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Permanence and periodicity of a delayed ratio-dependent predator–prey model with Holling type functional response and stage structure[☆]

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ABSTRACT

A periodic and delayed ratio-dependent predator–prey system with Holling type III functional response and stage structure for both prey and predator is investigated. It is assumed that immature predator and mature individuals of each species are divided by a fixed age, and immature predator do not have the ability to attack prey. Sufficient conditions are derived for the permanence and existence of positive periodic solution of the model. Numerical simulations are presented to illustrate the feasibility of our main results.

1. Introduction

Traditional Lotka–Volterra type predator–prey with Holling III functional response has received great attention from both theoretical and mathematical biologists, and has been well studied. The standard Lotka–Volterra type model is built by assuming that the per capita rate of predation depends on the prey numbers only. Recently, the traditional prey-dependent predator–prey model has been challenged by several biologists (see, for example [1–5]). There is growing explicit biological and physiological evidence [1–5] that in some situations, especially when predator have to search for food (and therefore have to share or compete for food), a more suitable general predator–prey model should be based on the ratio-dependent theory. This roughly states that the per capita predator growth rate should be a function of the ratio of the prey to predator abundance. This is strongly supported by numerous field and laboratory experiments and observations [1,3,5].

Based on the Michaelis–Menten or Holling type II function, Arditi and Ginzburg [6] proposed a ratio-dependent predator–prey function of the form

$$P\left(\frac{x}{y}\right) = \frac{c(x/y)}{m + (x/y)} = \frac{cx}{my + x}$$

and the following ratio-dependent predator–prey model:

$$\begin{cases} \dot{x} = x(a - bx) - \dfrac{cxy}{my + x}, \\ \dot{y} = y\left(-d + \dfrac{fx}{my + x}\right). \end{cases} \tag{1.1}$$

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Here $x(t)$ and $y(t)$ represent the densities of the prey and the predator at time t, respectively. a/b is the carrying capacity, $d > 0$ is the death rate of the predator, and a, c, m and f/c are positive constants that stand for the intrinsic growth rate of the prey, capturing rate, half saturation constant and conversion rate of the predator, respectively.

The ratio-dependent predator–prey models with or without time delays have been studied by many researchers recently and very rich dynamics have been observed (see [7–17,37] and references cited therein). However, it is assumed in the classical ratio-dependent predator–prey model that each individual predator admits the same ability to attack prey. This assumption is obviously unrealistic for many animals. In the natural world, there are many species whose individuals have a life history that take them through two stage, immature and mature, where immature predator are raised by their parents, and the rate they attack at prey and the reproductive rate can be ignored.

Stage-structured models have received much attention in recent years (see [18–30,34–36,38]). In [18], a model of single species population growth incorporating stage structure as a reasonable generalization of the classical logistic model was derived and investigated. This model assumes an average age to maturity which appears as a constant time delay reflecting a delayed birth of immature and a reduced survival of immature to their maturity. The model takes the form

$$\begin{cases} \dot{x}_i(t) = \alpha x_m(t) - \gamma x_i(t) - \alpha e^{-\gamma\tau} x_m(t-\tau), \\ \dot{x}_m(t) = \alpha e^{-\gamma\tau} x_m(t-\tau) - \beta x_m^2(t), \quad t > \tau \end{cases} \tag{1.2}$$

where $x_i(t)$ represents the immature population density, $x_m(t)$ denotes the mature population density, $\alpha > 0$ represents the birth rate, $\gamma > 0$ is the immature death rate, $\beta > 0$ is the mature death and overcrowding rate, τ is the time to maturity. The term $\alpha e^{-\gamma\tau} x_m(t-\tau)$ represents the immature who were born at time $t-\tau$ and survive at time t (with the immature death rate γ), and therefore represents the transformation of immature to mature.

We note that any biological or environmental parameters are naturally subject to fluctuation in time. As Cushing [31] pointed out that it is necessary and important to consider models with periodic ecological parameters or perturbations which might be quite naturally exposed (for example, those due to seasonal effects of weather, food supply, mating habits, hunting or harvesting seasons, etc.). Thus, the assumption of the periodicity of the parameters is a way of incorporating the periodicity of the environment.

In the present paper, we consider the following delayed ratio-dependent predator–prey system with Holling type III functional response and stage structure.

$$\begin{cases} \dot{x}_1(t) = \alpha_1(t)x_2(t) - \gamma_1(t)x_1(t) - \alpha_1(t-\tau_1)e^{-\int_{t-\tau_1}^{t}\gamma_1(s)ds}x_2(t-\tau_1), \\ \dot{x}_2(t) = \alpha_1(t-\tau_1)e^{-\int_{t-\tau_1}^{t}\gamma_1(s)ds}x_2(t-\tau_1) - \beta_1(t)x_2^2(t) - \dfrac{a_1(t)x_2^2(t)y_2(t)}{m^2y_2^2(t)+x_2^2(t)}, \\ \dot{y}_1(t) = \alpha_2(t)\dfrac{x_2^2(t)y_2(t)}{m^2y_2^2(t)+x_2^2(t)} - \gamma_2(t)y_1(t) \\ \qquad - \alpha_2(t-\tau_2)e^{-\int_{t-\tau_2}^{t}\gamma_2(s)ds}\frac{x_2^2(t-\tau_2)y_2(t-\tau_2)}{m^2y_2^2(t-\tau_2)+x_2^2(t-\tau_2)}, \\ \dot{y}_2(t) = \alpha_2(t-\tau_2)e^{-\int_{t-\tau_2}^{t}\gamma_2(s)ds}\dfrac{x_2^2(t-\tau_2)y_2(t-\tau_2)}{m^2y_2^2(t-\tau_2)+x_2^2(t-\tau_2)} - \beta_2(t)y_2(t) \end{cases} \tag{1.3}$$

where $x_1(t)$ and $x_2(t)$ denote the densities of immature and mature individual preys at time t, respectively; $y_1(t)$ and $y_2(t)$ represent the densities of immature and mature individual predators at time t, respectively. $\alpha_1(t)$, $\alpha_2(t)$, $\beta_1(t)$, $\beta_2(t)$, $\gamma_1(t)$, $\gamma_2(t)$, and $a_1(t)$ are continuously positive periodic functions with period ω, and $x^2(t)/(m^2y_2^2(t)+x_2^2(t))$ here denotes the mature predator response function, which reflects the capture ability of the mature predator. The model is derived under the following assumptions.

(H_1) The prey population: the birth rate of the immature population is proportional to the existing mature population with a proportionality $\alpha_1(t) > 0$; the death rate of the immature population is proportional to the existing immature population with a proportionality $\gamma_1(t) > 0$; the death rate of the mature population is a logistic nature, i. e., it is proportional to square of the population with a proportionality $\beta_1(t)$. The term

$$\alpha_1(t-\tau_1)e^{-\int_{t-\tau_1}^{t}\gamma_1(s)ds}x_2(t-\tau_1)$$

represents the number of immature preys that were born at time $t-\tau_1$ which still survive at time t and are transferred from the immature stage to the mature stage at time t. We refer to the article of Liu et al. [23]. The mature predators feed on the mature prey only.

(H_2) The predator population: the death rate of the immature population is proportional to the existing immature population with a proportionality $\gamma_2(t) > 0$; $a_1(t)$ is the capturing rate of mature predator, m is the half capturing saturation constant, $\alpha_2(t)/a_1(t)$ is the rate of conversion of nutrients into the reproduction of the mature predator, $\beta_2(t)$ is the death rate of mate predators. The term

$$\alpha_2(t-\tau_2)e^{-\int_{t-\tau_2}^{t}\gamma_2(s)ds}\frac{x_2^2(t-\tau_2)y_2(t-\tau_2)}{m^2y_2^2(t-\tau_2)+x_2^2(t-\tau_2)}$$

represents the number of immature preys that were born at time $t - \tau_2$ which still survive at time t and are transferred from the immature stage to the mature stage at time t. It is assumed in (1.3) that immature individual predators do not feed on prey and do not have the ability to reproduce.

The initial conditions for system (1.3) take the form of

$$\begin{cases} x_i(\theta) = \phi_i(\theta) \geq 0, \\ y_i(\theta) = \psi_i(\theta) > 0, \\ \phi_i(0) > 0, \quad i = 1, 2, \theta \in [-\tau, 0]. \end{cases} \tag{1.4}$$

where $\tau = \max\{\tau_1, \tau_2\}$, $(\phi_1(\theta), \phi_2(\theta), \psi_1(\theta), \psi_2(\theta)) \in \mathbf{C}([-\tau, 0], \mathbf{R}^4_{+0})$, the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbf{R}^4_{+0}$, where we define

$$\mathbf{R}^4_{+0} = \{(x_1, x_2, x_3, x_4) : x_i \geq 0, i = 1, 2, 3, 4\},$$

and the interior of $\mathbf{R}^4_{+}$,

$$\mathbf{R}^4_{+} = \{(x_1, x_2, x_3, x_4) : x_i > 0, i = 1, 2, 3, 4\}.$$

For continuity of initial conditions, we require

$$\begin{cases} x_1(0) = \displaystyle\int_{-\tau_1}^{0} \alpha_1(s) e^{-\int_s^0 \gamma_1(u)du} \phi_2(s) ds, \\ y_1(0) = \displaystyle\int_{-\tau_2}^{0} \alpha_2(s) e^{-\int_s^0 \gamma_2(u)du} \frac{\phi_2^2(s)\psi_2(s)}{m^2\psi_2^2(s) + \phi_2^2(s)} ds. \end{cases} \tag{1.5}$$

We adopt the following notations throughout this paper:

$$\bar{f} = \frac{1}{\omega}\int_0^{\omega} f(t)dt, \qquad f^L = \min_{t\in[0,\omega]} |f(t)|, \qquad f^M = \max_{t\in[0,\omega]} |f(t)|,$$

where f is a continuous ω-periodic function.

