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Analysis of some mixed elements for the Stokes problem [1]

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Abstract

In this paper we discuss some mixed finite element methods related to the reduced integration penalty method for solving the Stokes problem. We prove optimal order error estimates for bilinear–constant and biquadratic–bilinear velocity–pressure finite element solutions. The result for the biquadratic–bilinear element is new, while that for the bilinear-constant element improves the convergence analysis of Johnson and Pitkäranta (1982). In the degenerate case when the penalty parameter is set to be zero, our results reduce to some related known results proved in by Brezzi and Fortin (1991) for the bilinear-constant element, and Bercovier and Pironneau (1979) for the biquadratic–bilinear element. Our theoretical results are consistent with the numerical results reported by Carey and Krishnan (1982) and Oden et al. (1982).

Keywords: Stokes problem; Mixed finite elements; Reduced integration penalty method; Optimal order error estimates.

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1. Introduction

In this paper, we consider the numerical solution of Stokes problem for viscous incompressible fluid flows by the finite element method. The model problem to be studied is to find the velocity $\boldsymbol{u} \in [H_0^1(\Omega)]^2$ and the pressure $p \in L_0^2(\Omega)$ such that

$$\begin{aligned} &(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (\operatorname{div} \boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v}), && \forall\, \boldsymbol{v} \in [H_0^1(\Omega)]^2, \\ &(\operatorname{div} \boldsymbol{u}, q) = 0, && \forall\, q \in L_0^2(\Omega). \end{aligned} \tag{1.1}$$

Here, $\Omega \subset \mathbb{R}^2$ is a bounded domain, $\boldsymbol{f}$ is the given body force. The symbol $(\cdot,\cdot)$ denotes the usual inner product in $L^2(\Omega)$ or $[L^2(\Omega)]^2$. The space $L_0^2(\Omega)$ consists of all the $L^2(\Omega)$-functions whose

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mean values in Ω are zero. The Sobolev space $H_0^1(\Omega)$ is the set of all the $L^2(\Omega)$-functions whose first order partial derivatives are also in $L^2(\Omega)$, and whose traces on the boundary $\partial\Omega$ vanish. For $\boldsymbol{f} \in [H^{-1}(\Omega)]^2$, the Stokes problem (1.1) has a unique solution, cf. [10]. The problem (1.1) is equivalent to a constrained minimization problem

$$\tfrac{1}{2}(\nabla \boldsymbol{v}, \nabla \boldsymbol{v}) - (\boldsymbol{f}, \nabla \boldsymbol{v}) \to \inf, \quad \boldsymbol{v} \in [H_0^1(\Omega)]^2, \ \operatorname{div} \boldsymbol{v} = 0. \tag{1.2}$$

Given finite element spaces $V_h \subset [H_0^1(\Omega)]^2$ and $Q_h \subset L_0^2(\Omega)$, a mixed finite element method can be developed based on the formulation (1.1). We need to find unknowns $\boldsymbol{u}_h \in V_h$ and $p_h \in Q_h$, such that

$$\begin{aligned} &(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) - (\operatorname{div} \boldsymbol{v}_h, p_h) = (\boldsymbol{f}, \boldsymbol{v}_h), && \forall \boldsymbol{v}_h \in V_h, \\ &(\operatorname{div} \boldsymbol{u}_h, q_h) = 0, && \forall q_h \in Q_h. \end{aligned} \tag{1.3}$$

Another approach to develop numerical schemes is to use the formulation (1.2). Here we need a finite element space of divergence free functions in $[H_0^1(\Omega)]^2$. Let W_h be such a space. Then a finite element method based on the formulation (1.2) reads as follows:

$$\boldsymbol{u}_h \in W_h, \quad (\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in W_h. \tag{1.4}$$

An advantage of the scheme (1.4) is it involves fewer unknowns than in the mixed method (1.3). The major disadvantage is that usually it is impractical to use divergence free finite element functions. To circumvent the difficulty caused by the divergence free constraint, it is natural to use the idea of penalty methods. Let $\varepsilon > 0$ be a small parameter. Then the constrained minimization problem (1.2) is approximated by a penalized problem: find $\boldsymbol{u}^\varepsilon \in [H_0^1(\Omega)]^2$, such that

$$(\nabla \boldsymbol{u}^\varepsilon, \nabla \boldsymbol{v}) + \frac{1}{\varepsilon}(\operatorname{div} \boldsymbol{u}^\varepsilon, \operatorname{div} \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in [H_0^1(\Omega)]^2. \tag{1.5}$$

The convergence of the penalty method is proved in several papers, e.g., [2, 15]. Now let V_h be a finite element space in $[H_0^1(\Omega)]^2$. A finite element method based on the formulation (1.5) is to find $\boldsymbol{u}_h^\varepsilon \in V_h$ such that

$$(\nabla \boldsymbol{u}_h^\varepsilon, \nabla \boldsymbol{v}_h) + \frac{1}{\varepsilon}(\operatorname{div} \boldsymbol{u}_h^\varepsilon, \operatorname{div} \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in V_h. \tag{1.6}$$

Theoretically, to obtain more accurate approximations, one needs to use smaller values for the penalty parameter ε. However, as $\varepsilon \to 0+$, locking phenomenon arises due to overconstraining. A cure for this difficulty is the use of a reduced numerical quadrature for computing terms involving the parameter ε [14, 2], and the numerical scheme is to find $\boldsymbol{u}_h^\varepsilon \in V_h$ such that

$$(\nabla \boldsymbol{u}_h^\varepsilon, \nabla \boldsymbol{v}_h) + \frac{1}{\varepsilon} I(\operatorname{div} \boldsymbol{u}_h^\varepsilon, \operatorname{div} \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in V_h, \tag{1.7}$$

where I is a numerical integration operator. Under certain assumptions, the reduced integration penalty method (1.7) is equivalent to the following mixed finite element method: find $(\boldsymbol{u}_h^\varepsilon, p_h^\varepsilon) \in V_h \times Q_h$ such that

$$\begin{aligned} &(\nabla \boldsymbol{u}_h^\varepsilon, \nabla \boldsymbol{v}_h) - (\operatorname{div} \boldsymbol{v}_h, p_h^\varepsilon) = (\boldsymbol{f}, \boldsymbol{v}_h), && \forall \boldsymbol{v}_h \in V_h, \\ &\varepsilon(p_h^\varepsilon, q_h) + (\operatorname{div} \boldsymbol{u}_h^\varepsilon, q_h) = 0, && \forall q_h \in Q_h, \end{aligned} \tag{1.8}$$

for some finite element space $Q_h \subset L_0^2(\Omega)$. For example, bilinear velocity quadrilateral elements with a one-point Gaussian quadrature rule for (1.7) results in an equivalent bilinear-constant velocity-pressure finite element method (1.8) [6]. As another example, biquadratic velocity quadrilateral elements with a four-point Gaussian quadrature rule for (1.7) is equivalent to a biquadratic–bilinear velocity–pressure finite element method (1.8).

