n}\rangle \in s_{q_n}$, and since $q_n \preccurlyeq p_n$, we have $K_{y_n} \in B_{1/(n+1)}(K_y)$, i.e., $d(K_y, K_{y_n}) < 1/(n+1)$. Since $x_n \in K_{y_n}$, we also have $d(K_y, \{x_n\}) < 1/(n+1)$. It follows that the set $\{x_n\colon n \in \omega\}$ has some cluster point x in K_y (by compactness of K_y), and therefore $x \in M$ (since M is closed in X' and $K_y \subseteq X'$). Now $f(x) = y$. We have thus shown that $f \mid X'$ is a closed map. $\square$

4. Generalization of a result by Ostrovsky

The main result of this section is the following.

Theorem 4.0. *Suppose $f: X \to Y$ is a tri-quotient map of a regular space X onto a paracompact space Y. If there exists a perfect extension $f^+ : X^+ \to Y$ of f such that X is a W_δ-set in X^+, then f is inductively perfect.*

It should be noted that in Theorem 4.0 we do not require X^+ to have any separation properties whatsoever.

A surjective map $f: X \to Y$ is called *tri-quotient* [5, Definition 6.1] if one can assign to each open U in X an open U^* in Y such that:

(a) $U^* \subseteq f[U]$,

(b) $X^* = Y$,

(c) $U \subseteq V$ implies $U^* \subseteq V^*$,

(d) if $y \in U^*$ and $\mathscr{W}$ is a cover of $f^{-1}\{y\} \cap U$ by open subsets of X, then there is a finite $\mathscr{F} \subseteq \mathscr{W}$ such that $y \in (\bigcup \mathscr{F})^*$.

The correspondence $U \mapsto U^*$ is called a *t-assignment for f.*

The formal definition of a W_δ-set will make its natural appearance at the beginning of the proof of Theorem 4.0. Let us first put Theorem 4.0 into the perspective of related theorems. Substituting in Theorem 4.0 "G_δ" for "W_δ", one obtains Ostrovsky's theorem [9, Theorem 1]. It will be evident from the definition of a W_δ-set that every G_δ-set is W_δ, but the converse is false.

Ostrovsky obtains as a corollary to his theorem the following result of Michael [5, Theorem 1.6]:

A tri-quotient map $f: X \to Y$ of a Čech-complete regular space X onto a paracompact space Y is inductively perfect.

However, as Ostrovsky mentioned in his paper, the following result of Michael [5, Theorem 6.6] is not a corollary of [9, Theorem 1].

Theorem 4.1. *A tri-quotient map $f: X \to Y$ of a sieve-complete regular space X onto a paracompact space Y is inductively perfect.*

We shall show later in this paper that Theorem 4.1 is in fact a consequence of Theorem 4.0. Let us now turn to the proof of Theorem 4.0. We shall use the convenient terminology of sieves developed in [1] to express concepts originally introduced by Wicke and Worrell under different names (see e.g. [11] or the bibliography to [1]).

Proof of Theorem 4.0. Assume X, f, f^+ and X^+ are as stated in the hypothesis. Let $\langle \mathscr{U}_n, \mathscr{A}_n, \pi_n \rangle_{n \in \omega}$ be a strong W-sieve of X in X^+, with $\mathscr{U}_0 = \{X^+\}$. Thus each $\mathscr{U}_n = \{U_\alpha \colon \alpha \in A_n\}$ is an open collection in X^+ which covers X (i.e., $\bigcup \mathscr{U}_n \supseteq X$), and $\pi_n : A_{n+1} \to A_n$ is such that

(1) If $\alpha \in A_n$, then $X \cap U_\alpha = \bigcup \{X \cap U_\beta \colon \beta \in \pi_n^{-1}\{\alpha\}\}$, and

(2) if $\alpha \in A_{n+1}$, then $\mathrm{cl}_{X^+}(U_\alpha) \subseteq U_{\pi_n(\alpha)}$.

A sequence $(\alpha_n \colon n \in \omega)$ will be called a *π-chain for* $\langle \mathscr{U}_n, \mathscr{A}_n, \pi_n \rangle_{n \in \omega}$ iff $\alpha_n \in A_n$ and $\pi_n(\alpha_{n+1}) = \alpha_n$ for all $n \in \omega$. That $\langle \mathscr{U}_n, \mathscr{A}_n, \pi_n \rangle_{n \in \omega}$ is a strong W-sieve means that for every π-chain $(\alpha_n \colon n \in \omega)$ we have $\mathrm{cl}_{X^+}(U_{\alpha_{n+1}}) \subseteq U_{\alpha_n}$ and if $x \in \bigcap_{n \in \omega} U_{\alpha_n}$, then $x \in X$. X is a W_δ-set in X^+ iff there exists a strong W-sieve of X in X^+.

