Application of Taylor series in obtaining the orthogonal operational matrix

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A R T I C L E   I N F O

Article history:
Received 3 July 2010
Received in revised form 1 March 2011
Accepted 1 March 2011

Keywords:
Operational matrices
Jacobi orthogonal polynomials
Rational Jacobi functions
Taylor series
Orthogonal expansion
Spectral methods

A B S T R A C T

In this research first we explicitly obtain the relation between the coefficients of the Taylor series and Jacobi polynomial expansions. Then we present a new method for computing classical operational matrices (derivative, integral and product) for general Jacobi orthogonal functions (polynomial and rational). This method can be used for many classes of orthogonal functions.

1. Introduction

As we know orthogonal series play an important role in problems of mathematical engineering and physics [1]. According to the theory of orthogonal polynomials [2] the sequence \(\{\phi_k\}_{k=0}^{\infty}\) is called a sequence of orthogonal polynomials with respect to the weight function \(w(x)\) on \([a, b]\) if and only if it satisfies the following condition:

\[
\langle \phi_m(x), \phi_n(x) \rangle_w = \int_a^b w(x)\phi_m(x)\phi_n(x)dx = \gamma_m\delta_{m,n}, \tag{1}
\]

where \(\langle ., . \rangle_w\) is called the inner product and

\[
\gamma_m = \|\phi_m\|^2_w = \int_a^b w(x)\phi_m^2(x)dx, \tag{2}
\]

and \(\delta_{m,n}\) (Kronecker Delta) is defined in the following:

\[
\delta_{m,n} = \begin{cases} 
0 & \text{if } m \neq n, \\
1 & \text{if } m = n.
\end{cases} \tag{3}
\]

Let us start with the following theorem for orthogonal polynomials which is taken from [3].
Theorem 1. Let \( \phi_k(x) \) be an orthogonal polynomial. If \( f(x) \) is an arbitrary function that satisfies the following form:

\[
f(x) = \sum_{k=0}^{\infty} C_k \phi_k(x),
\]

then

\[
C_k = \frac{\int_{a}^{b} w(x) f(x) \phi_k(x) dx}{\| \phi_k(x) \|_w^2}.
\]

On the other hand if \( f(x) \) is an infinitely differentiable function in a neighborhood of arbitrary \( \lambda \) then it has a Taylor series [4] in the form:

\[
f(x) = \sum_{k=0}^{\infty} B_k (x - \lambda)^k,
\]

where

\[
B_k = \frac{f^{(k)}(\lambda)}{k!}.
\]

In this paper by comparing the expansion (4) for the Jacobi family of orthogonal polynomials and the Taylor series (6) we obtain a relation between the coefficients of these two expansions. Then we use these results for obtaining a new method for computing explicitly the operational matrices (derivative, integral and product) for orthogonal Jacobi polynomials and rational functions. It is worth pointing out that this method can be used for many classes of orthogonal functions. For this purpose let us first introduce Jacobi polynomials.

1.1. Jacobi polynomials

The Jacobi polynomials [3] \( P^{(\alpha,\beta)}_n (x) \), are defined as the orthogonal polynomials with respect to the weight function \( w^{(\alpha,\beta)}(x) = (1-x)^\alpha (1+x)^\beta (\alpha > -1, \beta > -1) \) on \((-1, 1)\). It is proved that the Jacobi polynomials satisfy the following relation.

\[
P^{(\alpha,\beta)}_n (x) = \sum_{k=0}^{n} b^{(\alpha,\beta,n)}_k (x - 1)^k; \quad \alpha, \beta > -1,
\]

where

\[
b^{(\alpha,\beta,n)}_k = 2^{-k} \binom{n+\alpha+\beta+k}{k} \binom{n+\alpha}{n-k}; \quad k = 0, 1, 2, \ldots, n,
\]

or

\[
P^{(\alpha,\beta)}_n (x) = \sum_{j=0}^{n} e^{(\alpha,\beta,n)}_j x^j; \quad \alpha, \beta > -1,
\]

with

\[
e^{(\alpha,\beta,n)}_j = \sum_{k=j}^{n} (-1)^{j-k} b^{(\alpha,\beta,n)}_k \binom{j}{k}; \quad \alpha, \beta > -1, 0 \leq j \leq n,
\]

and

\[
\|P_m^{(\alpha,\beta)}\|_{w^{(\alpha,\beta)}}^2 = \lambda^{(\alpha,\beta)}_m = \frac{2^{\alpha+\beta+1} \Gamma(m+\alpha+1) \Gamma(m+\beta+1)}{m!(2m+\alpha+\beta+1) \Gamma(m+\alpha+\beta+1)}.
\]