The organization of this paper is as follows. In the next section, sufficient conditions are established for the positivity of solutions and the persistence of system (1.3) with initial conditions (1.4) and (1.5). In Section 3, by using Gaines and Mawhin's continuation theorem of coincidence degree theory, we show the existence of positive ω-periodic solutions of (1.3) with initial conditions (1.4)–(1.5). In Section 4, our main results are illustrated by numerical simulations. A brief discussion is given in Section 5.

2. Uniform persistence

In this section, we will perform analysis on the permanence and extinction of system (1.3) with initial conditions (1.4) and (1.5).

Definition. System (1.3) is said to be permanent if there exists a compact region $\mathbf{D} \subset \operatorname{Int} \mathbf{R}^4_{+}$ such that every solution $z(t)$ of (1.3) with initial conditions (1.4) and (1.5) eventually enters and remains in the region $\mathbf{D}$.

Lemma 2.1. *Solutions of system (1.3) with initial conditions (1.4) and (1.5) are positive for all $t \geq 0$.*

Proof. Let $(x_1(t), x_2(t), y_1(t), y_2(t))$ be a solution of system (1.3) with initial conditions (1.4) and (1.5). Set $\tau^* = \min\{\tau_1, \tau_2\}$. Let us first consider $y_2(t)$ for $t \in [0, \tau^*]$. It follows from the fourth equation of system (1.3) that

$$\begin{aligned} \dot{y}_2(t) &= \alpha_2(t-\tau_2) e^{-\int_{t-\tau_2}^{t} \gamma_2(s)ds} \frac{\phi_2^2(t-\tau_2)\psi_2(t-\tau_2)}{m^2\psi_2^2(t-\tau_2) + \phi_2^2(t-\tau_2)} - \beta_2(t)y_2(t) \\ &\geq -\beta_2(t)y_2(t) \end{aligned} \tag{2.1}$$

since $\phi_2(\theta) \geq 0$, $\psi_2(\theta) > 0$ for $\theta \in [-\tau^*, 0]$. Therefore, a standard comparison argument shows that

$$y_2(t) \geq y_2(0) e^{-\int_0^t \beta_2(s)ds},$$

i.e., $y_2(t) > 0$ for $t \in [0, \tau^*]$.

By the second equation of system (1.3), for $t \in [0, \tau^*]$, we derive

$$\begin{aligned} \dot{x}_2(t) &= \alpha_1(t-\tau_1) e^{-\int_{t-\tau_1}^{t} \gamma_1(s)ds} \phi_2(t-\tau_1) - \beta_1(t)x_2^2(t) - \frac{a_1(t)x_2^2(t)y_2(t)}{m^2y_2^2(t) + x_2^2(t)} \\ &\geq -\beta_1(t)x_2^2(t) - \frac{a_1(t)x_2^2(t)y_2(t)}{m^2y_2^2(t) + x_2^2(t)} \end{aligned} \tag{2.2}$$

since $\phi_2(\theta) \geq 0$ for $\theta \in [-\tau^*, 0]$. Therefore, a standard comparison argument show

$$x_2(t) \geq \frac{1}{\frac{1}{x_2(0)} + \int_0^t (\beta_1(s) + \frac{a_1(s)y_2(s)}{m^2y_2^2(s)+x_2^2(s)})\mathrm{d}s} > 0 \quad \text{for } t \in [0, \tau^*]. \tag{2.3}$$

By (1.5) and the first and the third equations of system (1.3), one can rewrite $x_1(t)$ and $y_1(t)$ as follows:

$$\begin{cases} x_1(t) = \displaystyle\int_{t-\tau_1}^{t} \alpha_1(s)\mathrm{e}^{-\int_s^t \gamma_1(u)\mathrm{d}u} x_2(s)\mathrm{d}s, \\ y_1(t) = \displaystyle\int_{t-\tau_2}^{t} \alpha_2(s)\mathrm{e}^{-\int_s^t \gamma_2(u)\mathrm{d}u} \frac{x_2^2(s)y_2(s)}{m^2y_2^2(s)+x_2^2(s)}\mathrm{d}s. \end{cases} \tag{2.4}$$

Hence the positivity of $x_2(t), y_2(t)$ on $[-\tau^*, \tau^*]$ implies that of $x_1(t)$ and $y_1(t)$ for $t \in [0, \tau^*]$.

In a similar way we treat the intervals $[\tau^*, 2\tau^*], \ldots, [n\tau^*, (n+1)\tau^*]$, $n \in N$. Thus, $x_1(t) > 0, x_2(t) > 0, y_1(t) > 0$, $y_2(t) > 0$ for all $t \geq 0$. This completes the proof. □

Lemma 2.2 ([26]). *Consider the following equation:*

$$\dot{x}(t) = ax(t-\tau) - bx(t) - cx^2(t),$$

where a, b, c and τ are positive constants, $x(t) > 0$ for $t \in [-\tau, 0]$. We have
(i) *if $a > b$, then $\lim_{t\to+\infty} x(t) = \frac{a-b}{c}$;*
(ii) *if $a < b$, then $\lim_{t\to+\infty} x(t) = 0$.*

Lemma 2.3 ([32]). *Consider the following equation:*

$$\dot{x}(t) = f\left(\int_{-\tau}^{-\delta} x(t+s)\mathrm{d}u(s)\right) - g(x(t)),$$

assume that (a_1) and (a_2) hold and $x_M > x^*$, then $x(t) = x^*$ is absolutely globally asymptotically stable. Where

(a_1) $f(0) = 0$; there is an $x_M > 0$, such that $f(\cdot)$ is strictly increasing in $[0, x_M]$ and strictly decreasing in $[x_M, +\infty)$; $\lim_{x\to+\infty} f(x) \geq 0$. There is a unique $x^* > 0$ such that $f(x) > g(x)$ for $x \in (0, x^*)$ and $f(x) < g(x)$ for $x > x^*$.

(a_2) $g(x)$ is strictly increasing, $g(0) = 0$, $\lim_{x\to+\infty} g(x) = +\infty$.

Lemma 2.4. *Positive solutions of system (1.3) with initial conditions (1.4) and (1.5) are ultimately bounded.*

Proof. Suppose $z(t) = (x_1(t), x_2(t), y_1(t), y_2(t))$ is any positive solution of system (1.3) with initial conditions (1.4) and (1.5). Let

$$\rho(t) = x_1(t) + x_2(t) + y_1(t) + y_2(t).$$

Calculating the derivative of $\rho(t)$ along positive solutions of (1.3), we obtain

$$\begin{aligned}\dot{\rho}(t) &= \alpha_1(t)x_2(t) - \gamma_1(t)x_1(t) - \beta_1(t)x_2^2(t) - \frac{a_1(t)x_2^2(t)y_2(t)}{m^2y_2^2(t)+x_2^2(t)} \\ &\quad + \alpha_2(t)\frac{x_2^2(t)y_2(t)}{m^2y_2^2(t)+x_2^2(t)} - \gamma_2(t)y_1(t) - \beta_2(t)y_2(t).\end{aligned}$$

By using the inequalities $a^2 + b^2 \geq 2ab$, where $a \geq 0, b \geq 0$, we have

$$\dot{\rho}(t) \leq -\gamma_1^L x_1(t) + \left(\alpha_1^M + \frac{\alpha_2^M}{2m}\right)x_2(t) - \beta_1^L x_2^2(t) - \gamma_2^L y_1(t) - \beta_2^L y_2(t). \tag{2.5}$$

For a positive constant ϵ ($\epsilon < \min\{\gamma_1^L, \gamma_2^L, \beta_2^L\}$), it follows from (2.5) that

$$\dot{\rho}(t) + \epsilon\rho(t) \leq \left(\epsilon + \alpha_1^M + \frac{\alpha_2^M}{2m}\right)x_2(t) - \beta_1^L x_2^2(t).$$

Therefore, there exists a positive constant A such that

$$\dot{\rho}(t) + \epsilon\rho(t) < A,$$

which yields

$$\rho(t) < \frac{A}{\epsilon} + \left(\rho(0) - \frac{A}{\epsilon}\right)\mathrm{e}^{-\epsilon t}.$$

Hence, positive solutions of (1.3) with initial conditions (1.4) and (1.5) are ultimately bounded, i.e., there exist positive constants T_1 and M_i $(i = 1, 2, 3, 4)$ such that $x_i(t) \leq M_i$, $y_i(t) \leq M_{i+2}$ $(i = 1, 2)$ for $t > T_1$. □

Lemma 2.5. *Consider the following equation:*

$$\dot{x}(t) = ax(t - \tau) - bx^2(t),$$

where a, b *and* τ *are positive constants,* $x(t) > 0$ *for* $t \in [-\tau, 0]$, *then* $\lim_{t\to+\infty} x(t) = a/b$.

Proof. It is easy to show that $x(t)$ is positive and bounded for all $t > 0$. Clearly $x^* = a/b$ is the unique equilibrium of equation $\dot{x}(t) = ax(t - \tau) - bx^2(t)$. Suppose that $x(t)$ is eventually monotonic, then $\lim_{t\to+\infty} x(t)$ exists. Denote $X = \lim_{t\to+\infty} x(t)$, now we prove $X = x^*$. Otherwise if $X > x^*$, then

$$\lim_{x\to+\infty} \dot{x} = aX - bX^2 = bX(x^* - X) < 0,$$

which implies $\lim_{t\to+\infty} x(t) = -\infty$, this is a contradiction, therefore $X = x^*$.