It is well known that for the method to work, the spaces V_h and Q_h cannot be chosen arbitrarily. The method can be expected to behave well if the Babuška–Brezzi condition is satisfied, see [1, 5]. In practical computations, however, many popular elements (including the bilinear-constant elements and the biquadratic–bilinear elements) do not satisfy the Babuška–Brezzi condition, yet the numerical experiments in [15, 7] show that these elements work well for the velocity and for the pressure after filtering out spurious pressure mode. Indeed, for the bilinear-constant element for the Stokes problem, it is proved in [4] that the inf–sup constant is exactly of the order of the meshsize. For the biquadratic–bilinear element for the Stokes problem, the same conclusion is proved in [12]. A direct application of the classical saddle point approximation theory (cf. [6]) would then predict a degenerate convergence order for the approximation of the velocity, and no convergence for the approximation of the pressure. Thus, to obtain optimal order error estimates, we cannot apply the classical saddle point approximation theory directly. In 1982, Johnson and Pitkäranta [13] proved an optimal order error estimate for the bilinear-constant velocity–pressure rectangular element, under an extra smoothness assumption on the solution. Cheng [8] extended this result to a general quadrilateral mesh by using the techniques of the reduced integration penalty directly. See also [16, 6] for a discussion of the optimal order error estimate of the bilinear-constant element for the method (1.3). For the biquadratic–bilinear velocity–pressure finite element, an optimal order error analysis seems not available in the literature.

The main purpose of the paper is to prove optimal order error estimates for the bilinear–constant and biquadratic–bilinear elements for the Stokes problem, without extra smoothness assumptions on the solution. In particular, for the bilinear–constant element, we improve an error estimate result of [13]. Our optimal order error estimates are confirmed by the numerical results reported in [15, 7].

The organization of the paper is as follows. In the next section, we prove an error estimate for the approximation of an abstract problem, which includes the Stokes problem as a special case. In Section 3, we comment on the macroelement technique used in later sections. The last two sections are devoted to deriving optimal order error estimates for the bilinear–constant velocity–pressure element and the biquadratic–bilinear velocity–pressure element, respectively.

We remark that our result are more general than some related known results. Our proofs go through also when the penalty parameter is zero, and in this case our results reduce to the related results proved in [6] for the bilinear-constant element, and [3] for the biquadratic–bilinear element.

2. An abstract error estimate

In this section, we provide an error estimate in an abstract framework which contains the Stokes problem and its numerical approximations as a special case. Following [6], let V and Q be two Hilbert spaces. Let $a : V \times V \to R$ be a continuous, V-elliptic, bilinear form, $b : V \times Q$ a continuous bilinear form. By V-ellipticity of a, we mean the inequality

$$a(v,v) \geqslant \alpha \|v\|_V^2, \quad \forall v \in V \tag{2.1}$$

holds for some constant $\alpha > 0$. Associated with the bilinear form b, we introduce a linear operator $B : V \to Q'$ and its transpose $B^{\mathrm{T}} : Q \to V'$ through the relation

$$b(v,q) = \langle Bv, q\rangle_{Q'\times Q} = \langle v, B^{\mathrm{T}}q\rangle_{V\times V'}, \quad \forall v \in V, \ \forall q \in Q.$$

The range of B is denoted by $\operatorname{Im} B$. We assume the kernel space

$$\operatorname{Ker} B^{\mathrm{T}} = \{q \in Q \colon b(v,q) = 0 \ \forall v \in V\} = \{0\}. \tag{2.2}$$

We may as well develop a theory for the more general case where $\operatorname{Ker} B^{\mathrm{T}} \neq \{0\}$. However, such a generalization is not needed in applications in later sections.

Given $f \in V'$ and $g \in \operatorname{Im} B \subset V$, the abstract problem is to find $u \in V$ and $p \in Q$, such that

$$\begin{aligned} a(u,v) + b(v,p) &= \langle f, v\rangle_{V'\times V}, && \forall v \in V, \\ b(u,q) &= \langle g, q\rangle_{Q'\times Q}, && \forall q \in Q. \end{aligned} \tag{2.3}$$

Under the assumptions made on the data, together with the inf–sup inequality

$$\sup_{v\in V} \frac{b(v,q)}{\|v\|_V} \geqslant k_0 \|q\|_Q, \quad \forall q \in Q \tag{2.4}$$

for some $k_0 > 0$, the problem (2.3) has a unique solution (u, p).

Let V_h and Q_h be finite dimensional subspaces of V and Q. Consider the following approximation scheme: find $u_h^\varepsilon \in V_h$ and $p_h^\varepsilon \in Q_h$, such that

$$\begin{aligned} a(u_h^\varepsilon, v_h) + b(v_h, p_h^\varepsilon) &= \langle f, v_h\rangle_{V'\times V}, && \forall v_h \in V_h, \\ b(u_h^\varepsilon, q_h) - \varepsilon\,(p_h^\varepsilon, q_h)_Q &= \langle g, q_h\rangle_{Q'\times Q}, && \forall q_h \in Q_h, \end{aligned} \tag{2.5}$$

where $\varepsilon > 0$ is a small parameter. By Proposition II.1.4 of [6], the discrete problem (2.5) has a unique solution $(u_h^\varepsilon, p_h^\varepsilon) \in V_h \times Q_h$. The goal of the section is to derive an error estimate for the approximation method (2.5). To do this, we assume there exist subspaces $\hat{V}_h \subset V_h$ and $\hat{Q}_h \subset Q_h$ such that

$$\sup_{\hat{v}_h \in \hat{V}_h} \frac{b(\hat{v}_h, \hat{q}_h)}{\|\hat{v}_h\|_V} \geqslant \beta_0 \|\hat{q}_h\|_Q, \quad \forall \hat{q}_h \in \hat{Q}_h \tag{2.6}$$

where $\beta_0 > 0$ is a constant independent of h. Let $\tilde{Q}_h$ be the orthogonal complement of $\hat{Q}_h$ in Q_h. We shall assume

$$b(\hat{v}_h, \tilde{q}_h) = 0, \quad \forall \tilde{q}_h \in \tilde{Q}_h, \quad \forall \hat{v}_h \in \hat{V}_h. \tag{2.7}$$

Theorem 2.1. *We write $p_h^\varepsilon = \hat{p}_h^\varepsilon + \tilde{p}_h$, with $\hat{p}_h^\varepsilon \in \hat{Q}_h$ and $\tilde{p}_h^\varepsilon \in \tilde{Q}_h$. Then, under the given assumptions, we have the following error estimates*:

$$\|u - u_h^\varepsilon\|_V \leqslant C \left\{ \inf_{v_h \in V_h} \{\|u - v_h\|_V + |b(u - v_h, \tilde{p}_h^\varepsilon)|^{1/2}\} + \inf_{\hat{q}_h \in \hat{Q}_h} \{\|p - \hat{q}_h\|_Q + \varepsilon \|\hat{q}_h\|_Q\} \right\}, \tag{2.8}$$

$$\|p - \hat{p}_h^\varepsilon\|_Q \leqslant C\{\|u - u_h^\varepsilon\|_V + \inf_{\hat{q}_h \in \hat{Q}_h} \|p - \hat{q}_h\|_0\}. \tag{2.9}$$

Proof. From (2.3) and (2.5), we get the error relations

$$a(u - u_h^\varepsilon, v_h) + b(v_h, p - p_h^\varepsilon) = 0, \quad \forall v_h \in V_h, \tag{2.10}$$

and

$$b(u - u_h^\varepsilon, q_h) + \varepsilon (p_h^\varepsilon, q_h)_Q = 0, \quad \forall q_h \in Q_h. \tag{2.11}$$

For any $v_h \in V_h$, we have

$$\|u - u_h^\varepsilon\|_V \leqslant \|u - v_h\|_V + \|u_h^\varepsilon - v_h\|_V. \tag{2.12}$$

Applying the V-ellipticity of a (cf. (2.1)), we get

$$\alpha \|u_h^\varepsilon - v_h\|_V^2 \leqslant a(u_h^\varepsilon - v_h, u_h^\varepsilon - v_h) = a(u_h^\varepsilon - u, u_h^\varepsilon - v_h) + a(u - v_h, u_h^\varepsilon - v_h).$$