Michael in [5] calls a sieve *finitely additive* if every collection $\mathscr{U}_n$ and every collection $\{U_\beta\colon \beta \in \pi_n^{-1}(\alpha)\}$ is closed under finite unions. We form the finitely additive strong sieve $\langle \mathscr{U}_n^F, \mathscr{B}_n, p_n \rangle_{n \in \omega}$ associated with $\langle \mathscr{U}_n, \mathscr{A}_n, \pi_n \rangle_{n \in \omega}$ as in [5, Lemma 2.3]:

Let $\mathscr{B}_n = \{B \in [\mathscr{A}_n]^{<\omega}\colon B \neq \emptyset\}$, and for $B \in \mathscr{B}_n$, let $U_B = \bigcup\{U_\alpha\colon \alpha \in B\}$. Define $p_n\colon \mathscr{B}_{n+1} \to \mathscr{B}_n$ by $p_n(B) = \{\pi_n(\alpha)\colon \alpha \in B\}$. Let $\mathscr{U}^F \cap X$ denote the family of coverings $(\mathscr{U}^F \cap X)_n = \{U_B \cap X\colon B \in \mathscr{B}_n\}$, with $\mathscr{B}$ and p as just defined. Then $\langle (\mathscr{U}^F \cap X)_n, \mathscr{B}_n, p_n \rangle_{n \in \omega}$ is a sieve of X in X. We may use the tri-quotient map f with the t-assignment $U \mapsto U^*$ to obtain the sieve $\langle (\mathscr{U}^F \cap X)_n^*, \mathscr{B}_n, p_n \rangle_{n \in \omega}$ on Y, where $(\mathscr{U}^F \cap X)_n^* = \{(U_B \cap X)^*\colon B \in \mathscr{B}_n\}$ (see [5, Lemma 6.2] for details).

By the paracompactness of Y, there is a locally finite sieve $\langle \mathscr{W}_n, \mathscr{B}_n, p_n \rangle_{n \in \omega}$ on Y such that $\mathrm{cl}_Y(W_B) \subseteq (U_B \cap X)^*$ for all $B \in \mathscr{B}_n$ and all $n \in \omega$ (see [5, Lemma 2.2]).

For all $n \in \omega$ and $B \in \mathscr{B}_n$, define $V_B = U_B \cap f^{-1}W_B$. Then $V_B \subseteq U_{p_n(B)} \cap f^{-1}W_{p_n(B)} = V_{p_n(B)}$ for each $n > 0$ and $B \in \mathscr{B}_{n+1}$.

Let $X_n^+ = \bigcup\{\mathrm{cl}_{X^+}(V_B)\colon B \in \mathscr{B}_n\}$. Then X_n^+ is closed in X^+ since $\{f^{-1}W_B\colon B \in \mathscr{B}_n\}$ is locally finite in X^+, as it is the inverse image of the locally finite collection $\mathscr{W}_n$.

If $y \in Y$, there is $B \in \mathscr{B}_n$ such that $y \in W_B$. Since $W_B \subseteq U_B^* \subseteq f[U_B]$, the intersection $f^{-1}\{y\} \cap V_B \neq \emptyset$. Thus $f_n^+ = f^+ \mid X_n^+$ maps X_n^+ onto Y. Moreover, f_n^+ is perfect, since X_n^+ is a closed subspace of X^+.

If $x \in X_{n+1}^+$, then $x \in \mathrm{cl}_{X^+}(V_B)$ for some $B \in \mathscr{B}_{n+1}$. Thus $x \in \mathrm{cl}_{X^+}(V_{p_n(B)})$. Hence $X_{n+1}^+ \subseteq X_n^+$.

Note that for all $n \in \omega$ and $y \in Y$ the inclusion $(f_{n+1}^+)^{-1}\{y\} \subseteq (f_n^+)^{-1}\{y\}$ holds. Since these sets are compact and nonempty,

$$\bigcap_{n \in \omega} (f_n^+)^{-1}\{y\} \neq \emptyset \quad \text{for all } y \in Y. \tag{$*$}$$

Let $X' = \bigcap_{n \in \omega} X_n^+$. Then X' is closed in X^+ and $(*)$ implies that also the map $f^+ \mid X' = \bigcap_{n \in \omega} f_n^+$ is a perfect map of X' onto Y.