Furthermore Jacobi polynomials have the following important properties:

\[
P^{(\alpha,\beta)}_n (1) = \binom{n+\alpha}{n}.
\]

\[
\frac{d^k}{dx^k} P^{(\alpha,\beta)}_n (x) = \frac{\Gamma(\alpha+\beta+n+1+k)}{2^k \Gamma(\alpha+\beta+n+1)} P^{(\alpha+k,\beta+k)}_{n-k}(x).
\]

\[
P^{(\alpha,\beta)}_n (x) = (-1)^n p^{(\beta,\alpha)}_n (-x).
\]
2. Taylor series and orthogonal expansions

Suppose in (4) we take \( \phi_k(x) = P_k^{(\alpha, \beta)}(x) \) then we have:

\[
f(x) = \sum_{k=0}^{\infty} C_k^{(\alpha, \beta)} P_k^{(\alpha, \beta)}(x), \tag{16}
\]

where

\[
C_k^{(\alpha, \beta)} = \left( P_k^{(\alpha, \beta)}, f \right)_{\alpha, \beta} / \left\| P_k^{(\alpha, \beta)} \right\|_{\alpha, \beta}^2.
\]

The formula (16) is the Jacobi expansion of \( f(x) \). Note that (16) can be written as

\[
f(x) = \sum_{k=0}^{\infty} C_k^{(\alpha, \beta)} B_k^{(\alpha, \beta)}(x) = \left( \sum_{i=0}^{\infty} C_i^{(\alpha, \beta)} B_i^{(\alpha, \beta)}(x), P_k^{(\alpha, \beta)} \right) (x - 1)^k.
\]

This is the Taylor series of \( f(x) \) at \( x = 1 \). Therefore we have:

\[
\sum_{i=0}^{\infty} C_i^{(\alpha, \beta)} B_i^{(\alpha, \beta)} = \sum_{i=0}^{\infty} C_{i+k}^{(\alpha, \beta)} B_k^{(\alpha, \beta)} = \frac{f^{(k)}(1)}{k!}; \quad k = 0, 1, 2, \ldots.
\]

Plug (9) and (17) into (19) which gives:

\[
\sum_{i=0}^{\infty} (2i + 2k + \alpha + \beta + 1) \binom{i + k}{k} \binom{i + \alpha + \beta + 2k}{\alpha + k} A_{i+k}^{(\alpha, \beta)} = \frac{2k^{\alpha+\beta+1}}{k!} f^{(k)}(1),
\]

where

\[
A_{i+k}^{(\alpha, \beta)} = \left( P_{i+k}^{(\alpha, \beta)}, f \right)_{\alpha, \beta}.
\]

Now we have proved the following theorem:

**Theorem 2.** If we present \( f(x) \) in the form (16) where the \( P_k^{(\alpha, \beta)}(x) \) are Jacobi polynomials with parameters \( (\alpha, \beta) \) of degree \( k \) then we have:

\[
\sum_{i=0}^{\infty} C_i^{(\alpha, \beta)} B_i^{(\alpha, \beta)} = \sum_{i=0}^{\infty} C_{i+k}^{(\alpha, \beta)} B_k^{(\alpha, \beta)} = \frac{f^{(k)}(1)}{k!}; \quad k = 0, 1, 2, \ldots,
\]

where \( C_i^{(\alpha, \beta)} \) and \( B_k^{(\alpha, \beta)} \) are defined respectively in (17) and (9). In other words we have:

\[
\sum_{i=0}^{\infty} (2i + 2k + \alpha + \beta + 1)(k + 1)_{\alpha + \beta + 2k + 1} A_{i+k}^{(\alpha, \beta)} = \frac{2k^{\alpha+\beta+1} \Gamma(\alpha + k + 1) \Gamma(\beta + k + 1)}{\Gamma(\alpha + \beta + 2k + 1)} f^{(k)}(1),
\]

where \( (a)_{i} = a(a-1)(a-2) \cdots (a-i+1) \) and \( A_{i+k}^{(\alpha, \beta)} \) are defined in (21).