Using (2.10) with v_h replaced by $u_h^\varepsilon - v_h$, we then get

$$\alpha \|u_h^\varepsilon - v_h\|_V^2 \leqslant b(u_h^\varepsilon - v_h, p - p_h^\varepsilon) + a(u - v_h, u_h^\varepsilon - v_h). \tag{2.13}$$

For any $\hat{q}_h \in \hat{Q}_h$, we have

$$b(u_h^\varepsilon - v_h, p - p_h^\varepsilon) = b(u_h^\varepsilon - v_h, p - \hat{q}_h) + b(u_h^\varepsilon - v_h, \hat{q}_h - p_h^\varepsilon). \tag{2.14}$$

Using the relation (2.11), we have

$$\begin{aligned} & b(u_h^\varepsilon - v_h, \hat{q}_h - p_h^\varepsilon) \\ & \quad = b(u_h^\varepsilon - u, \hat{q}_h - p_h^\varepsilon) + b(u - v_h, \hat{q}_h - p_h^\varepsilon) \\ & \quad = \varepsilon (p_h^\varepsilon, \hat{q}_h - p_h^\varepsilon) + b(u - v_h, \hat{q}_h - p_h^\varepsilon) \\ & \quad = \varepsilon (p_h^\varepsilon, \hat{q}_h - p_h^\varepsilon) + b(u - v_h, \hat{q}_h - \hat{p}_h^\varepsilon) - b(u - v_h, \tilde{p}_h^\varepsilon), \end{aligned}$$

which, combined with (2.13) and (2.14), implies

$$\begin{aligned} & \alpha \|u_h^\varepsilon - v_h\|_V^2 \\ & \quad \leqslant a(u - v_h, u_h^\varepsilon - v_h) + b(u_h^\varepsilon - v_h, p - \hat{q}_h) \\ & \qquad + b(u - v_h, \hat{q}_h - \hat{p}_h^\varepsilon) - b(u - v_h, \tilde{p}_h^\varepsilon) + \varepsilon (p_h^\varepsilon, \hat{q}_h - p_h^\varepsilon). \end{aligned}$$

Thus,

$$
\begin{aligned}
&\alpha \|u_h^\varepsilon - v_h\|_V^2 \\
&\quad \leqslant c \left(\|u - v_h\|_V \|u_h^\varepsilon - v_h\|_V + \|u_h^\varepsilon - v_h\|_V \|p - \hat{q}_h\|_Q + \|u - v_h\|_V \|\hat{q}_h - \hat{p}_h^\varepsilon\|_Q \right) \\
&\qquad + |b(u - v_h, \tilde{p}_h^\varepsilon)| + \varepsilon\, (p_h^\varepsilon, \hat{q}_h - p_h^\varepsilon).
\end{aligned}
$$

A simple manipulation then shows that

$$
\begin{aligned}
&\|u_h^\varepsilon - v_h\|_V^2 + \varepsilon\, \|\hat{q}_h - p_h^\varepsilon\|_Q^2 \\
&\quad \leqslant c \left(\|u - v_h\|_V^2 + \|p - \hat{q}_h\|_Q^2 + \|u - v_h\|_V \|\hat{q}_h - \hat{p}_h^\varepsilon\|_Q \right) \\
&\qquad + |b(u - v_h, \tilde{p}_h^\varepsilon)| + \varepsilon\, (\hat{q}_h, \hat{q}_h - p_h^\varepsilon)_Q.
\end{aligned} \tag{2.15}
$$

To estimate the term $\|\hat{q}_h - \hat{p}_h^\varepsilon\|_Q$, we use (2.6), (2.7) and (2.10) to obtain

$$
\begin{aligned}
&\|\hat{q}_h - \hat{p}_h^\varepsilon\|_Q \\
&\quad \leqslant c \sup_{\hat{v}_h \in \hat{V}_h} \frac{b(\hat{v}_h, \hat{q}_h - \hat{p}_h^\varepsilon)}{\|\hat{v}_h\|_V} \\
&\quad = c \sup_{\hat{v}_h \in \hat{V}_h} \frac{b(\hat{v}_h, \hat{q}_h - p) + b(\hat{v}_h, p - p_h^\varepsilon)}{\|\hat{v}_h\|_V} \\
&\quad = c \sup_{\hat{v}_h \in \hat{V}_h} \frac{b(\hat{v}_h, \hat{q}_h - p) - a(u - u_h^\varepsilon, \hat{v}_h)}{\|\hat{v}_h\|_V}.
\end{aligned}
$$

Hence,

$$
\|\hat{q}_h - \hat{p}_h^\varepsilon\|_Q \leqslant c \left(\|p - \hat{q}_h\|_Q + \|u - u_h^\varepsilon\|_V \right). \tag{2.16}
$$

Also, we have

$$
(\hat{q}_h, \hat{q}_h - p_h^\varepsilon)_Q = (\hat{q}_h, \hat{q}_h - \hat{p}_h^\varepsilon)_Q \leqslant c\, \|\hat{q}_h\|_Q \|\hat{q}_h - \hat{p}_h^\varepsilon\|_Q. \tag{2.17}
$$

It is easy to see, from (2.12), (2.15), (2.16) and (2.17), that

$$
\begin{aligned}
&\|u_h^\varepsilon - v_h\|_V^2 + \varepsilon\, \|\hat{q}_h - p_h^\varepsilon\|_Q^2 \\
&\quad \leqslant c \left\{ \|u - v_h\|_V^2 + |b(u - v_h, \tilde{p}_h^\varepsilon)| + \|p - \hat{q}_h\|_Q^2 + \varepsilon^2 \|\hat{q}_h\|_Q^2 \right\}.
\end{aligned} \tag{2.18}
$$

By the arbitrariness of $v_h \in V_h$ and $\hat{q}_h \in \hat{Q}_h$, we have thus proved the estimate (2.8). The estimate (2.9) follows from (2.16). □

3. Macroelement technique

We first specialize the result of Theorem 2.1 to the numerical method (1.8) for the Stokes problem (1.1). As in [6], we choose two subspaces $\hat{V} \subset V_h$ and $\hat{Q}_h \subset Q_h$ so that

$$
\beta_0 \|\hat{q}_h\|_0 \leqslant \sup_{0 \neq \hat{v}_h \in \hat{V}_h} \frac{(\operatorname{div} \hat{v}_h, \hat{q}_h)}{\|\hat{v}_h\|_1}, \quad \forall \hat{q}_h \in \hat{Q}_h, \tag{3.1}
$$

for a positive constant β_0 independent of h and

$$(\operatorname{div} \hat{\boldsymbol{v}}_h, \tilde{q}_h) = 0, \quad \forall \tilde{q}_h \in \tilde{Q}_h, \ \forall \hat{\boldsymbol{v}}_h \in \hat{V}_h, \tag{3.2}$$

where $\tilde{Q}_h$ is the orthogonal complement of $\hat{Q}_h$ in Q_h. From Theorem 2.1 and the fact that $\operatorname{div} \boldsymbol{u} = 0$, we obtain the following result.