It remains to show that $X' \subseteq X$. So suppose $x \in X'$. Because $\{f^{-1}W_B\colon B \in \mathscr{B}_n\}$ is locally finite in X^+ for all $n \in \omega$, the collections $\mathscr{C}_n = \{B \in \mathscr{B}_n\colon x \in \mathrm{cl}_{X^+}(V_B)\}$ are nonempty and finite; and if $B \in \mathscr{C}_{n+1}$, then $p_n(B) \in \mathscr{C}_n$.

By König's lemma, there is a p-chain $\langle B_n\colon n \in \omega \rangle$ for $\langle \mathscr{U}_n^F, \mathscr{B}_n, p_n \rangle_{n \in \omega}$ such that $x \in \mathrm{cl}_{X^+}(V_{B_n})$ for all $n \in \omega$. For all $B \in \mathscr{B}_n$ we have $\mathrm{cl}_{X^+}(V_B) \subseteq \mathrm{cl}_{X^+}(U_B) \subseteq U_{p_n(B)}$, since $\langle \mathscr{U}_n^F, \mathscr{B}_n, p_n \rangle_{n \in \omega}$ is a strong sieve. Hence $x \in \bigcap_{n \in \omega} U_{B_n}$. For each $n \in \omega$, let $\mathscr{M}_n = \{U_\alpha \in \mathscr{U}_n\colon \alpha \in B_n \text{ and } x \in U_\alpha\}$. Then $\mathscr{M}_n$ is finite and nonempty. If $U_\alpha \in \mathscr{M}_{n+1}$, then $U_{\pi_n(\alpha)} \in \mathscr{M}_n$, since $B_n = p_n(B_{n+1}) = \{\pi_n(\alpha)\colon \alpha \in B_{n+1}\}$. Thus by König's lemma again, there is a π-chain $\langle \alpha_n\colon n \in \omega \rangle$ for $\langle \mathscr{U}_n, \mathscr{A}_n, \pi_n \rangle_{n \in \omega}$ such that $x \in U_{\alpha_n}$. Since $\langle \mathscr{U}_n, \mathscr{A}_n, \pi_n \rangle_{n \in \omega}$ is a W-sieve of X in X^+, this implies that $x \in X$. Thus $X' \subseteq X$, and we have shown that $f \mid X' = f^+ \mid X'$ is a perfect map of $X' \subseteq X$ onto Y. $\square$

Now we want to show that Theorem 4.1 is indeed a consequence of Theorem 4.0. We shall work with the *Wallman extension* wX of a T_1-space X. As a reference for the basic facts about wX we recommend [3, pp. 176–178]. For the convenience of the reader we use here the terminology from the above-mentioned source.

Thus for a given space X, by $\mathscr{D}(X)$ (or simply $\mathscr{D}$, since our space will always be the same) we denote the family of nonempty closed subsets of X, by $F(X)$ (or simply F) the family of all ultrafilters on $\mathscr{D}$. For each nonempty open $U \subseteq X$ we denote $U^* = \{\mathscr{F} \in F\colon (\exists A \in \mathscr{F})[A \subseteq U]\}$.

The family $\{U^*\colon U$ is nonempty open in $X\}$ is a base for wX. We need the following lemma.

Lemma 4.2. *Suppose U, V are nonempty open subsets of X such that $\mathrm{cl}_X(V) \subseteq U$. Then $\mathrm{cl}_{wX}(V^*) \subseteq U^*$.*

Proof. We show the contrapositive. Suppose $\mathscr{F} \notin U^*$. Then no closed subset of U is in $\mathscr{F}$, in particular, $\mathrm{cl}_X(V) \notin \mathscr{F}$. It follows now from [3, p. 177(3)] that there is $B \in \mathscr{F}$ such that $B \cap \mathrm{cl}_X(V) = \emptyset$. Fix such B. Let $W = X \setminus \mathrm{cl}_X(V)$. Then $B \subseteq W$, and hence $\mathscr{F} \in W^*$. Note that $V \cap W = \emptyset$, and hence by [3, p. 177(9)], also $V^* \cap W^* = \emptyset$. It follows that $\mathscr{F} \notin \mathrm{cl}_{wX}(V)$, and we have proved the lemma. $\square$

Proof of Theorem 4.1 from Theorem 4.0. Let f, X and Y be as in the assumptions of Theorem 4.1, and note that Y is Tychonoff. We may treat f as a map from X into βY. By [3, p. 178, Theorem 3.6.21], there exists a continuous extension $F\colon wX \to \beta Y$ of f. Fix such F, and let $f^+ = F \mid F^{-1}Y$. Then f^+ is a perfect extension of f, and by Theorem 4.0, we shall be done if we prove the following:

Lemma 4.3. *Let f, f^+, X, Y be as above, and denote $X^+ = F^{-1}Y$. Then X is a W_δ-subset of X^+.*

Proof. This is similar to a result from [11]. Let $\mathscr{S} = \langle \mathscr{U}_n, \mathscr{A}_n, \pi_n \rangle_{n \in \omega}$ be a strong sieve (i.e., for every π-chain $(\alpha_n\colon n \in \omega)$ and every $n \in \omega$, the inclusion $\mathrm{cl}_X U_{\alpha_{n+1}} \subseteq U_{\alpha_n}$ holds) for X in X that witnesses sieve-completeness. Thus every filter $\mathscr{F}$ of closed subsets of X that meshes with a π-chain of $\mathscr{S}$ clusters in X.

Now form a sieve $\mathscr{S}^+$ in X^+ as follows: $\mathscr{S}^+ = \langle \mathscr{U}_n^+, \mathscr{A}_n, \pi_n \rangle_{n \in \omega}$, where $\mathscr{U}_n^+ = \{U_A^+\colon A \in \mathscr{A}_n\}$ for all $n \in \omega$, with $U_A^+ = (U_A)^* \cap X^+$ (The $*$-operation is the one from the description of the Wallman extension, not the t-assignment for f). By Lemma 4.2, this is a strong sieve. Now it suffices to show that it is a W-sieve for X. So suppose we are given a π-chain $(A_n\colon n \in \omega)$, and that $\mathscr{F} \in \bigcap_{n \in \omega} U_{A_n}^+$. We need to show that $\mathscr{F} \in X$. If not, then $\mathscr{F}$ is a free ultrafilter of closed subsets of X. Since for every n and every $B \in \mathscr{F}$ we have $\mathrm{cl}_{wX}(U_{A_n}^*) \cap B \neq \emptyset$, it follows from the proof of Lemma 4.2 that $\mathrm{cl}_X(U_{A_n}) \cap B \neq \emptyset$ for every $B \in \mathscr{F}$. In other words, the filter $\mathscr{F}$ meshes with $\{U_{A_n}\colon n \in \omega\}$, and therefore, by the choice of $\mathscr{S}$, the filter $\mathscr{F}$ clusters at some $x \in X$. But since $\mathscr{F}$ is a filter of closed sets, this means that $x \in \bigcap \mathscr{F}$. The latter contradicts the assumption that $\mathscr{F}$ is a free ultrafilter on $\mathscr{D}$, so we are done. $\square$

Remark 4.4. The assertion of Lemma 4.3 cannot be strengthened to "X is a G_δ-subset of X^+". The simplest counterexample is a map from any sieve-complete, not Čech-complete Tychonoff space X onto the one-point space Y (see, e.g., [1] or [11] for such an X).

5. Partition-completeness does not suffice

It has been open whether the result of Michael which we stated and reproved in Theorem 4.1 in the previous section remains true if "sieve-complete" is weakened to "partition-complete" (see [8, Section 4]). In the present section we show that this is not the case. Our example has interest independent of the study of partition-completeness. The following fact summarizes all the information about this concept needed in the present section.

Fact 5.0. (a) *Each sieve-complete space is partition-complete.*
(b) *Every scattered space is partition-complete.*

We refer the reader interested in the definition of partition-completeness or the proof of Fact 5.0 to [10] and the references given therein.

Since open maps are among the chief examples of tri-quotient maps, and a map with compact range is inductively perfect iff it is compact-covering, the following answers the question from [8].

Example 5.1. There exists an open map $f : X \to Y$ from a Tychonoff space X with one nonisolated point onto a compact Hausdorff space Y which is not compact-covering.

Note that a space X as in Example 5.1 is necessarily scattered.

Remark 5.2. All fibres of the map we are going to construct are compact, and f is countable-compact-covering.

Construction 5.3. Let κ be an uncountable cardinal, and let $A(\kappa) = \kappa \cup \{x^*\}$, where every ordinal in $A(\kappa)$ is isolated, and every neighborhood of x^* contains all but finitely many of the ordinals in κ. In other words, $A(\kappa)$ is the one-point compactification of the discrete space of size κ.

Now let A consist of all pairs (β, α) such that

- either $\beta = \alpha = x^*$,
- or $\alpha, \beta \in \kappa$ and $\beta = \alpha + \omega$,
- or $\alpha \in \kappa$, $\beta \in \omega$, and $\alpha = \lambda + \beta$ for some limit ordinal λ.

Let f be the projection of A on the second coordinate, i.e., $f(\beta, \alpha) = \alpha$, $f(x^*, x^*) = x^*$.