3. The operational matrix

It is known that operational matrices are employed for solving many engineering and physical problems such as dynamical systems [5], optimal control systems [6–8], robotic systems [9] etc. Furthermore they are used in several areas of numerical analysis, and they hold particular importance in various subjects such as integral equations [10], differential equations [11,12], calculus of variations [13], partial differential equations [14], integro-differential equations [15,16] etc. Also many textbooks and papers have employed the operational matrix for spectral methods [17,18]. Let us start this section by introducing operational matrices. Suppose

\[
\Phi(t) = \left[ \phi_0(t), \phi_1(t), \ldots, \phi_n(t) \right]^T,
\]

where the elements \( \phi_0(t), \phi_1(t), \ldots, \phi_n(t) \) are the basis functions on the given interval \([a, b] \). The matrices \( E_{n \times n} \) and \( F_{n \times n} \) are named respectively as the operational matrices of derivatives and integrals if and only if

\[
\frac{d}{dt} \Phi(t) \simeq E \Phi(t), \tag{25}
\]

\[
\int_a^x \Phi(t) dt \simeq F \Phi(x). \tag{26}
\]
Furthermore assume \( g = [g_0, g_1, \ldots, g_{n-1}] \), \( G_g(n \times n) \) is named as the operational matrix of the product if and only if
\[
\Phi(x)\Phi^T(x)g \simeq G_g \Phi(x).
\] (27)

In other words to obtain the operational matrix of a product it is sufficient to find \( c_{i,j} \) in the following relation
\[
\phi_i(x)\phi_j(x) \simeq \sum_{k=0}^{n-1} g_{i,j,k} \phi_k(x)
\] (28)

which is called the linearization formula.

3.1. Operational matrix for general Jacobi polynomials

In this section we introduce a new method for computing the operational matrix for general Jacobi polynomials. Let us first obtain the operational matrix of the derivative for Jacobi polynomials by the new method. Suppose in (24) we define \( \phi(t) = P_{\alpha,\beta}(t) \). Using properties of Jacobi polynomials and (8) we have:
\[
\frac{d}{dx} P_i^{\alpha,\beta}(x) = \sum_{j=0}^{i-1} c_{i,j} P_j^{\alpha,\beta}(x) = \sum_{j=0}^{i-1} \sum_{k=0}^{j} c_{i,j}^D P_k^{\alpha,\beta,j}(x-1)^k; \quad i = 1, 2, 3, \ldots.
\] (29)

On the other hand by defining \( P_i^{\alpha,\beta}(t) = P_i(t) \) and \( f^{(k-1)}(x) = \frac{d^k}{dx^k} P_i(x) = P_i^{(k)}(x) \) in Theorem 2 we conclude:
\[
\sum_{j=k}^{i-1} c_{i,j}^D P_k^{\alpha,\beta,j} = \frac{f^{(k)}(1)}{k!} = \frac{P_i^{(k+1)}(1)}{k!}; \quad k = 0, 1, \ldots, i - 1, i = 1, 2, 3, \ldots.
\] (30)

The use of (13) and (14) leads to:
\[
P_i^{(k+1)}(1) = \frac{\Gamma(\alpha + \beta + i + 2 + k)}{2^{k+1} \Gamma(\alpha + \beta + i + 1)} P_i^{(\alpha+k+1,\beta+k+1)} (1) = \frac{\Gamma(\alpha + \beta + i + 2 + k)}{2^{k+1} \Gamma(\alpha + \beta + i + 1)} (i + \alpha - i - k - 1).
\] (31)