Theorem 3.1. *Let $\hat{V}_h$ and $\hat{Q}_h$ satisfy hypotheses* (3.1) *and* (3.2). *Let* $(\boldsymbol{u}, p)$ *and* $(\boldsymbol{u}_h^\varepsilon, p_h^\varepsilon)$ *be the solutions of the problems* (1.1) *and* (1.8), *respectively. We write $p_h^\varepsilon = \hat{p}_h^\varepsilon + \tilde{p}_h$, with $\hat{p}_h^\varepsilon \in \hat{Q}_h$ and $\tilde{p}_h^\varepsilon \in \tilde{Q}_h$. Then*

$$\|\boldsymbol{u} - \boldsymbol{u}_h^\varepsilon\|_1 \leqslant C \left\{ \inf_{\boldsymbol{v}_h \in V_h} \{\|\boldsymbol{u} - \boldsymbol{v}_h\|_1 + |(\operatorname{div} \boldsymbol{v}_h, \tilde{p}_h^\varepsilon)|^{1/2}\} + \inf_{\hat{q}_h \in \hat{Q}_h} \{\|p - \hat{q}_h\|_0 + \varepsilon \|\hat{q}_h\|_0\} \right\}, \tag{3.3}$$

$$\|p - \hat{p}_h^\varepsilon\|_0 \leqslant C\{\|\boldsymbol{u} - \boldsymbol{u}_h^\varepsilon\|_1 + \inf_{\hat{q}_h \in \hat{Q}_h} \|p - \hat{q}_h\|_0\}. \tag{3.4}$$

The key assumptions of the above result are hypotheses (3.1) and (3.2). To choose the subspaces $\hat{V}_h$ and $\hat{Q}_h$ properly, we use the macroelement technique.

Let $\Im_H$ be a nested coarse mesh of $\Im_h$. We call the element of $\Im_H$ the macroelement which is the union of a fixed number of adjacent elements of $\Im_h$. Consider a general macroelement

$$M = \bigcup_{j=1}^{m} K_j,$$

where $K_j \in \Im_h$. We define the spaces

$$V_{0,M} = \{\boldsymbol{v}_h : \boldsymbol{v}_h \in V_h, \ \boldsymbol{v}_h = 0 \text{ in } \Omega \backslash M\}, \tag{3.5}$$

$$Q_M = Q_h|_M, \tag{3.6}$$

$$N_M = \left\{ q_h \in Q_M : \int_M q_h \operatorname{div} \boldsymbol{v}_h \, \mathrm{d}x \, \mathrm{d}y = 0, \ \forall \boldsymbol{v}_h \in V_{0,M} \right\}. \tag{3.7}$$

For the discontinuous pressure elements, we quote the following results obtained in [17, 6].

Theorem 3.2. *Let us suppose on Ω a partition into macroelements such that*

$$N_M \text{ is one-dimensional.} \tag{3.8}$$

Suppose, moreover, that there exists an operator $\Pi_h : [H_0^1(\Omega)]^2 \to V_h$ such that one has

$$\int_M \operatorname{div} (\Pi_h \boldsymbol{u} - \boldsymbol{u}) \, \mathrm{d}x \, \mathrm{d}y = 0, \ \forall M \tag{3.9}$$

$$\|\Pi_h \boldsymbol{u}\|_1 \leqslant C \, \|\boldsymbol{u}\|_1. \tag{3.10}$$

Then for some constant $\beta_0 > 0$ independent of h, we have

$$\sup_{\boldsymbol{v}_h \in V_h} \frac{(\operatorname{div} \boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_1} \geqslant \beta_0 \|q_h\|_0, \quad \forall q_h \in Q_h.$$

Now we present our ideas for choosing the subspaces $\hat{V}_h$ and $\hat{Q}_h$. Let us write

$$N_M = \{\text{constants on } M\} \oplus N_M^*. \tag{3.11}$$

Then we define

$$\tilde{Q}_h = \bigcup_M N_M^*, \quad Q_h = \hat{Q}_h \oplus \tilde{Q}_h, \tag{3.12}$$

where $\hat{Q}_h$ is the orthogonal complement of $\tilde{Q}_h$ in Q_h. And for some $V_h^* \subset V_h$, we define

$$\hat{V}_h = V_h^* + \bigcup_M V_{0,M}. \tag{3.13}$$

Theorem 3.3. *If the spaces* V_h^* *and* $\tilde{Q}_h$ *satisfy the conditions* (3.2), (3.9), *and* (3.10), *then the spaces* $\hat{V}_h$, $\hat{Q}_h$ *and* $\tilde{Q}_h$ *satisfy* (3.1) *and* (3.2).

Proof. Similar to (3.5)–(3.7), we define

$$\hat{V}_{0,M} = \{\hat{\boldsymbol{v}}_h : \hat{\boldsymbol{v}}_h \in \hat{V}_h,\ \hat{\boldsymbol{v}}_h = 0 \text{ in } \Omega \backslash M\},$$
$$\hat{Q}_M = \hat{Q}_h|_M,$$
$$\hat{N}_M = \left\{\hat{q}_h \in \hat{Q}_M : \int_M \hat{q}_h \operatorname{div} \hat{\boldsymbol{v}}_h \, dx\, dy = 0,\ \forall\, \hat{\boldsymbol{v}}_h \in \hat{V}_{0,M}\right\}.$$

We have $\hat{V}_{0,M} = V_{0,M}$, and $\hat{N}_M = \{\text{constants on } M\}$. By Theorem 3.2, we know that (3.1) is valid if we can choose V_h^* to satisfy the relations (3.9) and (3.10). By definition (3.12), we know that (3.2) holds for $(\hat{V}_h, \tilde{Q}_h)$ if (3.2) holds for $(V_h^*, \tilde{Q}_h)$. □

4. Bilinear-constant velocity–pressure element

For simplicity we assume Ω is a rectangle $\{(x, y): 0<x<x_0,\ 0<y<y_0\}$, x_0 and y_0 being given positive numbers. We consider partitions of rectangle Ω into rectangular elements with uniform partitions in the x- and y-direction. Denote

$$\Im_h = \{K_{ij}: 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2\},$$
$$K_{ij} = \{(x, y) \in \mathbb{R}^2 : (i-1)h_1 \leqslant x \leqslant ih_1,\ (j-1)h_2 \leqslant y \leqslant jh_2\},$$

where $m_1 = x_0/h_1$ and $m_2 = y_0/h_2$ are even integers. We shall assume that h_1 and h_2 depend on the mesh parameter h in such a way that h_1/h_2 is bounded by positive constants from below and above independent of h.

We discuss the bilinear-constant $(Q_1 - Q_0)$ velocity–pressure element. This is probably the most popular of all elements for incompressible flow problems. The finite spaces V_h and Q_h are defined as follows:

$$V_h = \{\boldsymbol{v}_h \in [H_0^1(\Omega)]^2 : \boldsymbol{v}_h|_K \in [Q_1(K)]^2,\ \forall K \in \Im_h\}, \tag{4.1}$$
$$Q_h = \{q_h \in L_0^2(\Omega) : q_h|_K \in Q_0(K),\ \forall K \in \Im_h\}. \tag{4.2}$$

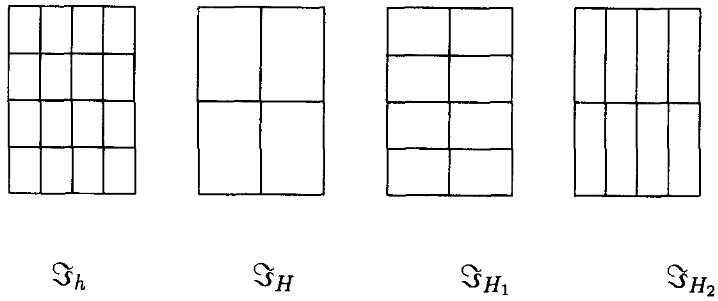

Fig. 1.

-1	+1
+1	-1

Fig. 2. ϕ_{ij}^M

Here and below, for a nonnegative integer l, $Q_l(K)$ denotes the space of polynomials of degree less than or equal to l in each variable on K.

