Symmetry Groups of Linear Partial Differential Equations and Representation Theory: The Laplace and Axially Symmetric Wave Equations

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Received June 28, 1999; revised October 25, 1999

We examine the Lie point symmetry groups of two important equations of mathematics and mathematical physics. We establish that the action of the symmetry groups are in fact equivalent to principal series representations of the underlying group. Some applications are given.

Key Words: symmetry groups; PDEs; harmonic analysis.

1. INTRODUCTION

The origins of the theory of Lie groups are to be found in Lie's attempt to develop a method of integrating differential equations using group properties. Lie's analysis led him to consider one parameter groups of transformations of PDEs which preserve solutions. These groups usually turn out to be what are now known as Lie groups.

A symmetry of a differential equation is an automorphism of the solution space. In other words a map which takes a solution of the differential equation and transforms it to another solution. Most work on symmetries of differential equations involves the study of groups of symmetries, though one can also consider symmetries which do not form groups. For example, if we have a solution \( u(x, t) \) of the heat equation \( u_{xx} = u_t \), then the map \( \tau_a: u(x, t) \mapsto u(x + a, t) \) is a symmetry, since \( u(x + a, t) \) is also a solution. We regard \( \tau_a \) as defining a one parameter group. There are many more symmetries of the heat equation which can be obtained by Lie's prolongation algorithm.

1The author thanks Professor A. H. Dooley and the Australian Research Council. Some of this work first appeared in the author's Ph.D. thesis. E-mail: Mark.Craddock@uts.edu.
Briefly, the prolongation algorithm works as follows. Consider a PDE in \( n \) variables

\[
\mathcal{L}(x, D^*) u(x) = f(x),
\]

where \( D^* = \frac{\partial^{[\alpha]} x_1 \cdots x_n}{\partial x_1 \cdots \partial x_n} \)

\( \alpha \) a multindex, and a vector field

\[
v = \sum_{k=1}^{n} \tilde{e}_k(x, u) \frac{\partial}{\partial x_k} + \phi(x, u) \frac{\partial}{\partial t}.
\]

The transformations generated by (1.2) are Lie point symmetries of (1.1) if and only if the following condition holds.

\[
pr^nu[\mathcal{L}(x, D^*) u(x) - f] = 0
\]

whenever \( u \) is a solution of (1.1). \( pr^nu \) is called the \( n \)th prolongation of \( v \) (hence prolongation algorithm). It is defined in such a way as to ensure that the group of transformations it generates take \( u \) and its \( n \)th derivatives to the \( n \)th derivatives of \( u^* \), where \( u^* \) is the function \( u \) transformed under the group generated by \( v \). Essentially we just insist that the chain rule hold.

The rigorous justification of this method and an explicit formula for \( pr^nu \) is contained in Olver’s book [9]. Olver [9] provides probably the best modern account of Lie’s theory and its many applications. It also has numerous examples.

The important point to keep in mind is that computation of Lie point symmetries is in practice, an entirely straightforward procedure. Indeed many computer packages exist for determining point symmetries.

The symmetries that Lie’s prolongation algorithm produces are frequently quite complex and unexpected. As such they yield valuable information about the PDE in question. However, it has usually been thought that they can only be defined as local groups of symmetries and do not lie within the realm of representation theory. The purpose of this paper, and its predecessor is to demonstrate that there is a strong connection between the group of point symmetries that Lie’s method produces and the representation theory of the group in which the symmetries are contained. In fact Lie’s prolongation algorithm very often yields representations of the underlying groups.

In a previous paper [4] we studied the symmetry groups of three important linear partial differential equations of mathematical physics. These were the Schrödinger, heat and a particular Fokker–Planck equation.
In the case of the Schrödinger equation we showed that the group of point symmetries is in fact a semi direct product of the double cover of $SL(2, \mathbb{R})$ and the three dimensional Heisenberg group. Moreover $G = H_3 \rtimes SL(2, \mathbb{R})$ is a global group of symmetries, and if we denote the symmetry operator that Lie’s prolongation algorithm gives by $\sigma$, then $\sigma$ is a genuine irreducible representation of $G$. (We refer to $\sigma$ as the Lie representation of $G$). Further there exists a unitary representation $\{ T_1, L^2(\mathbb{R}) \}$ of $G$, and a unitary operator $A$ (built essentially from the fundamental solution of the Schrödinger equation) such that

$$\sigma(g)Af(x, t) = (AT_1(g)f)(x, t)$$

(1.4)

for all $g \in G$ and $f \in L^2(\mathbb{R})$.

For the Fokker–Planck and heat equations analogous results hold, although $\sigma$ is no longer unitary and the representation space is now a space of distributions rather than $L^2(\mathbb{R})$. Analogous results hold for the $n$-dimensional versions of the heat and Schrödinger equations, and Schrödinger’s equation for several important nonzero potentials.