But noting (31), the coefficients \( c_{i,j}^D \) in (30) are obtained from the following upper triangular system:
\[
\begin{bmatrix}
P_0^{(\alpha,\beta,0)} & \cdots & B_0^{(\alpha,\beta,i-3)} & B_0^{(\alpha,\beta,i-2)} & B_0^{(\alpha,\beta,i-1)} \\
0 & B_1^{(\alpha,\beta,1)} & \cdots & B_1^{(\alpha,\beta,i-2)} & B_1^{(\alpha,\beta,i-1)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & B_{i-2}^{(\alpha,\beta,i-2)} & B_{i-2}^{(\alpha,\beta,i-1)} \\
0 & 0 & \cdots & 0 & B_{i-1}^{(\alpha,\beta,i-1)}
\end{bmatrix}
\begin{bmatrix}
c_{i,0}^D \\
c_{i,1}^D \\
\vdots \\
c_{i,i-2}^D \\
c_{i,i-1}^D
\end{bmatrix}
= \begin{bmatrix}
P_i^{(1)/0!} \\
P_i^{(1)/1!} \\
\vdots \\
P_i^{(i-1)/(i-2)!} \\
P_i^{(i)/i!}
\end{bmatrix},
\] (32)

where \( B_k^{(\alpha,\beta,j)} \) and \( P_i^{(k+1)}(1) \) are obtained in (9) and (31) respectively. Solving this linear system concludes:
\[
c_{i,i-1}^D = \frac{P_i^{(i+1)}(1)}{(i+1)!B_{i-1}^{(\alpha,\beta,i-1)}} = \frac{\Gamma(\alpha + \beta + 2i)}{2^{i+1} \Gamma(\alpha + \beta + i)}.
\] (33)

\[
c_{i,j}^D = \frac{P_i^{(j+1)}(1) - j! \sum_{k=j+1}^{i+1} B_j^{(\alpha,\beta,k,j)} c_{i,k}^D}{j!B_j^{(\alpha,\beta,j)}}; \quad j = 0, 1, \ldots, i - 2.
\] (34)

For example if we use the special values \( \alpha = \beta = 0 \) or \( \alpha = \beta = -1/2 \) the operational matrices of the derivative will be obtained for the Legendre and Chebyshev orthogonal polynomials respectively [19,20].

Therefore up to now we have obtained the operational matrix of derivative for Jacobi polynomials with parameter \( \alpha, \beta \). One can use this method to obtain the operational matrix of integral for Jacobi polynomials. For this purpose we use Theorem 2 and define \( f(x) = \int_0^x P_i(t) dt \). Therefore there exist \( c_{i,j}^D \) so that satisfy in following relation:
\[
f(x) = \sum_{j=0}^{i+1} c_{i,j}^D P_i^{(\alpha,\beta,j)}(x) = \sum_{j=0}^{i+1} \sum_{k=0}^{j} c_{i,j}^D P_k^{(\alpha,\beta,j)}(x-1)^k; \quad i = 1, 2, 3, \ldots.
\] (35)
But similar to (30) we must solve the following upper triangular system for the unknowns $c_i^{(1)}$:  
\[
\sum_{j=k}^{i+1} c_{ij}^{(1)} P_{k}^{(\alpha, \beta)}(x) = \frac{f^{(k)}(1)}{k!} = \begin{cases} \frac{P_i^{(k-1)}(1)}{k!} & k = 1, \ldots, i+1, i = 0, 1, 2, 3, \ldots, \\ \int_{-1}^{1} P_i(t) dt & k = 0; i = 0, 1, 2, 3, \ldots, \end{cases} \tag{36}
\]
where by integrating from both sides of (8) we have:
\[
f(1) = \int_{-1}^{1} P_i(t) dt = 2 \times \sum_{k=0}^{i} \left( \frac{i + \alpha + \beta + k}{k+1} \right) (-1)^k, \quad \alpha, \beta > -1, \tag{37}
\]
and $P_i^{(k-1)}(1)$ is obtained from (31).

**Remark 1.** This technique can be employed for computing operational matrices (derivative, integral and product) for other orthogonal functions such as Laguerre [21–23], Hermite [24], Rational Legendre [25], Rational Chebyshev [26], Walsh [27, Bessel [28], Haar [9,11], wavelets [9,13], Bernstein [29,30], Block-pulse [31], and Fourier [32–35] functions. For example in the next section we study this technique for obtaining the operational matrix for general rational Jacobi functions. Also the interested reader can see [36,37].