Now suppose that $x(t)$ is not eventually monotonic, since $x(t)$ is bounded, let $\delta = \lim_{t\to+\infty} \sup |x(t) - x^*|$, hence, δ is bounded, we now show that $\delta = 0$, otherwise, if $\delta > 0$, then there exists a sequence $x(t_i)(t_i > t_{i-1}, \lim_{i\to+\infty} t_i = +\infty)$ such that $\lim_{i\to+\infty} x(t_i) = x^* + \delta$, or $\lim_{i\to+\infty} x(t_i) = x^* - \delta$, $(x^* > \delta)$. Without loss of generalization, we only consider the case $\lim_{i\to+\infty} x(t_i) = x^* + \delta$. Then there is an ϵ (here $0 < \epsilon < (3a + 2b\delta - \sqrt{9a^2 + 8ab\delta})/(2b)$) such that

$$a(x^* + \delta + \epsilon) - b(x^* + \delta - \epsilon)^2 < 0.$$

For this ϵ, there exists a $T = T(\epsilon) > \tau$ such that for $t_i > T - \tau$, we have $x(t_i) < x^* + \delta + \epsilon$. We also know that there is a $\overline{t_i} > T$ such that $\dot{x}(\overline{t_i}) = 0$, $x(\overline{t_i}) - x^* > \delta - \epsilon$. This implies that

$$ax(\overline{t_i} - \tau) = bx^2(\overline{t_i}).$$

Thus $ax(\overline{t_i} - \tau) > b(x^* + \delta - \epsilon)^2$, by $a(x^* + \delta + \epsilon) - b(x^* + \delta - \epsilon)^2 < 0$, we have $ax(\overline{t_i} - \tau) > a(x^* + \delta + \epsilon)$. Hence $x(\overline{t_i} - \tau) > x^* + \delta + \epsilon$, this is a contradiction to $x(t_i) < x^* + \delta + \epsilon$, then $\delta = 0$, that is $\lim_{t\to+\infty} x(t) = x^*$. □

Theorem 2.1. *System* (1.3) *with initial conditions* (1.4) *and* (1.5) *is permanent provided that*

(H_3) $2m\alpha_1^L e^{-\gamma_1^M\tau_1} > a_1^M$, $\beta_2^M < \alpha_2^L e^{-\gamma_2^M\tau_2} < 2\beta_2^M$.

Proof. Suppose $z(t) = (x_1(t), x_2(t), y_1(t), y_2(t))$ is any solution of system (1.3) with initial conditions (1.4)–(1.5).

It follows from the second equation of system (1.3) that for $t > \tau_1$,

$$\dot{x}_2(t) \geq \alpha_1^L e^{-\gamma_1^M\tau_1} x_2(t - \tau_1) - \beta_1^M x_2^2(t) - \frac{a_1^M}{2m} x_2(t).$$

We consider the following auxiliary equation:

$$\dot{u}(t) = \alpha_1^L e^{-\gamma_1^M\tau_1} u(t - \tau_1) - \beta_1^M u^2(t) - \frac{a_1^M}{2m} u(t).$$

By Lemma 2.2 we derive

$$\lim_{t\to+\infty} u(t) = \frac{\alpha_1^L e^{-\gamma_1^M\tau_1} - \frac{a_1^M}{2m}}{\beta_1^M} := m_2^*.$$

By comparison, there exist a $T_2 > \tau_1$ and a positive constant $m_2 < m_2^*$ such that $x_2(t) > m_2$ for $t \geq T_2$. As a consequence, from the fourth equation of system (1.3) we derive for $t > T_2 + \tau_2$ that

$$\dot{y}_2(t) > \alpha_2^L e^{-\gamma_2^M\tau_2} \frac{m_2^2 y_2(t - \tau_2)}{m^2 y_2^2(t - \tau_2) + m_2^2} - \beta_2^M y_2(t).$$

Consider the following auxiliary equation:

$$\dot{u}(t) = \alpha_2^L e^{-\gamma_2^M\tau_2} \frac{m_2^2 u(t - \tau_2)}{m^2 u^2(t - \tau_2) + m_2^2} - \beta_2^M u(t).$$

In order to apply Lemma 2.3, we now prove that the equation

$$\dot{u}(t) = \alpha_2^L e^{-\gamma_2^M\tau_2} \frac{m_2^2 u(t - \tau_2)}{m^2 u^2(t - \tau_2) + m_2^2} - \beta_2^M u(t)$$

satisfies conditions (a_1), (a_2) of Lemma 2.3. We let

$$f(u(t-\tau_2)) = \alpha_2^L e^{-\gamma_2^M \tau_2} \frac{m_2^2 u(t-\tau_2)}{m^2 u^2(t-\tau_2) + m_2^2}, \qquad g(u) = \beta_2^M u(t).$$

Clearly, $f(0) = 0$, there is an $u_M = m_2/m > 0$ such that $f(\cdot)$ is strictly increasing in $[0, u_M]$ and strictly decreasing in $[u_M, +\infty)$; obviously, $\lim_{u\to+\infty} f(u) = 0$. By (H_3) and the equation

$$\dot{u}(t) = \alpha_2^L e^{-\gamma_2^M \tau_2} \frac{m_2^2 u(t-\tau_2)}{m^2 u^2(t-\tau_2) + m_2^2} - \beta_2^M u(t),$$

we can obtain that there is a unique

$$u^* = \frac{m_2}{m} \sqrt{\frac{\alpha_2^L e^{-\gamma_2^M \tau_2} - \beta_2^M}{\beta_2^M}} > 0$$

such that $f(u) > g(u)$ for $x \in (0, u^*)$ and $f(u) < g(u)$ for $u > u^*$. Hence, the condition (a_1) of Lemma 2.3 holds. Since $g(u) = \beta_2^M u(t)$ is strictly increasing, and $g(0) = 0$, $\lim_{u\to+\infty} g(u) = +\infty$, the condition (a_2) of Lemma 2.3 holds. By $\alpha_2^L e^{-\gamma_2^M \tau_2} < 2\beta_2^M$ of (H_3), we can also know that $u_M > u^*$. Hence, all conditions of Lemma 2.3 hold.

By Lemma 2.3, we now derive

$$\lim_{t\to+\infty} u(t) = \frac{m_2}{m} \sqrt{\frac{\alpha_2^L e^{-\gamma_2^M \tau_2} - \beta_2^M}{\beta_2^M}}.$$

By the comparison principle, we have

$$\liminf_{t\to+\infty} y_2(t) \geq \frac{m_2}{m} \sqrt{\frac{\alpha_2^L e^{-\gamma_2^M \tau_2} - \beta_2^M}{\beta_2^M}} := m_4^*.$$

Hence, there exist a $T_3 > T_2 + \tau_2$ and a positive constant $m_4 < m_4^*$ such that $y_2(t) > m_4$ for $t \geq T_3$.

By the second equation of system (1.3), for $t > \tau_1$, we derive

$$\dot{x}_2(t) \leq \alpha_1^M e^{-\gamma_1^L \tau_1} x_2(t-\tau_1) - \beta_1^L x_2^2(t).$$

We consider the following auxiliary equation:

$$\dot{u}(t) = \alpha_1^M e^{-\gamma_1^L \tau_1} u(t-\tau_1) - \beta_1^L u^2(t).$$

By Lemma 2.5 we derive

$$\lim_{t\to\infty} u(t) = \frac{\alpha_1^M e^{-\gamma_1^L \tau_1}}{\beta_1^L} := M_2^*.$$

By comparison, there exists positive constants $T_1 > \tau_1$ and $M_2 > M_2^*$ such that $x_2(t) < M_2$ for $t > T_1$. At the same time, by the fourth equation of system (1.3), for $t > \tau_1$, we derive

$$\dot{y}_2(t) \leq \frac{\alpha_2^M e^{-\gamma_2^L \tau_2} M_2}{2m} - \beta_2^L y_2(t).$$

We consider the following auxiliary equation:

$$\dot{u}(t) = \frac{\alpha_2^M e^{-\gamma_2^L \tau_2} M_2}{2m} - \beta_2^L u(t).$$

By the equation, we obtain

$$u(t) \leq \frac{\alpha_2^M e^{-\gamma_2^L \tau_2} M_2}{2m\beta_2^L} := M_4^*.$$

By comparison, there exists positive constants $T_4 > \tau_2$ and $M_4 > M_4^*$ such that $y_2(t) < M_4$ for $t > T_4$.

It follows from (2.4) that for $t \geq \max\{T_3 + \tau, T_4\}$,

$$\begin{aligned} x_1(t) &= \int_{t-\tau_1}^{t} \alpha_1(s) e^{-\int_s^t \gamma_1(u)\mathrm{d}u} x_2(s)\mathrm{d}s \geq \int_{t-\tau_1}^{t} \alpha_1^L m_2 e^{-\gamma_1^M (t-s)}\mathrm{d}s \\ &= \frac{\alpha_1^L m_2}{\gamma_1^M}(1 - e^{-\gamma_1^M \tau_1}) := m_1, \end{aligned}$$

and

$$\begin{aligned} y_1(t) &= \int_{t-\tau_2}^{t} \alpha_2(s)\mathrm{e}^{-\int_s^t \gamma_2(u)\mathrm{d}u} \frac{x_2^2(s)y_2(s)}{m^2y_2^2(s)+x_2^2(s)}\mathrm{d}s \ge \int_{t-\tau_2}^{t} \alpha_2^L \mathrm{e}^{-\gamma_2^M(t-s)} \frac{m_2^2y_2(s)}{m^2y_2^2(s)+m_2^2}\mathrm{d}s \\ &\ge \int_{t-\tau_2}^{t} \alpha_2^L \mathrm{e}^{-\gamma_2^M(t-s)} \frac{m_2^2m_4}{m^2y_2^2(s)+m_2^2}\mathrm{d}s \ge \int_{t-\tau_2}^{t} \alpha_2^L \mathrm{e}^{-\gamma_2^M(t-s)} \frac{m_2^2m_4}{m^2M_4^2+m_2^2}\mathrm{d}s \\ &= \frac{\alpha_2^L m_2^2 m_4}{\gamma_2^M(m^2M_4^2+m_2^2)}(1-\mathrm{e}^{-\gamma_2^M\tau_2}) := m_3. \end{aligned}$$

We now let

$$\mathbf{D} = \{(x_1,x_2,y_1,y_2) | m_i \le x_i \le M_i, m_{i+2} \le y_i \le M_{i+2}, i = 1, 2\}.$$

Then $\mathbf{D}$ is a bounded compact region in $\mathbf{R}_+^4$ which has positive distance from coordinate hyper-planes. From what has been discussed above, we obtain that there exists a $T > T_3 + \tau_2$ such that if $t > T$, every positive solution of system (1.3) with initial conditions (1.4) and (1.5) eventually enters and remains in the region $\mathbf{D}$. The proof is complete. □

In the following, by using a similar method in the proof of Theorem 2.1 in Wang et al. [29], we present a simple result for extinction of the predator.