Let $\Im_H$ $(H = 2h)$ be a nested coarse mesh of $\Im_h$. Each element of $\Im_H$ consists of four elements of $\Im_h$.

$$\Im_H = \{M_{ij}: 1 \leqslant i \leqslant m_1/2,\ 1 \leqslant j \leqslant m_2/2\},$$
$$M_{ij} = \{(x,y) \in \mathbb{R}^2: 2(i-1)h_1 \leqslant x \leqslant 2ih_1,\ 2(j-1)h_2 \leqslant y \leqslant 2jh_2\}.$$

And we introduce $\Im_{H_1}$ and $\Im_{H_2}$, two auxiliary nested coarse meshes of $\Im_h$, as follows (cf. [11]).

$$\Im_{H_1} = \{T_{ij}: 1 \leqslant i \leqslant m_1/2,\ 1 \leqslant j \leqslant m_2\},$$
$$T_{ij} = \{(x,y) \in \mathbb{R}^2: 2(i-1)h_1 \leqslant x \leqslant 2ih_1,\ (j-1)h_2 \leqslant y \leqslant jh_2\},$$

and

$$\Im_{H_2} = \left\{S_{ij} : 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant \frac{m_2}{2}\right\},$$
$$S_{ij} = \{(x,y) \in \mathbb{R}^2 : (i-1)h_1 \leqslant x \leqslant ih_1,\ 2(j-1)h_2 \leqslant y \leqslant 2jh_2\}.$$

See Fig. 1 for the case $m_1 = m_2 = 4$.

By (3.12), we have

$$\tilde{Q}_h = \bigcup_{M_{ij} \in \Im_H} \operatorname{span}\{\phi_{ij}^M\}, \tag{4.3}$$

where ϕ_{ij}^M are defined on the element M_{ij} as in Fig. 2.

Let $\hat{Q}_h$ be the orthogonal complement of $\tilde{Q}_h$ in Q_h. Then

$$Q_h = \hat{Q}_h \oplus \tilde{Q}_h. \tag{4.4}$$

Define

$$V_{0,M} = \{\boldsymbol{v}_h : \boldsymbol{v}_h|_M \in V_h,\ \boldsymbol{v}_h = 0 \text{ in } \Omega \backslash M\}, \quad \forall M \in \Im_H, \tag{4.5}$$

$$\hat{V}_h = V_h^* + \bigcup_{M \in \Im_H} V_{0,M}, \tag{4.6}$$

where $V_h^* = (X_{1h}, X_{2h})$, and X_{1h} and X_{2h} are the bilinear finite element spaces on the meshes $\Im_{H_1}$ and $\Im_{H_2}$:

$$X_{lh} = \{\boldsymbol{v}_h \in [H_0^1(\Omega)]^2 : \boldsymbol{v}_h|_K \in [Q_1(K)]^2,\ \forall K \in \Im_{H_l}\}, \quad l = 1,2. \tag{4.7}$$

Lemma 4.1. *The spaces $\hat{V}_h$, $\hat{Q}_h$ and $\tilde{Q}_h$ defined above satisfy the conditions* (3.1) *and* (3.2).

Proof. We apply Theorem 3.3. By Lemma 2.1 in [11], there exists an interpolation operator $\Pi_h : [H_0^1(\Omega)]^2 \longrightarrow V_h^*$ such that (3.9) and (3.10) hold.

For every element $T_{ij} = K_{2i-1,j} + K_{2i,j} \in \Im_{H_1}$, with $K_{2i-1,j}, K_{2i,j} \in \Im_h$, we have

$$\int_{T_{ij}} \frac{\partial v_1}{\partial x} \phi_{ij}^M \,\mathrm{d}x\,\mathrm{d}y = \int_{K_{2i-1,j}} \frac{\partial v_1}{\partial x} \,\mathrm{d}x\,\mathrm{d}y - \int_{K_{2i,j}} \frac{\partial v_1}{\partial x} \,\mathrm{d}x\,\mathrm{d}y = 0, \quad \forall v_1 \in X_{1h},$$

since $\partial v_1/\partial x$ is independent of x. Similarly, for every element $S_{ij} = K_{i,2j-1} + K_{i,2j} \in \Im_{H_2}$, with $K_{i,2j-1}, K_{i,2j} \in \Im_h$, we have

$$\int_{S_{ij}} \frac{\partial v_2}{\partial y} \phi_{ij}^M \,\mathrm{d}x\,\mathrm{d}y = \int_{K_{i,2j-1}} \frac{\partial v_2}{\partial y} \,\mathrm{d}x\,\mathrm{d}y - \int_{K_{i,2j}} \frac{\partial v_2}{\partial y} \,\mathrm{d}x\,\mathrm{d}y = 0, \quad \forall v_2 \in X_{2h}.$$

Thus,

$$(\tilde{q}_h, \operatorname{div} \boldsymbol{v}_h) = 0, \quad \forall \tilde{q}_h \in \tilde{Q}_h,\ \forall \boldsymbol{v}_h \in V_h^*. \tag{4.8}$$

The result now follows from an application of Theorem 3.3. □

The main result of the section is the following theorem on an optimal order error estimate.

Theorem 4.2. *Let $(\boldsymbol{u}, p) \in [H^2(\Omega) \cap H_0^1(\Omega)]^2 \times H^1(\Omega)$ be the solution of the problem* (1.1), *$(\boldsymbol{u}_h^\varepsilon, p_h^\varepsilon) \in V_h \times Q_h$ be the solution of the discrete problem* (1.8) *using the finite element spaces V_h and Q_h defined by* (4.1) *and* (4.2). *Denote $\hat{p}_h^\varepsilon$ the projection of p_h^ε on $\hat{Q}_h$. If $0 < \varepsilon \leqslant \alpha h$, then*

$$\|\boldsymbol{u} - \boldsymbol{u}_h^\varepsilon\|_1 + \|p - \hat{p}_h^\varepsilon\|_0 \leqslant C h (|\boldsymbol{u}|_2 + \|p\|_1), \tag{4.9}$$

where C is a constant independent of h.

Proof. By the standard finite element interpolation theory (cf. [9]), we have

$$\inf_{\boldsymbol{v}_h \in V_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_1 \leqslant \inf_{\boldsymbol{v}_h \in \hat{V}_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_1 \leqslant Ch|\boldsymbol{u}|_2,$$

$$\inf_{\hat{q}_h \in \hat{Q}_h} \|p - \hat{q}_h\|_0 + \varepsilon\|\hat{q}_h\|_0 \leqslant Ch\|p\|_1.$$

It is easy to see that $\tilde{p}_h^\varepsilon = p_h^\varepsilon - \hat{p}_h^\varepsilon$ is the projection of p_h^ε on $\tilde{Q}_h$, and satisfies

$$(\tilde{p}_h^\varepsilon, \operatorname{div} \boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in \hat{V}_h.$$

From Lemma 4.1 and Theorem 3.1, we get the optimal order error estimate (4.9). □

We remark that in [13], a similar error estimate is proved, but under the assumption $\boldsymbol{u} \in W^{3,s}(\Omega)$, $1<s<\infty$, and $0<\varepsilon \leqslant \alpha_1 h^2$. The next result shows that the difference between p_h^ε and its projection $\hat{p}_h^\varepsilon$ is small.