### 3.2. Operational matrix for general rational Jacobi functions

In this section first we study another orthogonal system of rational functions introduced by Jacobi polynomials. These functions are named as rational Jacobi functions (general) and can be given by [38]
\[
R_k^{(\alpha, \beta)}(x) = P_k^{(\alpha, \beta)} \left( \frac{x-1}{x+1} \right); \quad k = 0, 1, 2, \ldots, x \in (0, \infty), \alpha, \beta > -1. \tag{38}
\]
Furthermore these functions are orthogonal with respect to the weight function $\mu^{\alpha, \beta}(x) = x^{\alpha}(x+1)^{\alpha-\beta-2}(\alpha > -1, \beta > -1)$ on $[0, \infty)$. One of the important properties of the rational Jacobi functions is
\[
\|R_m^{(\alpha, \beta)}\|_{\mu^{\alpha, \beta}}^2 = k_m^{(\alpha, \beta)} = \frac{\Gamma(m+\alpha+1)\Gamma(m+\beta+1)}{(2m+\alpha+\beta+1)\Gamma(m+1)\Gamma(m+\alpha+\beta+1)}. \tag{39}
\]
Also if
\[
f(x) = \sum_{k=0}^{\infty} D_k^{(\alpha, \beta)} R_k^{(\alpha, \beta)}(x), \quad x \in (0, \infty), \tag{40}
\]
then we have
\[
D_k^{(\alpha, \beta)} = (k_k^{(\alpha, \beta)})^{-1} \langle f, R_k^{(\alpha, \beta)} \rangle_{\mu^{\alpha, \beta}}. \tag{41}
\]
Now if we use the change of variables $x \to \frac{x-1}{x+1}$ in (10) and employ (38) then we have:
\[
R_n^{(\alpha, \beta)}(x) = \sum_{k=0}^{n} E_k^{(\alpha, \beta, n)} \left( \frac{x-1}{x+1} \right)^k, \tag{42}
\]
where $E_k^{(\alpha, \beta, n)}$ is defined in (11).

But (40) can be rewritten as
\[
f(x) = \sum_{k=0}^{\infty} D_k^{(\alpha, \beta)} \left( \sum_{i=0}^{k} E_i^{(\alpha, \beta, k)} \left( \frac{x-1}{x+1} \right)^i \right) = \left( \sum_{i=0}^{\infty} D_i^{(\alpha, \beta)} E_i^{(\alpha, \beta, i)} \right) + \left( \sum_{i=1}^{\infty} D_i^{(\alpha, \beta)} E_1^{(\alpha, \beta, i)} \right) \left( \frac{x-1}{x+1} \right) + \cdots = \sum_{k=0}^{\infty} \left( \sum_{i=2}^{\infty} D_i^{(\alpha, \beta)} E_2^{(\alpha, \beta, i)} \right) \left( \frac{x-1}{x+1} \right)^2 + \cdots \tag{43}
\]
Now if we define
\[
g(x) = f \left( \frac{1+x}{1-x} \right) \Rightarrow f(x) = g \left( \frac{x-1}{x+1} \right), \quad \tag{44}
\]
then from the Maclaurin series for \( g(x) \) we have:

\[
g(x) = \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} x^k,
\]

(45)

or

\[
f(x) = g \left( \frac{x - 1}{x + 1} \right) = \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} \left( \frac{x - 1}{x + 1} \right)^k,
\]

(46)

where (44) gives:

\[
g^{(k)}(0) = \sum_{i=0}^{k} G_i^{(k)} f^{(i)}(1).
\]

(47)

But if we deeply see (47) we can conclude the following relations

\[
G_0^{(k)} = 1; \quad G_1^{(k)} = 2k!,
\]

(48)

\[
G_i^{(k)} = \frac{2(k - i + 1)}{(i - 1)!} G_{i-1}^{(k)}; \quad i = 2, 3, \ldots, k.
\]

(49)

Therefore from (48) and (49) we have:

\[
G_i^{(k)} = 2^i \binom{k}{i} \frac{(k - 1)!}{(i - 1)!}.
\]

(50)

Now comparing (43) and (46) we get:

\[
\sum_{j=k}^{\infty} D_j^{(\alpha, \beta)} E_k^{(\alpha, \beta, i)} = \frac{g^{(k)}(0)}{k!} = \frac{f(1) + \sum_{i=1}^{k} 2^i \binom{k}{i} \frac{(k - 1)!}{(i - 1)!} f^{(i)}(1)}{k!},
\]