Theorem 2.2. *Adult predator population will go to extinction if* $\alpha_2^M \mathrm{e}^{-\gamma_2^L\tau_2} < \beta_2^L$.

Proof. Let $(x_1(t), x_2(t), y_1(t), y_2(t))$ be a positive solution of system (1.3) with initial conditions (1.4) and (1.5). It follows from the fourth equation of system (1.3) that

$$\dot{y}_2(t) \le \alpha_2^M \mathrm{e}^{-\gamma_2^L\tau_2} y_2(t-\tau_2) - \beta_2^L y_2(t).$$

Consider the following auxiliary equation:

$$\dot{y}(t) = \alpha_2^M \mathrm{e}^{-\gamma_2^L\tau_2} y(t-\tau_2) - \beta_2^L y(t). \tag{2.6}$$

Since $\alpha_2^M \mathrm{e}^{-\gamma_2^L\tau_2} < \beta_2^L$, we can choose a positive constant $q > 1$ such that $q\alpha_2^M \mathrm{e}^{-\gamma_2^L\tau_2} < \beta_2^L$. Take $p(s) = q^2 s$, $V(y) = y^2$. Calculating the derivative of $V(y)$ along solutions of Eq. (2.6) we obtain

$$\dot{V}(y(t)) = 2(\alpha_2^M \mathrm{e}^{-\gamma_2^L\tau_2} y(t-\tau_2) - \beta_2^L y(t)). \tag{2.7}$$

If $p(V(y(t))) > V(y(t+\theta))$ for $-\tau_2 \le \theta \le 0$, we have $|qy(t)| > |y(t+\theta)|$. Therefore, it follows from (2.7) that

$$\dot{V}(y(t)) \le 2y^2(t)(q\alpha_2^M \mathrm{e}^{-\gamma_2^L\tau_2} - \beta_2^L)$$

if $p(V(y(t))) > V(y(t+\theta))$ for $-\tau_2 \le \theta \le 0$. Since $q\alpha_2^M \mathrm{e}^{-\gamma_2^L\tau_2} < \beta_2^L$, by Theorem 4.2 in Chapter 5 of [33], we can derive $\lim_{t\to\infty} y(t) = 0$. A standard comparison argument shows that $\lim_{t\to\infty} y_2(t) = 0$. This completes the proof. □

3. Existence of periodic solutions

In order to obtain the existence of positive periodic solutions of system (1.3), we first make the following preparations:

Let $\mathbf{X}$, $\mathbf{Y}$ be real Banach spaces, let $\mathbf{L} : \operatorname{Dom}\mathbf{L} \subset \mathbf{X} \to \mathbf{Y}$ be a linear mapping, and $\mathbf{N} : \mathbf{X} \to \mathbf{Y}$ be a continuous mapping. The mapping $\mathbf{L}$ is called a Fredholm mapping of index zero if $\dim\operatorname{Ker}\mathbf{L} = \operatorname{codim}\operatorname{Im}\mathbf{L} < +\infty$ and $\operatorname{Im}\mathbf{L}$ is closed in $\mathbf{Y}$. If $\mathbf{L}$ is a Fredholm mapping of index zero and there exist continuous projectors $\mathbf{P} : \mathbf{X} \to \mathbf{X}$, and $\mathbf{Q} : \mathbf{Y} \to \mathbf{Y}$ such that $\operatorname{Im}\mathbf{P} = \operatorname{Ker}\mathbf{L}$, $\operatorname{Ker}\mathbf{Q} = \operatorname{Im}\mathbf{L} = \operatorname{Im}(\mathbf{I}-\mathbf{Q})$, then the restriction $\mathbf{L_P}$ of $\mathbf{L}$ to $\operatorname{Dom}\mathbf{L} \bigcap \operatorname{Ker}\mathbf{P} : (\mathbf{I}-\mathbf{P})\mathbf{X} \to \operatorname{Im}\mathbf{L}$ is invertible. Denote the inverse of $\mathbf{L_P}$ by $\mathbf{K_P}$. If Ω is an open bounded subset of $\mathbf{X}$, the mapping $\mathbf{N}$ will be called $\mathbf{L}$-compact on $\overline{\Omega}$ if $\mathbf{QN}(\overline{\Omega})$ is bounded and $\mathbf{K_P}(\mathbf{I}-\mathbf{Q})\mathbf{N} : \overline{\Omega} \to \mathbf{X}$ is compact. Since $\operatorname{Im}\mathbf{Q}$ is isomorphic to $\operatorname{Ker}\mathbf{L}$, there exists an isomorphism $\mathbf{J} : \operatorname{Im}\mathbf{Q} \to \operatorname{Ker}\mathbf{L}$.

Lemma 3.1 ([22]). *Let* $\Omega \subset \mathbf{X}$ *be an open bounded set. Let* $\mathbf{L}$ *be a Fredholm mapping of index zero and* $\mathbf{N}$ *be* $\mathbf{L}$*-compact on* $\overline{\Omega}$. *Assume*

(b$_1$) *for each* $\lambda \in (0,1)$, $x \in \partial\Omega \bigcap \operatorname{Dom}\mathbf{L}$, $\mathbf{L}x \ne \lambda\mathbf{N}x$;
(b$_2$) *for each* $x \in \partial\Omega \bigcap \operatorname{Ker}\mathbf{L}$, $\mathbf{QN}x \ne 0$;
(b$_3$) $\deg\{\mathbf{JQN}, \Omega \bigcap \operatorname{Ker}\mathbf{L}, 0\} \ne 0$.
Then $\mathbf{L}x = \mathbf{N}x$ *has at least one solution in* $\overline{\Omega} \bigcap \operatorname{Dom}\mathbf{L}$.

We are now in a position to state and prove our result on the existence of positive periodic solutions of system (1.3).

Theorem 3.1. *Let* (H_3) *hold. Then system (1.3) with initial conditions (1.4) and (1.5) has at least one strictly positive ω-periodic solution.*

Proof. We first consider the following subsystem:

$$\begin{cases} \dot{x}_2(t) = \alpha_1(t-\tau_1)\mathrm{e}^{-\int_{t-\tau_1}^{t}\gamma_1(s)\mathrm{d}s}x_2(t-\tau_1) - \beta_1(t)x_2^2(t) - \dfrac{a_1(t)x_2^2(t)y_2(t)}{m^2y_2^2(t)+x_2^2(t)}, \\ \dot{y}_2(t) = \alpha_2(t-\tau_2)\mathrm{e}^{-\int_{t-\tau_2}^{t}\gamma_2(s)\mathrm{d}s}\dfrac{x_2^2(t-\tau_2)y_2(t-\tau_2)}{m^2y_2^2(t-\tau_2)+x_2^2(t-\tau_2)} - \beta_2(t)y_2(t) \end{cases} \tag{3.1}$$

with initial conditions

$$\begin{cases} x_2(\theta) = \phi_2(\theta), \quad y_2(\theta) = \psi_2(\theta), \\ \phi_2(\theta) \geq 0, \quad \psi_2(0) > 0, \quad \phi_2(0) > 0, \quad \theta_2 \in [-\tau, 0]. \end{cases} \tag{3.2}$$

Let

$$u_1(t) = \ln[x_2(t)], \qquad u_2(t) = \ln[y_2(t)]. \tag{3.3}$$

On substituting (3.3) into (3.1), we derive

$$\begin{cases} \dfrac{\mathrm{d}u_1(t)}{\mathrm{d}t} = \alpha_1(t-\tau_1)\mathrm{e}^{-\int_{t-\tau_1}^{t}\gamma_1(s)\mathrm{d}s}\mathrm{e}^{u_1(t-\tau_1)-u_1(t)} - \beta_1(t)\mathrm{e}^{u_1(t)} - \dfrac{a_1(t)\mathrm{e}^{u_1(t)+u_2(t)}}{m^2\mathrm{e}^{2u_2(t)}+\mathrm{e}^{2u_1(t)}}, \\ \dfrac{\mathrm{d}u_2(t)}{\mathrm{d}t} = \alpha_2(t-\tau_2)\mathrm{e}^{-\int_{t-\tau_2}^{t}\gamma_2(s)\mathrm{d}s}\dfrac{\mathrm{e}^{2u_1(t-\tau_2)+u_2(t-\tau_2)-u_2(t)}}{m^2\mathrm{e}^{2u_2(t-\tau_2)}+\mathrm{e}^{2u_1(t-\tau_2)}} - \beta_2(t). \end{cases} \tag{3.4}$$

It is easy to see that if system (3.4) has one ω-periodic solution $(u_1^*(t), u_2^*(t))^{\mathrm{T}}$, then $z^*(t) = (x_2^*(t), y_2^*(t))^{\mathrm{T}} = (\exp[u_1^*(t)], \exp[u_2^*(t)])^{\mathrm{T}}$ is a positive ω-periodic solution of system (3.1). Therefore, in the following we first prove that system (3.4) has at least one ω-periodic solution.