Proposition 4.3. *We keep the assumptions of Theorem* 4.2. *Let* $\tilde{p}_h^\varepsilon$ *be the projection of* p_h^ε *on* $\tilde{Q}_h$, $\varepsilon = \alpha h$. *Then*

$$\|\tilde{p}_h^\varepsilon\|_0 \leqslant C\sqrt{h}(|\boldsymbol{u}|_2 + \|p\|_1).$$

Proof. For any $\hat{q}_h \in \hat{Q}_h$, we have

$$\|\tilde{p}_h^\varepsilon\|_0 = \|p_h^\varepsilon - \hat{p}_h^\varepsilon\|_0 \leqslant \|p_h^\varepsilon - \hat{q}_h\|_0 + \|\hat{p}_h^\varepsilon - p\|_0 + \|p - \hat{q}_h\|_0.$$

By (2.18) and the relation $(\tilde{p}_h^\varepsilon, \operatorname{div} \boldsymbol{v}_h) = 0$, $\forall \boldsymbol{v}_h \in \hat{V}_h$, we see that (3.3) can be replaced by the following inequality:

$$\|\boldsymbol{u} - \boldsymbol{u}_h^\varepsilon\|_1 + \sqrt{\varepsilon}\|\hat{q}_h - p_h^\varepsilon\|_0 \leqslant C\{\|\boldsymbol{u} - \boldsymbol{v}_h\|_1 + \|p - \hat{q}_h\|_0 + \varepsilon\|\hat{q}_h\|_0\}, \quad \forall \boldsymbol{v}_h \in V_h,\ \forall \hat{q}_h \in \hat{Q}_h.$$

Thus with the assumed regularity of the solution,

$$\inf_{\hat{q}_h \in \hat{Q}_h} \{\sqrt{\varepsilon}\|p_h^\varepsilon - \hat{q}_h\|_0 + \|p - \hat{q}_h\|_0\} \leqslant C h(|\boldsymbol{u}|_2 + \|p\|_1).$$

Noticing the assumption $\varepsilon = \alpha h$, we then have the stated estimate. □

Like in the case with the conventional finite element method, it is possible to prove an optimal order error estimate for the velocity approximation in $L^2(\Omega)$-norm via a duality argument.

Theorem 4.4. *Under the conditions of Theorem* 4.2, *if* $0<\varepsilon \leqslant \alpha h^2$, *then*

$$\|\boldsymbol{u} - \boldsymbol{u}_h^\varepsilon\|_0 \leqslant Ch^2(|\boldsymbol{u}|_2 + \|p\|_1),$$

where C *is independent of* h.

Proof. We introduce a dual problem: for any $\boldsymbol{g} \in [L^2(\Omega)]^2$, find $(\boldsymbol{\psi}, \lambda) \in [H_0^1(\Omega)]^2 \times L_0^2(\Omega)$ such that

$$\begin{aligned} &(\nabla\boldsymbol{\psi}, \nabla\boldsymbol{\phi}) - (\lambda, \operatorname{div}\boldsymbol{\phi}) = (\boldsymbol{g}, \boldsymbol{\phi}), && \forall \boldsymbol{\phi} \in [H_0^1(\Omega)]^2, \\ &(\operatorname{div}\boldsymbol{\psi}, \mu) = 0, && \forall \mu \in L_0^2(\Omega). \end{aligned} \tag{4.10}$$

By [18], we have the regularity of solution $(\boldsymbol{\psi}, \lambda)$,

$$\|\boldsymbol{\psi}\|_2 + \|\lambda\|_1 \leqslant C\|\boldsymbol{g}\|_0. \tag{4.11}$$

Then for any $(\boldsymbol{\psi}_h, \hat{\lambda}_h) \in \hat{V}_h \times \hat{Q}_h$,

$$\begin{aligned}
&(\boldsymbol{g}, \boldsymbol{u} - \boldsymbol{u}_h^\varepsilon) \\
&\quad = (\nabla\boldsymbol{\psi}, \nabla(\boldsymbol{u} - \boldsymbol{u}_h^\varepsilon)) - (\lambda, \operatorname{div}(\boldsymbol{u} - \boldsymbol{u}_h^\varepsilon)) \\
&\quad = (\nabla(\boldsymbol{\psi} - \boldsymbol{\psi}_h), \nabla(\boldsymbol{u} - \boldsymbol{u}_h^\varepsilon)) + (p - p_h^\varepsilon, \operatorname{div}\boldsymbol{\psi}_h) \\
&\qquad +(\lambda - \hat{\lambda}_h, \operatorname{div}\boldsymbol{u}_h^\varepsilon) - \varepsilon(\hat{\lambda}_h, p_h^\varepsilon) \\
&\quad \leqslant C h(|\boldsymbol{u}|_2 + \|p\|_1)(\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_1 + \|\lambda - \hat{\lambda}_h\|_0) + \varepsilon\|\hat{\lambda}_h\|_0\|\hat{p}_h^\varepsilon\|_0.
\end{aligned}$$

From (4.11) and $0<\varepsilon \leqslant \alpha h^2$,

$$\|\boldsymbol{u} - \boldsymbol{u}_h^\varepsilon\|_0 = \sup_{\boldsymbol{g}\in[L^2(\Omega)]^2} \frac{(\boldsymbol{g}, \boldsymbol{u} - \boldsymbol{u}_h^\varepsilon)}{\|\boldsymbol{g}\|_0} \leqslant Ch^2(|\boldsymbol{u}|_2 + \|p\|_1).$$

The proof is completed. □

5. Biquadratic-bilinear velocity–pressure element

Let $\Im_h$ be defined as in the last section. We define the biquadratic–bilinear (Q_2–Q_1) velocity–pressure element as follows.

$$V_h = \{\boldsymbol{v}_h \in [H_0^1(\Omega)]^2 \colon \boldsymbol{v}_h|_K \in [Q_2(K)]^2,\ \forall K \in \Im_h\}, \tag{5.1}$$

$$Q_h = \{q_h \in L_0^2(\Omega) \colon q_h|_K \in Q_1(K),\ \forall K \in \Im_h\}. \tag{5.2}$$

Define

$$\phi_K(x, y) = (x - \bar{x}_K)(y - \bar{y}_K)/h_1h_2, \quad \forall K \in \Im_h, \tag{5.3}$$

where $(\bar{x}_K, \bar{y}_K)$ is the centre of element K. Let $\Im_H$ $(H = 2h)$ be defined as before. First let us determine the set $\tilde{Q}_h$ from the definition (3.12).

Lemma 5.1. *Associated with a macroelement $M = \bigcup_{i=1}^4 K_i$ with $K_i \in \Im_h$, we define a function $\phi_M(x, y)$ by $\phi_M(x, y)|_{K_i} = \phi_{K_i}(x, y)$, $1 \leqslant i \leqslant 4$. Then*

$$\tilde{Q}_h = \{q_h \in L_0^2(\Omega) \colon q_h|_M \in \operatorname{span}\{\phi_M\},\ \forall M \in \Im_H\}. \tag{5.4}$$

Proof. It suffices to prove the result for a reference macroelement $\hat{M} = [-1, 1] \times [-1, 1]$ which is the union of four elements $\hat{K}_1 = [0, 1] \times [0, 1]$, $\hat{K}_2 = [-1, 0] \times [0, 1]$, $\hat{K}_3 = [-1, 0] \times [-1, 0]$, and $K_4 = [0, 1] \times [-1, 0]$. Correspondingly, we use $V_{0,\hat{M}}$ to denote the space of piecewise biquadratic functions which vanish on the boundary of $\hat{M}$, $Q_{\hat{M}}$ the space of piecewise bilinear functions on $\hat{M}$, and

$$N_{\hat{M}} = \{q \in Q_{\hat{M}} \colon (q, \operatorname{div}\boldsymbol{v})_{\hat{M}} = 0,\ \forall \boldsymbol{v} \in V_{0,\hat{M}}\}.$$

Then we only need to show that any function in $N_{\hat{M}}$ can be written as the summation of a constant and a constant multiple of $\phi_{\hat{M}}$.