Clearly, f is continuous and each fibre of f has either one or two points, so the fibres are compact.

Claim 5.4. *f is not compact-covering.*

Proof. It suffices to show that there is no compact subspace $C \subseteq A$ such that $f[C] = A(\kappa)$.

Suppose there were such a C, and let

$$X = \{\alpha < \kappa\colon (\exists n \in \omega)[(n, \alpha) \in C]\}.$$

Case 1: *X is uncountable.*

Then there is some $n \in \omega$ such that $(n, \alpha) \in C$ for uncountably many α. But then $(n, x^*) \in \mathrm{cl}_{A(\kappa)^2}(C)$. However, $(n, x^*) \notin C$, so C is not closed in $A(\kappa)^2$, hence C is not compact, a contradiction.

Case 2: *X is countable.*

Let $\alpha < \kappa$ be a limit ordinal such that $[\alpha, \alpha + \omega) \cap X = \emptyset$. Since $f \mid C$ is a surjection onto $A(\kappa)$, for every $\gamma \in [\alpha, \alpha + \omega)$ we must either have $(\alpha + \omega, \gamma) \in C$, or $(n, \gamma) \in C$ for some $n \in \omega$. By the choice of α, the former will hold throughout the interval. But then C contains infinitely many elements of the form $(\alpha + \omega, \gamma)$, so the point $(\alpha + \omega, x^*) \in \mathrm{cl}_{A(\kappa)^2}(C)$, and we get a similar contradiction as in the first case. □

Claim 5.5. *The map f is open.*

Proof. It suffices to show that for every $a \in A$, there is a neighborhood base $\mathscr{B}(a)$ such that $f[U]$ is open in $A(\kappa)$ for every $U \in \mathscr{B}(a)$.

For $(\beta, \alpha) \in A \setminus \{(x^*, x^*)\}$, let $\mathscr{B}(\beta, \alpha) = \{\{(\beta, \alpha)\}\}$. Since α and β are isolated points in $A(\kappa)$, this works.

Let $\mathscr{B}(x^*, x^*) = \{((A(\kappa) \setminus F) \times (A(\kappa) \setminus G)) \cap A\colon F, G$ are finite subsets of $A(\kappa) \setminus \{x^*\}\}$.

Now let us see which α do *not* belong to

$$f[((A(\kappa) \setminus F) \times (A(\kappa) \setminus G)) \cap A].$$

These are precisely the α such that

$$\alpha \in G$$

or

$\alpha = \lambda + n$ for some limit ordinal λ, and *both* $n \in F$ *and* $\alpha + \omega \in F$.

One can now see that the complement of $f[((A(\kappa) \setminus F) \times (A(\kappa) \setminus G)) \cap A]$ in $A(\kappa)$ has a cardinality of at most $|G| + |F|^2$, and is thus finite. It follows that $f[((A(\kappa) \setminus F) \times (A(\kappa) \setminus G)) \cap A]$ is an open neighborhood of x^* in $A(\kappa)$. So the sets in $\mathscr{B}(x^*, x^*)$ have the desired property, and we have proved the claim. □

Finally, let us prove the second part of Remark 5.2, i.e., the following.

Claim 5.6. *f is countable-compact-covering.*

Proof. Let $E \subseteq A(\kappa)$ be countable and compact. If E is finite, there is nothing to prove, so assume E is infinite.

Then $x^* \in E$.

Let $L(E)$ be the set of all limit ordinals $\lambda < \kappa$ such that $[\lambda, \lambda + \omega) \cap E \neq \emptyset$. Arrange $L(E)$ into a sequence $(\lambda_k \colon k \in \omega)$, and let

$$C = \{(n, \lambda_k + n) \colon k \in \omega, \lambda_k + n \in E, n > k\}$$
$$\cup \{(\lambda_k + \omega, \lambda_k + n) \colon k \in \omega, \lambda_k + n \in E, n \leqslant k\} \cup \{(x^*, x^*)\}.$$

Clearly, $f[C] = E$.

To show that C is compact, it suffices to prove that no point of the form (β, x^*) is a cluster point of C, in other words, that for every $\beta < \kappa$ the set C contains only finitely many points of the form (β, α). Now, if $\beta = n$, then the only points with first coordinate β that may be contained in C are: $(n, \lambda_0 + n), \ldots, (n, \lambda_{n-1} + n)$. If $\beta = \lambda_k + \omega$ for some k, then the only points with first coordinate β that can belong to C are: $(\lambda_k + \omega, \lambda_k), \ldots, (\lambda_k + \omega, \lambda_k + k)$. In either case, there are only finitely many possibilities, so we are done. $\square$

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