To apply Lemma 3.1 to (3.4), we first define

$$\mathbf{X} = \mathbf{Y} = \{(u_1(t), u_2(t))^{\mathrm{T}} \in C(\mathbf{R}, \mathbf{R^2}) : u_i(t+\omega) = u_i(t), i = 1, 2\}$$

and

$$\|(u_1(t), u_2(t))^{\mathrm{T}}\| = \max_{t\in[0,\omega]}|u_1(t)| + \max_{t\in[0,\omega]}|u_2(t)|,$$

where $|\cdot|$ denote the Euclidean norm. Then it is easy to see that $\mathbf{X}$ and $\mathbf{Y}$ are Banach spaces with the norm $\|\cdot\|$. Let

$$\mathbf{L} : \mathrm{Dom}\,\mathbf{L}\bigcap\mathbf{X} \to \mathbf{X}, \qquad \mathbf{L}(u_1(t), u_2(t))^{\mathrm{T}} = \left(\frac{\mathrm{d}u_1(t)}{\mathrm{d}t}, \frac{\mathrm{d}u_2(t)}{\mathrm{d}t}\right)^{\mathrm{T}},$$

where $\mathrm{Dom}\,\mathbf{L} = \{(u_1(t), u_2(t))^{\mathrm{T}} \in C^1(\mathbf{R}, \mathbf{R^2})\}$ and $\mathbf{N} : \mathbf{X} \to \mathbf{X}$,

$$\mathbf{N}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \alpha_1(t-\tau_1)\mathrm{e}^{-\int_{t-\tau_1}^{t}\gamma_1(s)\mathrm{d}s}\mathrm{e}^{u_1(t-\tau_1)-u_1(t)} - \beta_1(t)\mathrm{e}^{u_1(t)} - \dfrac{a_1(t)\mathrm{e}^{u_1(t)+u_2(t)}}{m^2\mathrm{e}^{2u_2(t)}+\mathrm{e}^{2u_1(t)}} \\ \alpha_2(t-\tau_2)\mathrm{e}^{-\int_{t-\tau_2}^{t}\gamma_2(s)\mathrm{d}s}\dfrac{\mathrm{e}^{2u_1(t-\tau_2)+u_2(t-\tau_2)-u_2(t)}}{m^2\mathrm{e}^{2u_2(t-\tau_2)}+\mathrm{e}^{2u_1(t-\tau_2)}} - \beta_2(t) \end{pmatrix}.$$

Define

$$\mathbf{P}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \mathbf{Q}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\omega}\displaystyle\int_0^\omega u_1(t)\mathrm{d}t \\ \dfrac{1}{\omega}\displaystyle\int_0^\omega u_2(t)\mathrm{d}t \end{pmatrix}, \qquad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbf{X} = \mathbf{Y}.$$

It is not difficult to show that

$$\mathrm{Ker}\,\mathbf{L} = \{x | x \in \mathbf{X}, x = h, h \in \mathbf{R^2}\},$$

$$\mathrm{Im}\,\mathbf{L} = \left\{y | y \in \mathbf{Y}, \int_0^\omega y(t)\mathrm{d}t = 0\right\},$$

$\mathrm{Im}\,\mathbf{L}$ is closed in $\mathbf{Y}$, and

$$\dim \mathrm{Ker}\,\mathbf{L} = \mathrm{codim}\,\mathrm{Im}\mathbf{L} = 2,$$

and $\mathbf{P}$ and $\mathbf{Q}$ are continuous projectors such that

$$\operatorname{Im}\mathbf{P} = \operatorname{Ker}\mathbf{L}, \qquad \operatorname{Ker}\mathbf{Q} = \operatorname{Im}\mathbf{L} = (\mathbf{I}-\mathbf{Q}).$$

It follows that $\mathbf{L}$ is a Fredholm mapping of index zero. Furthermore, the inverse $\mathbf{K_P}$ of $\mathbf{L_P}$ exists and has the form $\mathbf{K_P} : \operatorname{Im}\mathbf{L} \to \operatorname{Dom}\mathbf{L} \bigcap \operatorname{Ker}\mathbf{P}$,

$$\mathbf{K_P}(y) = \int_0^t y(s)\mathrm{d}s - \frac{1}{\omega}\int_0^\omega \int_0^t y(s)\mathrm{d}s\mathrm{d}t.$$

Then $\mathbf{QN} : \mathbf{X} \to \mathbf{Y}$ and $\mathbf{K_P}(\mathbf{I}-\mathbf{Q})\mathbf{N} : \mathbf{X} \to \mathbf{X}$ are given respectively by

$$\mathbf{QN}x = \begin{pmatrix} \dfrac{1}{\omega}\displaystyle\int_0^\omega (\alpha_1(t-\tau_1)\mathrm{e}^{-\int_{t-\tau_1}^t \gamma_1(s)\mathrm{d}s}\mathrm{e}^{u_1(t-\tau_1)-u_1(t)} - \beta_1(t)\mathrm{e}^{u_1(t)} - \dfrac{a_1(t)\mathrm{e}^{u_1(t)+u_2(t)}}{m^2\mathrm{e}^{2u_2(t)}+\mathrm{e}^{2u_1(t)}})\mathrm{d}t \\ \dfrac{1}{\omega}\displaystyle\int_0^\omega (\alpha_2(t-\tau_2)\mathrm{e}^{-\int_{t-\tau_2}^t \gamma_2(s)\mathrm{d}s}\dfrac{\mathrm{e}^{2u_1(t-\tau_2)+u_2(t-\tau_2)-u_2(t)}}{m^2\mathrm{e}^{2u_2(t-\tau_2)}+\mathrm{e}^{2u_1(t-\tau_2)}} - \beta_2(t))\mathrm{d}t \end{pmatrix},$$

$$\mathbf{K_P}(\mathbf{I}-\mathbf{Q})\mathbf{N}x = \int_0^t \mathbf{N}x(s)\mathrm{d}s - \frac{1}{\omega}\int_0^\omega\int_0^t \mathbf{N}x(s)\mathrm{d}s\mathrm{d}t - \left(\frac{t}{\omega}-\frac{1}{2}\right)\int_0^\omega \mathbf{N}x(s)\mathrm{d}s.$$

Clearly, $\mathbf{QN}$ and $\mathbf{K_P}(\mathbf{I}-\mathbf{Q})\mathbf{N}$ are continuous.

In order to apply Lemma 3.1, we need to search for an appropriate open, bounded subset Ω.

Corresponding to the operator equation $\mathbf{L}x = \lambda\mathbf{N}x$, $\lambda \in (0, 1)$, we have

$$\begin{cases} \dfrac{\mathrm{d}u_1(t)}{\mathrm{d}t} = \lambda\left[\alpha_1(t-\tau_1)\mathrm{e}^{-\int_{t-\tau_1}^t \gamma_1(s)\mathrm{d}s}\mathrm{e}^{u_1(t-\tau_1)-u_1(t)} - \beta_1(t)\mathrm{e}^{u_1(t)} - \dfrac{a_1(t)\mathrm{e}^{u_1(t)+u_2(t)}}{m^2\mathrm{e}^{2u_2(t)}+\mathrm{e}^{2u_1(t)}}\right], \\ \dfrac{\mathrm{d}u_2(t)}{\mathrm{d}t} = \lambda\left[\alpha_2(t-\tau_2)\mathrm{e}^{-\int_{t-\tau_2}^t \gamma_2(s)\mathrm{d}s}\dfrac{\mathrm{e}^{2u_1(t-\tau_2)+u_2(t-\tau_2)-u_2(t)}}{m^2\mathrm{e}^{2u_2(t-\tau_2)}+\mathrm{e}^{2u_1(t-\tau_2)}} - \beta_2(t)\right]. \end{cases} \tag{3.5}$$

Suppose that $(u_1(t), u_2(t))^{\mathrm{T}} \in \mathbf{X}$ is a solution of (3.5) for a certain $\lambda \in (0, 1)$. Integrating (3.5) over the initial $[0, \omega]$ we obtain

$$\int_0^\omega \alpha_1(t-\tau_1)\mathrm{e}^{-\int_{t-\tau_1}^t \gamma_1(s)\mathrm{d}s}\mathrm{e}^{u_1(t-\tau_1)-u_1(t)}\mathrm{d}t = \int_0^\omega \beta_1(t)\mathrm{e}^{u_1(t)}\mathrm{d}t + \int_0^\omega \frac{a_1(t)\mathrm{e}^{u_1(t)+u_2(t)}}{m^2\mathrm{e}^{2u_2(t)}+\mathrm{e}^{2u_1(t)}}\mathrm{d}t, \tag{3.6}$$

$$\int_0^\omega \alpha_2(t-\tau_2)\mathrm{e}^{-\int_{t-\tau_2}^t \gamma_2(s)\mathrm{d}s}\frac{\mathrm{e}^{2u_1(t-\tau_2)+u_2(t-\tau_2)-u_2(t)}}{m^2\mathrm{e}^{2u_2(t-\tau_2)}+\mathrm{e}^{2u_1(t-\tau_2)}}\mathrm{d}t = \int_0^\omega \beta_2(t)\mathrm{d}t. \tag{3.7}$$

Since $(u_1(t), u_2(t))^{\mathrm{T}} \in \mathbf{X}$, there exist $\xi_i, \eta_i \in [0, \omega]$ such that

$$u_i(\xi_i) = \min_{t\in[0,\omega]} u_i(t), \qquad u_i(\eta_i) = \max_{t\in[0,\omega]} u_i(t), \quad i = 1, 2. \tag{3.8}$$

Multiplying the first equation of (3.5) by $\mathrm{e}^{u_1(t)}$ and integrating over $[0, \omega]$ give

$$\int_0^\omega \alpha_1(t-\tau_1)\mathrm{e}^{-\int_{t-\tau_1}^t \gamma_1(s)\mathrm{d}s}\mathrm{e}^{u_1(t-\tau_1)}\mathrm{d}t = \int_0^\omega \beta_1(t)\mathrm{e}^{2u_1(t)}\mathrm{d}t + \int_0^\omega \frac{a_1(t)\mathrm{e}^{2u_1(t)+u_2(t)}}{m^2\mathrm{e}^{2u_2(t)}+\mathrm{e}^{2u_1(t)}}\mathrm{d}t. \tag{3.9}$$