In this paper we present similar results for two different equations. Namely the axially symmetric wave equation and the Laplace equation on $\mathbb{R}^2$. These equations are respectively hyperbolic and elliptic. For the axially symmetric wave equation we have complete results. For Laplace’s equation we have only partial results. The essential reason for the incompleteness of the results for the Laplace equation on $\mathbb{R}^2$ is that the group of point symmetries is infinite dimensional, whereas the groups for the wave equation, heat equation, etc. are finite dimensional. Nevertheless the results we have suggest strongly that the phenomenon we are describing is in fact of a general nature. Namely that for linear PDEs the group of point symmetries that Lie’s algorithm produces is equivalent to the action of a representation of the underlying group on some locally convex topological vector space. This suggests that one may exploit the machinery of representation theory to study solutions of PDEs in a very direct manner. We will present some results which utilise the connection between the Lie symmetries and a representation of the underlying group, to illustrate the basic ideas.

2. THE AXIALLY SYMMETRIC WAVE EQUATION

As our first, and most complete case, we consider the axially symmetric wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{1}{x} \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial t^2} = 0.$$  (2.1)
Equation (2.1) is a hyperbolic PDE which arises in a number of areas of mathematical physics. It has a four dimensional Lie algebra of symmetries which may readily be computed. We state the result and refer the reader to Bluman and Kumei [1] for the details.

**Theorem 2.1.** The symmetry algebra of (2.1) is spanned by the vector fields

\[ v_1 = \frac{\partial}{\partial t} \]  
\[ v_2 = x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} - \frac{1}{2} u \frac{\partial}{\partial u} \]  
\[ v_3 = 2xt \frac{\partial}{\partial x} + (x^2 + t^2) \frac{\partial}{\partial t} - tu \frac{\partial}{\partial u} \]  
\[ v_4 = \frac{\partial}{\partial u} \]  
\[ v_5 = \varkappa(x, t) \frac{\partial}{\partial \varkappa} \]  

where \( \varkappa \) is any solution of (2.1).

We will be concerned with the vector fields \( \{ v_1, v_2, v_3, v_4 \} \). The infinite dimensional Lie algebra generated by the \( v_s \) reflects the fact that the equation is linear and one can add solutions together to form new solutions.

The next result is an elementary calculation with commutator tables.

**Lemma 2.2.** \( \{ v_1, v_2, v_3, v_4 \} \) generate a copy of the Lie algebra of \( \mathfrak{sl}_2(\mathbb{R}) \oplus \mathbb{R} \).

**Proof.** Fix the following basis for \( \mathfrak{sl}_2 \),

\[ X_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \]  

and let \( X_4 \) be any basis for the one dimensional Lie algebra of \( \mathbb{R} \). If one constructs the commutator tables for \( \{ X_1, X_2, X_3, X_4 \} \) and \( \{ v_1, v_2, v_3, v_4 \} \), we see that there is a Lie algebra isomorphism which send \( v_k \) to \( X_k \), \( k = 1, ..., 4 \).
The Lie group of symmetries spanned by \( v_1, \ldots, v_4 \) is locally isomorphic to \( SL(2, \mathbb{R}) \times \mathbb{R} \).

Our aim is to show that the symmetries generated by the above vector fields is in fact equivalent to an action of a representation of the principal series of \( SL(2, \mathbb{R}) \). In order to show this we need to know what the Lie point symmetries do to solutions of (2.1). This is the content of the next result. The action of the symmetries on solutions we refer to as the Lie representation.

**Proposition 2.4.** The Lie representation \( \tau \), obtained from exponentiating \( v_1, \ldots, v_4 \) is given by

\[
\begin{align*}
\tau(\exp \epsilon v_1) u(x, t) &= u(x, t - \epsilon) \\
\tau(\exp \epsilon v_2) u(x, t) &= e^{-(1/2)\epsilon} u(e^{-\epsilon}x, e^{-\epsilon}t) \\
\tau(\exp \epsilon v_3) u(x, t) &= \frac{1}{\sqrt{1 + 2\epsilon t - \epsilon^2(x^2 - t^2)}} \\
&\times u \left( \frac{x}{1 + 2\epsilon t - \epsilon^2(x^2 - t^2)}, \frac{t - \epsilon(x^2 - t^2)}{1 + 2\epsilon t - \epsilon^2(x^2 - t^2)} \right) \\
\tau(\exp \epsilon v_4) u(x, t) &= e^{\epsilon}u(x, t),
\end{align*}
\]

where \( u \) is any solution of (2.1).

The proof is once more straightforward. The content of this result is that if we have a solution of the axially symmetric wave equation, and we transform it according to Proposition 2.4, we will still have a solution, at least if \( \epsilon \) is not too large. In fact it will be an immediate consequence of what follows that one may choose \( \epsilon \) arbitrarily if we restrict the solutions \( u \) to lie in an appropriate Hilbert space.