(51)

where \( E_k^{(\alpha, \beta, i)} \) and \( D_j^{(\alpha, \beta)} \) are defined in (11) and (41). Therefore we have proved the following theorem:

**Theorem 3.** If we present \( f(x) \) in the form (40) where \( R_k^{(\alpha, \beta)}(x) \) are rational Jacobi functions with parameter \((\alpha, \beta)\) then we have:

\[
\sum_{j=k}^{\infty} D_j^{(\alpha, \beta)} E_k^{(\alpha, \beta, i)} = \mu_k,
\]

(52)

where

\[
\mu_k = \frac{f(1) + \sum_{i=1}^{k} 2^i \binom{k}{i} \frac{(k - 1)!}{(i - 1)!} f^{(i)}(1)}{k!},
\]

(53)

and \( E_k^{(\alpha, \beta, i)} \) and \( D_j^{(\alpha, \beta)} \) are defined respectively in (11) and (41).

Now we would like to use Theorem 3 for computing the operational matrix of derivative for rational Jacobi functions. To reach to this aim we know there exist \( c_{ij}^{RD} \) such that:

\[
\frac{(1 - x)^i P_j(x)}{2} = \sum_{j=0}^{i+1} c_{ij}^{RD} P_j(x).
\]

(54)

But if in (54) we apply the change of variables \( x \to \frac{x - 1}{x + 1} \) and define \( R_i^{(\alpha, \beta)}(x) = R_i(x) \) then we obtain:

\[
\frac{2}{(x + 1)^2} P_j' \left( \frac{x - 1}{x + 1} \right) = \sum_{j=0}^{i+1} c_{ij}^{RD} P_j' \left( \frac{x - 1}{x + 1} \right),
\]

(55)

or

\[
R_j'(x) = \sum_{j=0}^{i+1} c_{ij}^{RD} R_j(x).
\]

(56)
Combining (42) and (56) yields:

$$R_i'(x) = \sum_{j=0}^{i+1} \sum_{k=0}^{j} c_{ij}^{RD} E_k^{(\alpha, \beta, j)} \left( \frac{x - 1}{x + 1} \right)^k.$$  \hfill (57)

Now noting Theorem 3 in this case we get:

$$\sum_{j=k}^{i+1} c_{ij}^{RD} E_k^{(\alpha, \beta, j)} = \frac{f^{(1)} + \sum_{i=1}^{k} 2^i \binom{k}{i} \frac{f^{(k-1)}}{(x-1)!} f^{(1)}(1)}{k!}; \quad k = 0, 1, \ldots, i+1.$$  \hfill (58)

where \( f(x) = R'_i(x) \) and \( c_{ij}^{RD} \) are elements of the operational matrix of derivative. But for computing \( f^{(i)}(1) \) in (58), we have:

$$f(x) = R'_i(x) = \frac{2}{(1 + x)^2} P_i^{(1)} \left( \frac{x - 1}{x + 1} \right) \Rightarrow f(1) = R'_i(1) = \frac{1}{2} P_i^{(1)}(0).$$  \hfill (59)

From (59) we can write

$$f^{(k)}(1) = R^{(k+1)}_i(1) = \sum_{j=0}^{k+1} u_j^{(k+1)} P_j^{(j+1)}(0),$$  \hfill (60)

where

$$u_j^{(k+1)} = (-1)^{k+j+1} \frac{2^{k-j}}{2^{2k+2}} j! \binom{k+1}{j+1} \frac{k!}{j!} = (-1)^{k+j+1} \frac{2^{-k}}{j!}.$$  \hfill (61)