The general form for a function in $Q_{\hat{M}}$ is

$$q(\xi,\eta)=\begin{cases} a_{11}+a_{12}(\xi-1/2)+a_{13}(\eta-1/2)+a_{14}(\xi-1/2)(\eta-1/2) & \text{in } \hat{K}_1,\\ a_{21}+a_{22}(\xi+1/2)+a_{23}(\eta-1/2)+a_{24}(\xi+1/2)(\eta-1/2) & \text{in } \hat{K}_2,\\ a_{31}+a_{32}(\xi+1/2)+a_{33}(\eta+1/2)+a_{34}(\xi+1/2)(\eta+1/2) & \text{in } \hat{K}_3,\\ a_{41}+a_{42}(\xi-1/2)+a_{43}(\eta+1/2)+a_{44}(\xi-1/2)(\eta+1/2) & \text{in } \hat{K}_4. \end{cases}$$

Assume $q \in N_{\hat{M}}$. We will take functions of the forms $\boldsymbol{v}=(w,0)$ and $(0,w)$ from the set $V_{0,\hat{M}}$. First let us choose w to be

$$w=\begin{cases} 16\,\xi(1-\xi)\eta(1-\eta) \text{ in } \hat{K}_1,\\ 0 \qquad\qquad \text{otherwise,} \end{cases}$$

which can be viewed as an internal shape function associated with the point $(1/2,1/2)$. Then from the definition of the set $N_{\hat{M}}$, we have

$$\int_{\hat{K}_1} q\,\frac{\partial w}{\partial \xi}\,\mathrm{d}\xi\,\mathrm{d}\eta=\int_{\hat{K}_1} q\,\frac{\partial w}{\partial \eta}\,\mathrm{d}\xi\,\mathrm{d}\eta=0,$$

which imply

$$a_{12}=a_{13}=0.$$

Similarly, by choosing w to be internal shape functions associated with the points $(-1/2,1/2)$, $(-1/2,-1/2)$ and $(1/2,-1/2)$, it can be shown that

$$a_{22}=a_{23}=a_{32}=a_{33}=a_{42}=a_{43}=0.$$

Now let us choose w to be a side shape function associated with the point $(0,1/2)$:

$$w=\begin{cases} 8\,(1-\xi)(1/2-\xi)\eta(1-\eta) \text{ in } \hat{K}_1,\\ 8\,(1+\xi)(1/2+\xi)\eta(1-\eta) \text{ in } \hat{K}_2,\\ 0 \qquad\qquad \text{otherwise.} \end{cases}$$

Then from

$$\int_{\hat{K}_1\cup\hat{K}_2} q\,\frac{\partial w}{\partial \xi}\,\mathrm{d}\xi\,\mathrm{d}\eta=0,$$

we find that a_{11} and a_{21} are equal; let us use a for the common value. Similarly,

$$\int_{\hat{K}_1\cup\hat{K}_2} q\,\frac{\partial w}{\partial \eta}\,\mathrm{d}\xi\,\mathrm{d}\eta=0,$$

implies $a_{14}=a_{24}\equiv b$. By choosing w to be side shape functions associated with the points $(1/2,0)$, $(-1/2,0)$ and $(0,-1/2)$ in turn, we can show that

$$a_{i1}=a,\ a_{i4}=b,\quad 1\leqslant i\leqslant 4.$$

Thus, for $q\in N_{\hat{M}}$, $q=a+b\,\phi_{\hat{M}}$ for some constants a and b. $\square$

Now let $\hat{Q}_h$ be the orthogonal complement of $\tilde{Q}_h$ in Q_h,

$$Q_h = \hat{Q}_h \oplus \tilde{Q}_h \tag{5.5}$$

and

$$V_{0,M} = \{\boldsymbol{v}_h : \boldsymbol{v}_h|_M \in V_h,\ \boldsymbol{v}_h = 0 \text{ in } \Omega \backslash M\}, \quad \forall M \in \Im_H, \tag{5.6}$$

$$\hat{V}_h = V_h^* + \bigcup_{M \in \Im_H} V_{0,M}, \tag{5.7}$$

where $V_h^* = (X_{1h}, X_{2h})$ is defined as in (4.7).

Lemma 5.2. *The spaces $\hat{V}_h$, $\hat{Q}_h$ and $\tilde{Q}_h$ defined in* (5.4), (5.5) *and* (5.7) *satisfy the conditions* (3.1) *and* (3.2).

Proof. Like in the proof of Lemma 4.1, there exists an interpolation operator $\Pi_h : [H_0^1(\Omega)]^2 \longrightarrow V_h^*$ such that (3.9) and (3.10) hold.

For every $K \in \Im_h$, from definition of (5.3) and $\boldsymbol{v}_h = (v_1, v_2) \in V_h^*$,

$$\int_K \phi_K(x,y) \frac{\partial v_1}{\partial x} \,\mathrm{d}x\,\mathrm{d}y = \int (x - x_K)\,\mathrm{d}x \int (y - y_K) \frac{\partial v_1}{\partial x} \,\mathrm{d}y = 0,$$

since $\partial v_1/\partial x$ is a function independent of x. Similarly,

$$\int_K \phi_K(x,y) \frac{\partial v_2}{\partial y} \,\mathrm{d}x\,\mathrm{d}y = 0.$$

So

$$\int_K \phi_K(x,y) \operatorname{div} \boldsymbol{v}_h \,\mathrm{d}x\,\mathrm{d}y = 0,$$

and thus,

$$\int_M \phi_M(x,y) \operatorname{div} \boldsymbol{v}_h \,\mathrm{d}x\,\mathrm{d}y = 0.$$

That is

$$(\tilde{q}_h, \operatorname{div} \boldsymbol{v}_h) = 0, \quad \forall \tilde{q}_h \in \tilde{Q}_h,\ \forall \boldsymbol{v}_h \in V_h^*.$$

The proof is completed by an application of Theorem 3.3. □

We choose $\hat{K} = (-1,1) \times (-1,1)$ to be the reference element in o–ξ–η coordinate system. Define a mapping $\hat{\pi} : H^3(\hat{K}) \longrightarrow \hat{Q} \subset Q_2(\hat{K})$, $\forall \hat{w} \in H^3(\hat{K})$,

$$\hat{\pi}\hat{w}(\xi,\eta) = \sum_{i=1}^{4} \hat{w}_i \hat{\phi}_i(\xi,\eta) + \sum_{i=1}^{4} g_i(\hat{w}) \hat{\psi}_i(\xi,\eta), \tag{5.8}$$

where

$$\begin{aligned}
&\hat{w}_1 = \hat{w}(1,1), && \hat{\phi}_1(\xi,\eta) = \tfrac{1}{4}(1+\xi)(1+\eta),\\
&\hat{w}_2 = \hat{w}(-1,1), && \hat{\phi}_2(\xi,\eta) = \tfrac{1}{4}(1-\xi)(1+\eta),\\
&\hat{w}_3 = \hat{w}(-1,-1), && \hat{\phi}_3(\xi,\eta) = \tfrac{1}{4}(1-\xi)(1-\eta),\\
&\hat{w}_4 = \hat{w}(1,-1), && \hat{\phi}_4(\xi,\eta) = \tfrac{1}{4}(1+\xi)(1-\eta),
\end{aligned} \tag{5.9}$$