It follows from (3.9) that

$$\int_0^\omega \beta_1(t)\mathrm{e}^{2u_1(t)}\mathrm{d}t < \int_0^\omega \alpha_1(t-\tau_1)\mathrm{e}^{-\int_{t-\tau_1}^t \gamma_1(s)\mathrm{d}s}\mathrm{e}^{u_1(t-\tau_1)}\mathrm{d}t,$$

which yields

$$\beta_1^L\int_0^\omega \mathrm{e}^{2u_1(t)}\mathrm{d}t < \alpha_1^M\mathrm{e}^{-\gamma_1^L\tau_1}\int_0^\omega \mathrm{e}^{u_1(t-\tau_1)}\mathrm{d}t = \alpha_1^M\mathrm{e}^{-\gamma_1^L\tau_1}\int_0^\omega \mathrm{e}^{u_1(t)}\mathrm{d}t. \tag{3.10}$$

By using the inequalities

$$\left(\int_0^\omega \mathrm{e}^{u_1(t)}\mathrm{d}t\right)^2 \le \omega\int_0^\omega \mathrm{e}^{2u_1(t)}\mathrm{d}t,$$

we derive from (3.10) that

$$\beta_1^L\left(\int_0^\omega \mathrm{e}^{u_1(t)}\mathrm{d}t\right)^2 < \alpha_1^M\omega\mathrm{e}^{-\gamma_1^L\tau_1}\int_0^\omega \mathrm{e}^{u_1(t)}\mathrm{d}t,$$

which implies

$$\int_0^\omega e^{u_1(t)}dt \le \frac{\alpha_1^M \omega e^{-\gamma_1^L\tau_1}}{\beta_1^L}, \qquad u_1(\xi_1) \le \ln \frac{\alpha_1^M e^{-\gamma_1^L\tau_1}}{\beta_1^L}. \tag{3.11}$$

It follows from (3.5), (3.6) and (3.11) that

$$\begin{aligned}\int_0^\omega |u_1'(t)|dt &< \int_0^\omega \left[\alpha_1(t-\tau_1)e^{-\int_{t-\tau_1}^t \gamma_1(s)ds} e^{u_1(t-\tau_1)-u_1(t)} + \beta_1(t)e^{u_1(t)} + \frac{a_1(t)e^{u_1(t)+u_2(t)}}{m^2e^{2u_2(t)}+e^{2u_1(t)}}\right]dt \\ &= 2\int_0^\omega \left[\beta_1(t)e^{u_1(t)} + \frac{a_1(t)e^{u_1(t)+u_2(t)}}{m^2e^{2u_2(t)}+e^{2u_1(t)}}\right]dt \\ &\le 2\beta_1^M \int_0^\omega e^{u_1(t)}dt + \frac{\overline{a_1}\omega}{m} \\ &\le \frac{2\alpha_1^M\beta_1^M\omega e^{-\gamma_1^L\tau_1}}{\beta_1^L} + \frac{\overline{a_1}\omega}{m} := c_1. \end{aligned} \tag{3.12}$$

We derive from (3.11) and (3.12) that

$$u_1(t) \le u_1(\xi_1) + \int_0^\omega |u_1'(t)|dt \le \ln \frac{\alpha_1^M e^{-\gamma_1^L\tau_1}}{\beta_1^L} + c_1. \tag{3.13}$$

Noting that

$$\int_0^\omega \alpha_1(t-\tau_1)e^{-\int_{t-\tau_1}^t \gamma_1(s)ds} e^{u_1(t-\tau_1)}dt = \int_0^\omega \alpha_1(t)e^{-\int_t^{t+\tau_1}\gamma_1(s)ds}e^{u_1(t)}dt,$$

it follows from (3.9) that

$$\begin{aligned}\int_0^\omega \beta_1(t)e^{2u_1(t)}dt &= \int_0^\omega \alpha_1(t)e^{-\int_t^{t+\tau_1}\gamma_1(s)ds}e^{u_1(t)}dt - \int_0^\omega \frac{a_1(t)e^{2u_1(t)+u_2(t)}}{m^2e^{2u_2(t)}+e^{2u_1(t)}}dt \\ &\ge \alpha_1^L e^{-\gamma_1^M\tau_1}\int_0^\omega e^{u_1(t)}dt - \frac{a_1^M}{2m}\int_0^\omega e^{u_1(t)}dt,\end{aligned}$$

which yields

$$e^{u_1(\eta_1)} \ge \frac{\alpha_1^L e^{-\gamma_1^M\tau_1} - \frac{a_1^M}{2m}}{\beta_1^M},$$

i.e.,

$$u_1(\eta_1) \ge \ln \frac{\alpha_1^L e^{-\gamma_1^M\tau_1} - \frac{a_1^M}{2m}}{\beta_1^M}. \tag{3.14}$$

We derive from (3.12) and (3.14) that

$$u_1(t) \ge u_1(\eta_1) - \int_0^\omega |u_1'(t)|dt \ge \ln \frac{\alpha_1^L e^{-\gamma_1^M\tau_1} - \frac{a_1^M}{2m}}{\beta_1^M} - c_1. \tag{3.15}$$

This together with (3.13), leads to

$$\max_{t\in[0,\omega]} |u_1(t)| < \max\left\{\left|\ln \frac{\alpha_1^M e^{-\gamma_1^L\tau_1}}{\beta_1^L}\right| + c_1, \left|\ln \frac{\alpha_1^L e^{-\gamma_1^M\tau_1} - \frac{a_1^M}{2m}}{\beta_1^M}\right| + c_1\right\} := R_1. \tag{3.16}$$

Multiplying the second equation of (3.5) by $e^{u_2(t)}$ and integrating over $[0, \omega]$ gives

$$\begin{aligned}\int_0^\omega \beta_2(t)e^{u_2(t)}dt &= \int_0^\omega \alpha_2(t-\tau_2)e^{-\int_{t-\tau_2}^t \gamma_2(s)ds}\frac{e^{2u_1(t-\tau_2)+u_2(t-\tau_2)}}{m^2e^{2u_2(t-\tau_2)}+e^{2u_1(t-\tau_2)}}dt \\ &\le \frac{\alpha_2^M e^{-\gamma_2^L\tau_2}}{2m}\int_0^\omega e^{u_1(t-\tau_2)}dt = \frac{\alpha_2^M e^{-\gamma_2^L\tau_2}}{2m}\int_0^\omega e^{u_1(t)}dt,\end{aligned}$$

which, together with (3.11), implies

$$u_2(\xi_2) \leq \ln \frac{\alpha_1^M \alpha_2^M \mathrm{e}^{-(\gamma_1^L \tau_1 + \gamma_2^L \tau_2)}}{2m\beta_1^L \beta_2^L} := \ln d_1. \tag{3.17}$$

It follows from (3.5) and (3.7) that

$$\begin{aligned}\int_0^\omega |u_2'(t)|\mathrm{d}t &< \int_0^\omega \left[\alpha_2(t-\tau_2)\mathrm{e}^{-\int_{t-\tau_2}^t \gamma_2(s)\mathrm{d}s} \frac{\mathrm{e}^{2u_1(t-\tau_2)+u_2(t-\tau_2)-u_2(t)}}{m^2\mathrm{e}^{2u_2(t-\tau_2)} + \mathrm{e}^{2u_1(t-\tau_2)}} + \beta_2(t)\right]\mathrm{d}t \\ &= 2\overline{\beta_2}\omega. \end{aligned}\tag{3.18}$$

Thus, from (3.17) and (3.18) we can obtain

$$u_2(t) \leq u_2(\xi_2) + \int_0^\omega |u_2'(t)|\mathrm{d}t \leq \ln d_1 + 2\overline{\beta}\omega. \tag{3.19}$$

Multiplying the second equation of (3.5) by $\mathrm{e}^{u_2(t)}$ and integrating over $[0, \omega]$ again we derive

$$\begin{aligned}\int_0^\omega \beta_2(t)\mathrm{e}^{u_2(t)}\mathrm{d}t &= \int_0^\omega \alpha_2(t-\tau_2)\mathrm{e}^{-\int_{t-\tau_2}^t \gamma_2(s)\mathrm{d}s} \frac{\mathrm{e}^{2u_1(t-\tau_2)+u_2(t-\tau_2)}}{m^2\mathrm{e}^{2u_2(t-\tau_2)} + \mathrm{e}^{2u_1(t-\tau_2)}}\mathrm{d}t \\ &\geq \alpha_2^L \mathrm{e}^{-\gamma_2^M \tau_2} \int_0^\omega \frac{\mathrm{e}^{2u_1(\xi_1)+u_2(t-\tau_2)}}{m^2\mathrm{e}^{2u_2(\eta_2)} + \mathrm{e}^{2u_1(\xi_1)}}\mathrm{d}t \\ &= \frac{\alpha_2^L \mathrm{e}^{-\gamma_2^M \tau_2}\mathrm{e}^{2u_1(\xi_1)}}{m^2\mathrm{e}^{2u_2(\eta_2)} + \mathrm{e}^{2u_1(\xi_1)}} \int_0^\omega \mathrm{e}^{u_2(t-\tau_2)}\mathrm{d}t \\ &= \frac{\alpha_2^L \mathrm{e}^{-\gamma_2^M \tau_2}\mathrm{e}^{2u_1(\xi_1)}}{m^2\mathrm{e}^{2u_2(\eta_2)} + \mathrm{e}^{2u_1(\xi_1)}} \int_0^\omega \mathrm{e}^{u_2(t)}\mathrm{d}t \end{aligned}$$

which, together with (3.15), leads to

$$u_2(\eta_2) \geq \ln \left[\sqrt{\frac{\alpha_2^L \mathrm{e}^{-\gamma_2^M \tau_2} - \beta_2^M}{m^2 \beta_2^M} \cdot \frac{\alpha_1^L \mathrm{e}^{-\gamma_1^M \tau_1} - \frac{a_1^M}{2m}}{\beta_1^M \mathrm{e}^{c_1}}}\right] := \ln d_2. \tag{3.20}$$