But from (8) and (14) we obtain:

$$P_{n}^{(j+1)}(0) = \frac{\Gamma(\alpha + \beta + n + 2 + j)}{2^{j+1} \Gamma(\alpha + \beta + n + 1) \Gamma(n-j-1)} \rho_{n-j+1}^{(\alpha, \beta+1, j+1)}(0) = \frac{\Gamma(\alpha + \beta + n + 2 + j)}{2^{j+1} \Gamma(\alpha + \beta + n + 1) \Gamma(n-j-1)} \rho_{n-j+1}^{(\alpha, \beta+1, j+1)}(0) \times \sum_{i=0}^{n-j-1} \binom{n + \alpha + \beta + j + 1 + i}{i} \binom{\alpha + n}{n-j-1-i} \frac{1}{2^i} (-1)^i;$$  \hfill (62)

and finally we can write:

$$f^{(k)}(1) = \sum_{j=0}^{k+1} 2^{-k} \binom{k+1}{j+1} \frac{k!}{j!} \times \frac{\Gamma(\alpha + \beta + n + 2 + j)}{2^{j+1} \Gamma(\alpha + \beta + n + 1) \Gamma(n-j-1)} \rho_{n-j+1}^{(\alpha, \beta+1, j+1)}(0) \times \sum_{i=0}^{n-j-1} \binom{n + \alpha + \beta + j + 1 + i}{i} \binom{\alpha + n}{n-j-1-i} \frac{1}{2^i} (-1)^i.$$

\hfill (63)

Therefore the coefficients \( c_{ij}^{RD} \) in (58) can be obtained from the following upper triangular system:

$$
\begin{bmatrix}
E_0^{(\alpha, \beta, 0)} & \cdots & E_0^{(\alpha, \beta, i-1)} & E_0^{(\alpha, \beta, i)} & E_0^{(\alpha, \beta, i+1)} \\
0 & E_1^{(\alpha, \beta, 1)} & \cdots & E_1^{(\alpha, \beta, i)} & E_1^{(\alpha, \beta, i+1)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & E_i^{(\alpha, \beta, i)} & E_i^{(\alpha, \beta, i+1)} \\
0 & 0 & \cdots & 0 & E_{i+1}^{(\alpha, \beta, i+1)}
\end{bmatrix}
\begin{bmatrix}
c_{i,0}^{RD} \\
c_{i,1}^{RD} \\
\vdots \\
c_{i,i+1}^{RD}
\end{bmatrix}
= 
\begin{bmatrix}
\mu_0 \\
\mu_1 \\
\vdots \\
\mu_{i+1}
\end{bmatrix},$$  \hfill (64)

where \( E_k^{(\alpha, \beta, n)} \), \( \mu_k \), and \( f^{(k)}(1) \) are obtained previously. Solving the linear system (64) concludes:

$$c_{i,i+1}^{RD} = \frac{\mu_{i+1}}{E_{i+1}^{(\alpha, \beta, i+1)}},$$  \hfill (65)

$$c_{i,j}^{RD} = \frac{\mu_j - \sum_{k=j+1}^{i+1} E_j^{(\alpha, \beta, k)} c_{i,k}}{E_j^{(\alpha, \beta, j)}}; \quad j = 0, 1, \ldots, i.$$  \hfill (66)
One can use this method to obtain the operational matrix of the integral (product) for Jacobi rational functions. For this purpose it is obvious there exist \( c_{ij}^{RI} \) that satisfy the following relation.

\[
\int_{-1}^{u} \frac{(1+t)^2}{2} P_i(t) \, dt = \sum_{j=0}^{i+3} c_{ij}^{RI} P_j(u); \quad -1 \leq u \leq 1.
\]  

(67)

Employing the changing of variables \( t \rightarrow \frac{t-1}{t+1} \) on the right hand side of (67) gives:

\[
\int_{0}^{\frac{1-u}{1+u}} P_i \left( \frac{t-1}{t+1} \right) \, dt = \sum_{j=0}^{i+3} c_{ij}^{RI} P_j(u).
\]  

(68)

Now if we define \( x = \frac{1-u}{1+u} \) then (68) transforms to

\[
\int_{0}^{x} P_i \left( \frac{t-1}{t+1} \right) \, dt = \sum_{j=0}^{i+3} c_{ij}^{RI} P_j \left( \frac{x-1}{x+1} \right).
\]  

(69)

In other words we have

\[
\int_{0}^{x} R_i(t) \, dt = \sum_{j=0}^{i+3} c_{ij}^{RI} R_j(x).
\]  

(70)

Combining (43) and (70) yields

\[
\int_{0}^{x} R_i(t) \, dt = \sum_{j=0}^{i+3} \sum_{k=0}^{j} c_{ij}^{RI} E_{k}^{(x, \beta, \gamma)} \left( \frac{x-1}{x+1} \right)^k.
\]  