and

$$\begin{aligned}
&g_1(\hat{w}) = \int_{-1}^{1} \frac{\partial^2 \hat{w}(\xi,1)}{\partial \xi^2}\,\mathrm{d}\xi, && \hat{\psi}_1(\xi,\eta) = \tfrac{1}{8}(\xi^2-1)(1+\eta),\\
&g_2(\hat{w}) = \int_{-1}^{1} \frac{\partial^2 \hat{w}(-1,\eta)}{\partial \eta^2}\,\mathrm{d}\eta, && \hat{\psi}_2(\xi,\eta) = \tfrac{1}{8}(1-\xi)(\eta^2-1),\\
&g_3(\hat{w}) = \int_{-1}^{1} \frac{\partial^2 \hat{w}(\xi,-1)}{\partial \xi^2}\,\mathrm{d}\xi, && \hat{\psi}_3(\xi,\eta) = \tfrac{1}{8}(\xi^2-1)(1-\eta),\\
&g_4(\hat{w}) = \int_{-1}^{1} \frac{\partial^2 \hat{w}(1,\eta)}{\partial \xi^2}\,\mathrm{d}\eta, && \hat{\psi}_4(\xi,\eta) = \tfrac{1}{8}(1+\xi)(\eta^2-1).
\end{aligned} \tag{5.10}$$

Lemma 5.3. *Assume* $\hat{w} \in H^3(\hat{K})$. *Then*

$$|\hat{w} - \hat{\pi}\hat{w}|_{1,\hat{K}} \leqslant \hat{c}|\hat{w}|_{3,\hat{K}}. \tag{5.11}$$

Proof. It is easy to see that for any $\hat{w} \in P_2(\hat{K})$, $\hat{\pi}\hat{w} = \hat{w}$. The inequality (5.11) can be proved following a standard technique in finite element analysis (cf. [9]). □

Lemma 5.4. *Let* $\hat{u} = (\hat{u}_1, \hat{u}_2) \in [H^3(\hat{K})]^2$. *If* $\partial\hat{u}_1/\partial\xi + \partial\hat{u}_2/\partial\eta = 0$, *then*

$$\int_{\hat{K}} \xi\eta \left[\frac{\partial \hat{\pi}\hat{u}_1}{\partial \xi} + \frac{\partial \hat{\pi}\hat{u}_2}{\partial \eta}\right] \mathrm{d}\xi\,\mathrm{d}\eta = 0. \tag{5.12}$$

Proof. From a simple computation, we have

$$\begin{aligned}
&\int_{\hat{K}} \xi\eta \frac{\partial \hat{\pi}\hat{u}_1}{\partial \xi}\,\mathrm{d}\xi\,\mathrm{d}\eta\\
&\quad = \int_{\hat{K}} \xi\eta \left[\frac{\xi}{4}(1+\eta)g_1(\hat{u}_1) + \frac{\xi}{4}(1-\eta)g_3(\hat{u}_1)\right] \mathrm{d}\xi\,\mathrm{d}\eta\\
&\quad = \tfrac{1}{9}[g_1(\hat{u}_1) - g_3(\hat{u}_1)]\\
&\quad = \frac{1}{9}\int_{\hat{K}} \frac{\partial^3 \hat{u}_1(\xi,\eta)}{\partial^2\xi\partial\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta,
\end{aligned}$$

and

$$\int_{\hat{K}} \xi\eta \frac{\partial \hat{\pi}\hat{u}_2}{\partial \eta}\,\mathrm{d}\xi\,\mathrm{d}\eta = \frac{1}{9}\int_{\hat{K}} \frac{\partial^3 \hat{u}_2(\xi,\eta)}{\partial\xi\partial\eta^2}\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

Then

$$\int_{\hat{K}} \xi\eta \left[\frac{\partial \hat{\pi}\hat{u}_1}{\partial \xi} + \frac{\partial \hat{\pi}\hat{u}_2}{\partial \eta}\right] \mathrm{d}\xi\,\mathrm{d}\eta = \frac{1}{9}\int_{\hat{K}} \frac{\partial^2}{\partial\xi\partial\eta}\left[\frac{\partial \hat{u}_1}{\partial \xi} + \frac{\partial \hat{u}_2}{\partial \eta}\right] \mathrm{d}\xi\,\mathrm{d}\eta = 0,$$

and the proof is completed. □

For an element $K \in \Im_h$, let Ψ_K be the linear function mapping $\hat{K}$ to K. We define an interpolation operator on the element K by $\pi_K = \Psi_K \circ \hat{\pi} \circ \Psi_K^{-1}$. We then define a global mapping π_h through the relation: $\pi_h|_K = \pi_K$.

Lemma 5.5. *If* $\boldsymbol{u} \in [H^3(\Omega)]^3$ *and* $\operatorname{div} \boldsymbol{u} = 0$, *then*

$$|\boldsymbol{u} - \pi_h \boldsymbol{u}|_1 \leqslant C h^2 |\boldsymbol{u}|_3, \tag{5.13}$$

$$(\tilde{q}_h, \operatorname{div} \pi_h \boldsymbol{u}) = 0, \quad \forall \tilde{q}_h \in \tilde{Q}_h, \tag{5.14}$$

where $\tilde{Q}_h$ *is defined as in* (5.4).

Proof. First we can verify that $\forall \boldsymbol{u} \in [H^3(\Omega)]^2$, $\pi_h \boldsymbol{u} \in [C^0(\Omega)]^2$, and thus, $\pi_h \boldsymbol{u} \in V_h$. Then by the standard interpolation theory [9] and Lemma 5.3, we can prove (5.13). Obviously Lemma 5.4 implies (5.14). □

After the above preparations, we are ready to prove optimal order error estimates for the biquadratic–bilinear velocity–pressure element for solving the Stokes problem.

Theorem 5.6. *Let* $(\boldsymbol{u}, p)$ *be the solution of the problem* (1.1), $(\boldsymbol{u}_h^\varepsilon, p_h^\varepsilon)$ *be the solution of the problem* (1.8) *with* V_h *and* Q_h *defined as in* (5.1) *and* (5.2), $\hat{p}_h^\varepsilon$ *be the projection of* p_h^ε *on* $\hat{Q}_h$. *Assume* $0 < \varepsilon \leqslant \alpha h^2$. *Then*

$$\|\boldsymbol{u} - \boldsymbol{u}_h^\varepsilon\|_1 + \|p - \hat{p}_h^\varepsilon\|_0 \leqslant C h^2 (|\boldsymbol{u}|_3 + |p|_2 + \|p\|_0). \tag{5.15}$$

Furthermore, if $0 < \varepsilon \leqslant \alpha h^3$, *then*

$$\|\boldsymbol{u} - \boldsymbol{u}_h^\varepsilon\|_0 \leqslant C h^3 (|\boldsymbol{u}|_3 + |p|_2 + \|p\|_0). \tag{5.16}$$

Proof. From Lemma 5.2 and Theorem 3.1, we can choose $\boldsymbol{v}_h = \pi_h \boldsymbol{u} \in V_h$. Then we apply Lemma 5.5 and obtain (5.15) easily. Using a trick similar to that in proving Theorem 4.4, we can also obtain the L^2-norm error estimate (5.16). □

In [15, 7], results of many numerical experiments are reported. The numerical results are consistent with the error estimates proved in this paper.

As another remark, we observe that for $\varepsilon = 0$, the scheme (1.8) is the standard mixed finite element method for solving equation (1.1). By a careful examination, we find that Theorem 3.1 is valid also for $\varepsilon = 0$. Hence, all the error estimates in Theorems 4.2 and 5.6 are valid for the case $\varepsilon = 0$; then we get some related results proved in [6, 3]

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