It follows from (3.18) and (3.20) that

$$u_2(t) \geq u_2(\eta_2) - \int_0^\omega |u_2'(t)|\mathrm{d}t \geq \ln d_2 - 2\overline{\beta_2}\omega. \tag{3.21}$$

This, together with (3.19), leads to

$$\max_{t\in[0,\omega]} |u_2(t)| < \max\{|\ln d_1| + 2\overline{\beta_2}\omega, |\ln d_2| + 2\overline{\beta_2}\omega\} := R_2. \tag{3.22}$$

Clearly, R_1 and R_2 in (3.16) and (3.22) are independent of λ. Denote $\mathbf{M} = R_1 + R_2 + R_0$, where R_0 is taken sufficiently large such that the unique solution $(u^*, v^*)^{\mathrm{T}}$ of the system of algebraic equations

$$\begin{cases} \dfrac{1}{\omega}\displaystyle\int_0^\omega \alpha_1(t-\tau_1)\mathrm{e}^{-\int_{t-\tau_1}^t \gamma_1(s)\mathrm{d}s}\mathrm{d}t - \overline{\beta_1}\mathrm{e}^u - \dfrac{\overline{a_1}\mathrm{e}^{u+v}}{m^2\mathrm{e}^{2v} + \mathrm{e}^{2u}} = 0, \\ \dfrac{1}{\omega}\displaystyle\int_0^\omega \alpha_2(t-\tau_2)\mathrm{e}^{-\int_{t-\tau_2}^t \gamma_2(s)\mathrm{d}s}\mathrm{d}t \dfrac{\mathrm{e}^{2u}}{m^2\mathrm{e}^{2v} + \mathrm{e}^{2u}} - \overline{\beta_2} = 0 \end{cases} \tag{3.23}$$

satisfies $\|(u^*, v^*)^{\mathrm{T}}\| = |u^*| + |v^*| < \mathbf{M}$.

We now take $\Omega = \{(u_1(t), u_2(t))^{\mathrm{T}} \in \mathbf{X} : \|(u_1, u_2)^{\mathrm{T}}\| < \mathbf{M}\}$. This satisfies the condition (b_1) in Lemma 3.1. When $(u_1(t), u_2(t))^{\mathrm{T}} \in \partial\Omega \bigcap \operatorname{Ker}\mathbf{L} = \partial\Omega \bigcap \mathbf{R^2}$, $(u_1, u_2)^{\mathrm{T}}$ is a constant vector in $\mathbf{R^2}$ with $|u_1| + |u_2| = \mathbf{M}$. Thus, we have

$$\mathbf{QN}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\omega}\displaystyle\int_0^\omega \alpha_1(t-\tau_1)\mathrm{e}^{-\int_{t-\tau_1}^t \gamma_1(s)\mathrm{d}s}\mathrm{d}t - \overline{\beta_1}\mathrm{e}^{u_1} - \dfrac{\overline{a_1}\mathrm{e}^{u_1+u_2}}{m^2\mathrm{e}^{2u_2} + \mathrm{e}^{2u_1}} \\ \dfrac{1}{\omega}\displaystyle\int_0^\omega \alpha_2(t-\tau_2)\mathrm{e}^{-\int_{t-\tau_2}^t \gamma_2(s)\mathrm{d}s}\mathrm{d}t \dfrac{\mathrm{e}^{2u_1}}{m^2\mathrm{e}^{2u_2} + \mathrm{e}^{2u_1}} - \overline{\beta_2} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This proves that condition (b_2) in Lemma 3.1 is satisfied.

Taking $\mathbf{J} = \mathbf{I} : \operatorname{Im}\mathbf{Q} \to \operatorname{Ker}\mathbf{L}$, $(u_1, u_2)^{\mathrm{T}} \to (u_1^*, u_2^*)^{\mathrm{T}}$, a direct calculation shows that

$$\begin{aligned}
&\deg\left(\mathbf{JQN}(u_1, u_2)^{\mathrm{T}}, \Omega \bigcap \operatorname{Ker}\mathbf{L}, (0,0)^{\mathrm{T}}\right) \\
&= \deg\Bigg(\Bigg(\frac{1}{\omega}\int_0^{\omega} \alpha_1(t-\tau_1)\mathrm{e}^{-\int_{t-\tau_1}^{t}\gamma_1(s)\mathrm{d}s}\mathrm{d}t - \overline{\beta_1}\mathrm{e}^{u_1} - \frac{\overline{a_1}\mathrm{e}^{u_1+u_2}}{m^2\mathrm{e}^{2u_2}+\mathrm{e}^{2u_1}}, \\
&\quad \frac{1}{\omega}\int_0^{\omega} \alpha_2(t-\tau_2)\mathrm{e}^{-\int_{t-\tau_2}^{t}\gamma_2(s)\mathrm{d}s}\mathrm{d}t\frac{\mathrm{e}^{2u_1}}{m^2\mathrm{e}^{2u_2}+\mathrm{e}^{2u_1}} - \overline{\beta_2}\Bigg)^{\mathrm{T}}, \Omega \bigcap \operatorname{Ker}\mathbf{L}, (0,0)^{\mathrm{T}}\Bigg) \\
&= \operatorname{sgn}\left\{\frac{2m^2\mathrm{e}^{3u_1^*+2u_2^*}}{\omega(m^2\mathrm{e}^{2u_2^*}+\mathrm{e}^{u_1^*})^2}\int_0^{\omega}\alpha_2(t-\tau_2)\mathrm{e}^{-\int_{t-\tau_2}\gamma_2(s)\mathrm{d}s}\mathrm{d}t\right\} = 1,
\end{aligned}$$

where $(u_1^*, u_2^*)^{\mathrm{T}}$ is the unique solution of (3.23).

Finally, it is easy to show that the set $\{\mathbf{K_P}(\mathbf{I}-\mathbf{Q})\mathbf{N}x \mid x \in \overline{\Omega}\}$ is equicontinuous and uniformly bounded. By using the Arzela–Ascoli theorem, we see that $\mathbf{K_P}(\mathbf{I}-\mathbf{Q})\mathbf{N} : \overline{\Omega} \to \mathbf{X}$ is compact. Consequently, $\mathbf{N}$ is $\mathbf{L}$-compact.

By now we have proved that Ω satisfies all the requirement in Lemma 3.1. Hence, (3.4) has at least one ω-periodic solution of system (3.1). Accordingly, system (3.1) has at least one positive ω-periodic solution.

Let $(x_1^*(t), x_2^*(t), y_1^*(t), y_2^*(t))^{\mathrm{T}}$ be a positive ω-periodic solution of system (3.1). Then it is easy to verify that

$$x_1^*(t) = \int_{t-\tau_1}^{t} \alpha_1(s)\mathrm{e}^{-\int_s^t \gamma_1(u)\mathrm{d}u}x_2^*(s)\mathrm{d}s$$

and

$$y_1^*(t) = \int_{t-\tau_2}^{t} \alpha_2(s)\mathrm{e}^{-\int_s^t \gamma_2(u)\mathrm{d}u}\frac{(x_2^*(s))^2 y_2^*(s)}{m^2(y_2^*(s))^2+(x_2^*(s))^2}\mathrm{d}s$$

are also ω-periodic. Thus, $(x_1^*(t), x_2^*(t), y_1^*(t), y_2^*(t))^{\mathrm{T}}$ is a positive ω-periodic solution of system (1.3) with initial conditions (1.4) and (1.5). This completes the proof. □

4. Numerical simulations

In the section, we give some examples to illustrate the feasibility of our main results in Theorems 2.1, 2.2 and 3.1.

Example 1. In system (1.3), let $\alpha_1(t) = 3 + \sin t$, $\gamma_1 = 0.3$, $\beta_1 = 2$, $a_1 = 4$, $m = 2$, $\alpha_2(t) = 2 + \sin t$, $\gamma_2 = 0.1$, $\beta_2 = 0.6$, $\tau_1 = 0.5$, $\tau_2 = 0.3$. It is easy to verify that the coefficients of system (1.3) satisfy (H_3). By Theorem 2.1, system (1.3) is permanent; by Theorem 3.1, we know that system (1.3) has at least one strictly positive 2π-periodic solution. Taking

$$(\phi_1(\theta), \phi_2(\theta), \psi_1(\theta), \psi_2(\theta)) = (k_1, 0.6, k_2, 0.6), \tag{4.1}$$

where

$$\begin{cases} k_1 = 6(1-\mathrm{e}^{-0.15}) + 60[-1+\mathrm{e}^{-0.15}(0.3\sin 0.5+\cos 0.5)]/109, \\ k_2 = \dfrac{12}{5}(1-\mathrm{e}^{-0.03}) + 12[-1+\mathrm{e}^{-0.03}(\cos 0.3+0.1\sin 0.3)]/101. \end{cases} \tag{4.2}$$

Numerical integration of system (1.3) with above coefficients can now be carried out using standard algorithms. As shown in Fig. 1, numerical simulation also suggests that system (1.3) with the coefficients above admits at least one strictly positive 2π-periodic solution.

Example 2. In system (1.3), we let $\alpha_1(t) = 3 + \sin t$, $\gamma_1 = 0.3$, $\beta_1 = 2$, $a_1 = 4$, $m = 2$, $\alpha_2(t) = 2 + \sin t$, $\gamma_2 = 0.1$, $\beta_2 = 3.1$, $\tau_1 = 0.8$, $\tau_2 = 0.6$. In the case, by Theorem 2.2 we can see that the adult predator will go to extinction. Taking

$$(\phi_1(\theta), \phi_2(\theta), \psi_1(\theta), \psi_2(\theta)) = (k_3, 0.6, k_4, 0.6), \tag{4.3}$$

where

$$\begin{cases} k_3 = 6(1-\mathrm{e}^{-0.24}) + 60[-1+\mathrm{e}^{-0.24}(0.3\sin 0.8+\cos 0.8)]/109, \\ k_4 = \dfrac{12}{5}(1-\mathrm{e}^{-0.06}) + 12[-1+\mathrm{e}^{-0.06}(\cos 0.6+0.1\sin 0.6)]/101. \end{cases} \tag{4.4}$$

Numerical simulation also confirms that the adult predator population goes to extinction (see Fig. 2).