(71)

Now noting Theorem 3 we can write:

\[
\sum_{j=k}^{i+3} c_{ij}^{RI} E_{k}^{(x, \beta, \gamma)} \left( \frac{x-1}{x+1} \right)^k = \mu_k; \quad k = 0, 1, \ldots, i + 3,
\]  

(72)

where \( \mu_k \) is defined in (53) and

\[
f^{(k)}(1) = \begin{cases} 
\int_{0}^{1} R_i(t) \, dt; \quad k = 0; i = 0, 1, 2, 3, \ldots, \\
R_i^{(k-1)}(1); \quad k = 1, \ldots, i + 1, i = 0, 1, 2, 3, \ldots.
\end{cases}
\]  

(73)

where \( R_i^{(k-1)}(1); k = 1, \ldots, i + 1, i = 0, 1, 2, 3, \ldots, \) is defined in (61) and from (43) we can write

\[
\int_{0}^{1} R_i(x) \, dx = \sum_{k=0}^{i} E_{k}^{(x, \beta, \gamma)} \int_{0}^{1} \left( \frac{x-1}{x+1} \right)^k \, dx,
\]  

(74)

where \( E_{k}^{(x, \beta, \gamma, n)} \) is defined in (11) and for computing the value of \( \int_{0}^{1} \left( \frac{x-1}{x+1} \right)^k \, dx \) if we change the variable \( u = x + 1 \) then we have

\[
\int_{0}^{1} \left( \frac{x-1}{x+1} \right)^k \, dx = \int_{1}^{2} \left( 1 - \frac{2}{u} \right)^k \, du = \int_{1}^{2} \left( 1 - \frac{2}{u} \right)^k \, du = \sum_{j=0}^{k} \int_{1}^{2} (-1)^{k-j} \left( \begin{array}{c} k \\ j \end{array} \right) 2^{k-j} u^{-j} \, du
\]  

\[= \sum_{j=0}^{k} (-2)^{k-j} \left( \begin{array}{c} k \\ j \end{array} \right) \int_{1}^{2} \, u^{-k} \, du.
\]  

(75)

Now noting

\[
\int_{1}^{2} u^{-k} \, du = \begin{cases} 
\ln(2); \quad j = k - 1, \\
\left( -1 + 2^{1-j} \right) / (j - k + 1); \quad j < k - 1 \text{ or } j = k,
\end{cases}
\]  

(76)

we have

\[
\int_{0}^{1} \left( \frac{x-1}{x+1} \right)^k \, dx = \sum_{j=0}^{k} (-2)^{k-j} \left( \begin{array}{c} k \\ j \end{array} \right) \frac{(-1 + 2^{1-j+k})}{(j - k + 1)} - 2k \ln(2)
\]  

(77)
and finally we get:
\[
\int_0^1 R(t) \, dt = \sum_{k=0}^{n} E^{(\alpha, \beta, i)}_k \left( \sum_{j=1}^{k} (-2)^{k-j} \binom{k-1}{j-1} \left( -1 + 2^{j-k+1} \right) (j-k+1) - 2k \ln(2) \right).
\]

(78)

It is obvious that the coefficient \( e^{(\alpha, \beta, i)}_k \) in (70) is obtained from a linear system similar to (64). For example the rational Legendre \((\alpha = \beta = 0)\) and rational Chebyshev \((\alpha = \beta = -1/2)\) operational matrices are computed in [25, 26]. We refer the interested reader to [39] for application of the operational matrix in solving fractional differential equations.

4. Conclusion

In this research first we explicitly obtained the relation between coefficients of the Taylor series (derivatives of a function at a fixed point) and classical orthogonal series. Then we presented a new method for computing the operational matrices (derivative, integral) for general Jacobi orthogonal functions (polynomial and rational). Most papers and textbooks are limited to methods of obtaining operational matrices for Fourier, Chebyshev and sometimes Legendre but one of the advantages of the new method proposed in this paper is computing the operational matrices for Jacobi orthogonal functions for arbitrary parameters \((\alpha, \beta)\). Note that this method can be used for obtaining the operational matrices (derivative, integral, product) for many classes of orthogonal functions.

References