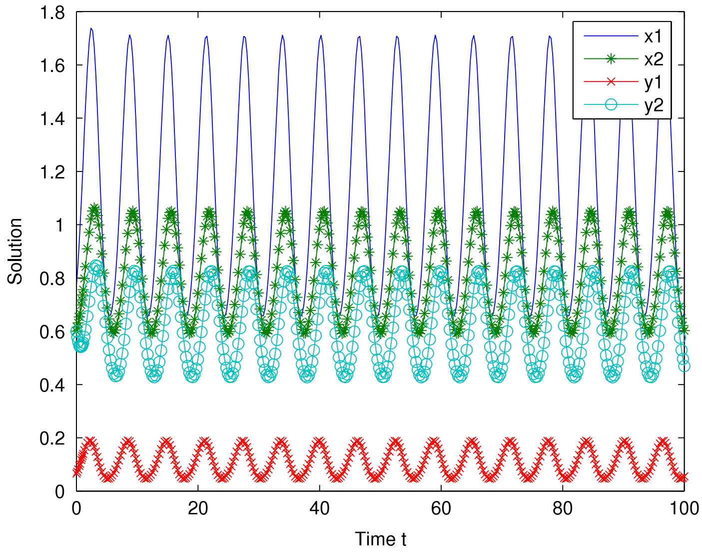

Fig. 1. The periodic solution found by numerical integration of system (1.3) with $\alpha_1(t) = 3 + \sin t, \gamma_1 = 0.3, \beta_1 = 2, a_1 = 4, m = 2, \alpha_2(t) = 2 + \sin t, \gamma_2 = 0.1, \beta_2 = 0.6, \tau_1 = 0.5, \tau_2 = 0.3, (\phi_1(\theta), \phi_2(\theta), \psi_1(\theta), \psi_2(\theta)) = (k_1, 0.6, k_2, 0.6)$, where k_1 and k_2 are defined in (4.2).

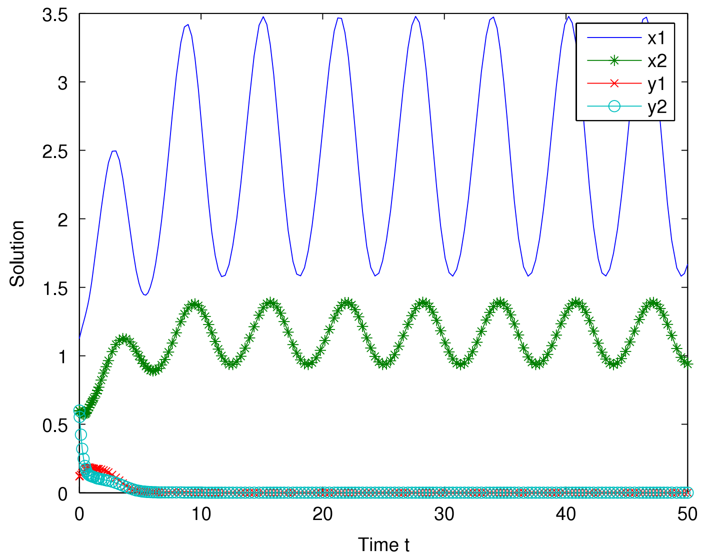

Fig. 2. The temporal solution found by numerical integration of system (1.3) with $\alpha_1(t) = 3 + \sin t, \gamma_1 = 0.3, \beta_1 = 2, a_1 = 4, m = 2, \alpha_2(t) = 2 + \sin t, \gamma_2 = 0.1, \beta_2 = 3.1, \tau_1 = 0.8, \tau_2 = 0.6, (\phi_1(\theta), \phi_2(\theta), \psi_1(\theta), \psi_2(\theta)) = (k_3, 0.6, k_4, 0.6)$, where k_3 and k_4 are defined in (4.4).

Example 3. In system (1.3), we let $\alpha_1(t) = 3 + \sin t, \gamma_1 = 0.3, \beta_1 = 2, a_1 = 4, m = 2, \alpha_2(t) = 2 + \sin t, \gamma_2 = 0.1, \beta_2 = 1, \tau_1 = 0.8, \tau_2 = 0.6$. It is easy to verity that (H_3) does not hold for system (1.3). In the case, we cannot get any information by Theorems 2.1 and 3.1. However, if we take $(\phi_1(\theta), \phi_2(\theta), \psi_1(\theta), \psi_2(\theta)) = (k_3, 0.6, k_4, 0.6)$, where k_3 and k_4 are defined in (4.4), numerical simulation suggests that system (1.3) with above coefficients is still permanent and admits at least one strictly positive 2π-periodic solution (see Fig. 3).

Example 4. In system (1.3), we let $\alpha_1(t) = 3 + \sin t, \gamma_1 = 0.3, \beta_1 = 2, a_1 = 4, m = 2, \alpha_2(t) = 2 + \sin t, \gamma_2 = 0.1, \beta_2 = 2.7, \tau_1 = 0.8, \tau_2 = 0.6$. It is easy to verity that $\alpha_2^M e^{-\gamma_2^L \tau_2} < \beta_2^L$ does not hold for system (1.3). In the case, we cannot get information by Theorem 2.2. However, if we take $(\phi_1(\theta), \phi_2(\theta), \psi_1(\theta), \psi_2(\theta)) = (k_3, 0.6, k_4, 0.6)$, where k_3 and k_4 are defined in (4.4), numerical simulation suggests that the adult predator population goes to extinction (see Fig. 4).

5. Conclusion

In this paper, we have studied the existence of positive periodic solutions and the permanence of system (1.3) in which the coefficients are periodic and we also give the sufficient condition that the adult predator population goes to extinction.

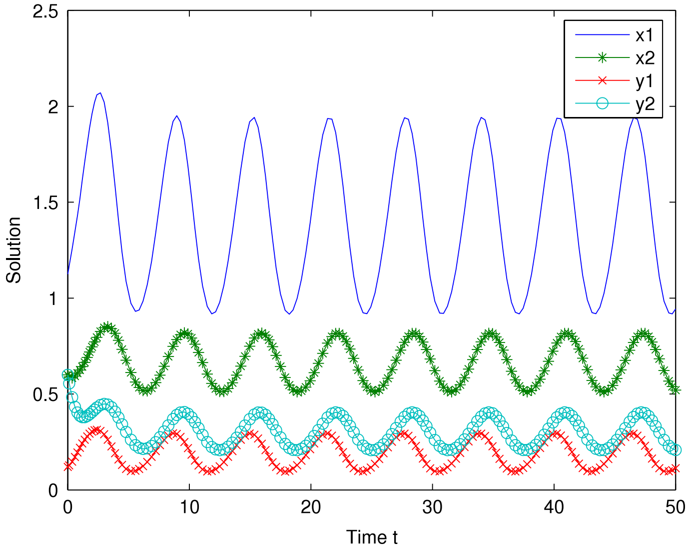

Fig. 3. The periodic solution found by numerical integration of system (1.3) with $\alpha_1(t) = 3 + \sin t, \gamma_1 = 0.3, \beta_1 = 2, a_1 = 4, m = 2, \alpha_2(t) = 2 + \sin t, \gamma_2 = 0.1, \beta_2 = 1, \tau_1 = 0.8, \tau_2 = 0.6, (\phi_1(\theta), \phi_2(\theta), \psi_1(\theta), \psi_2(\theta)) = (k_3, 0.6, k_4, 0.6)$, where k_3 and k_4 are defined in (4.4).

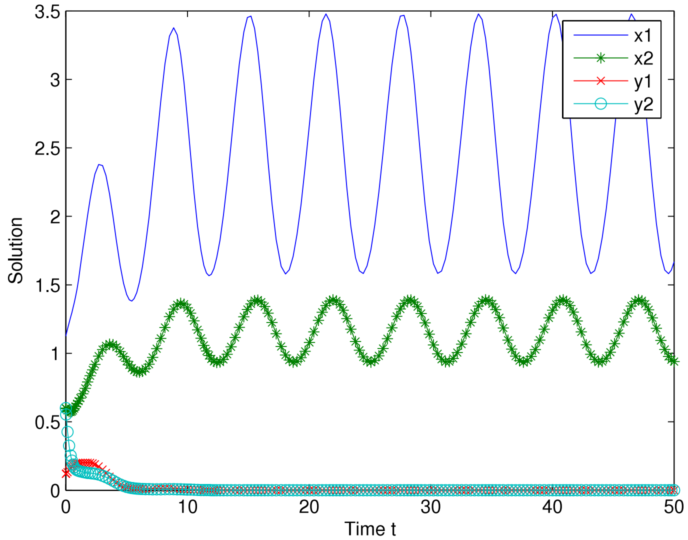

Fig. 4. The temporal solution found by numerical integration of system (1.3) with $\alpha_1(t) = 3 + \sin t, \gamma_1 = 0.3, \beta_1 = 2, a_1 = 4, m = 2, \alpha_2(t) = 2 + \sin t, \gamma_2 = 0.1, \beta_2 = 2.7, \tau_1 = 0.8, \tau_2 = 0.6, (\phi_1(\theta), \phi_2(\theta), \psi_1(\theta), \psi_2(\theta)) = (k_3, 0.6, k_4, 0.6)$, where k_3 and k_4 are defined in (4.4).

By using some comparison technique, we have presented some results on the permanence and extinction of the system. Using Gaines and Mawhin's continuation theorem of coincidence degree theory, we prove that for system (1.3) there exist positive periodic solutions and they are also permanent.

On the one hand, numerical simulations indicate that our results are right; On the other hand, we would like to mention here that Examples 3 and 4 show that our results in Theorems 2.1, 2.2 and 3.1 have room for improvement.

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