In our previous work \([4]\) we saw that the Lie symmetries of certain linear PDEs can be obtained from the representations of the underlying group. In this case we are interested in \( SL(2, \mathbb{R}) \), and so we need to know its representation theory if we are to obtain the analogous results here. The representations we need come from the principal series. These are well known and we take as our reference Knapp \([7, \text{p.} 35]\).

**Theorem 2.5.** The full nonunitary principal series of \( SL(2, \mathbb{R}) \) is indexed by pairs \( (\pm, w) \), where \( w = u + iv \in \mathbb{C} \).
The Hilbert space is $L^2(\mathbb{R}, (1 + x^2)^{\Re u} \, dx)$. \( \mathcal{P}^{\pm, w} \) is not unitary except for the case when \( u = 0 \). For $0 < u < 1$ it can be renormed to become unitary. This gives the complementary series.

**Remark.** Taking the tensor product of a character $\chi_\lambda(r) = e^{i\lambda r}, \lambda \in \mathbb{C}$ of $\mathbb{R}$ with $\mathcal{P}^{\pm, w}$, yields the complete irreducible representations of $G = SL(2, \mathbb{R}) \times \mathbb{R}$. Set $\mathcal{P}^{\pm, w} = \chi_\lambda \otimes \mathcal{P}^{\pm, w}$.

We will be interested in the case $w = 0, \lambda = 1$, and “+.” It will turn out that the Lie representation $\tau$ given in Proposition 2.4 and $\mathcal{P}^{+, 0, 1}$ are equivalent representations. For convenience we will refer to $\mathcal{P}^{+, 0, 1}$ as $\mathcal{P}$. We need the following simple observation.

**Corollary 2.6.** The action of $\mathcal{P}$ on functions in $L^2(\mathbb{R})$ for the one parameter groups $\exp(\varepsilon X_k), k = 1, \ldots, 4$ is

\[
\begin{align*}
\mathcal{P}(\exp \varepsilon X_1) f(x) &= f(x - \varepsilon) \quad (2.9) \\
\mathcal{P}(\exp \varepsilon X_2) f(x) &= e^{(1/2) \varepsilon^2} f(e^\varepsilon x) \quad (2.10) \\
\mathcal{P}(\exp \varepsilon X_3) f(x) &= \frac{1}{|1 + \varepsilon x|} f\left(\frac{x}{|1 + \varepsilon x|}\right) \quad (2.11) \\
\mathcal{P}(\exp \varepsilon X_4) f(x) &= e^\varepsilon f(x). \quad (2.12)
\end{align*}
\]

We now construct a representation of $G$ which is unitarily equivalent to $\mathcal{P}$, by conjugating with the Fourier transform. It is this new representation which we will finally be interested in.

**Definition 2.7.** The representation $\mathcal{F}$ of $G$ is defined by

\[
\mathcal{F}(g) f(y) = (\mathcal{F} \mathcal{P}(g) \mathcal{F}^{-1} f)(y) \quad (2.13)
\]

for $f \in L^2(\mathbb{R})$, where the Fourier transform is given by

\[
\mathcal{F} f(y) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x) e^{-2\pi i y x} \, dx
\]

We immediately have the following easy result.
Lemma 2.8. \( \{ \mathcal{H}, L^2(\mathbb{R}) \} \), has the following properties.

\[(\mathcal{H}(\exp \alpha x_1) f)(y) = e^{-2\alpha y} f(y)\]  
\[(\mathcal{H}(\exp \alpha x_2) f)(y) = e^{i(1/2)\alpha} f(\alpha y)\]  
\[(\mathcal{H}(\exp \alpha x_3) f)(y) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{1 + \epsilon s} f(\epsilon y) e^{-2\alpha y} dy\]  
\[(\mathcal{H}(\exp \alpha x_4) f)(y) = e^{\alpha} f(y)\]

and \( f^\vee \) denotes the inverse Fourier transform. Moreover \( \mathcal{H} \) is an irreducible representation whose restriction to \( SL(2, \mathbb{R}) \) is unitary.

The representation \( \mathcal{H} \) is equivalent to the Lie representation \( \tau \). In order to establish this we need an intertwining operator. To construct such an operator we require a map from \( L^2(\mathbb{R}) \) to the solution space of (2.1). This is provided by straightforward Fourier analysis.

Theorem 2.9. Let \( f \in L^2(\mathbb{R}) \) and let \( J_0 \) be the zeroth order Bessel function of the first kind. Then a solution of

\[ xu_{tt} = xu_{xx} + u_x \]

bounded at \( x = 0 \), with \( u(x, 0) = f(x) \) is given by

\[ u(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(y) J_0(2xy) e^{2\alpha y} dy. \]  

We shall write \( u(x, t) = (Af)(x, t) \) for (2.18).

Proof. We take the Fourier transform of (2.1) in \( t \) which yields the zeroth order Bessel equation

\[ x^2 \hat{u}_{xx} + xu_x + x^2 y^2 \hat{u} = 0. \]  

Solving, taking the solution bounded at the origin, then applying the inverse Fourier transform yields (2.18).

We may construct a Hilbert space of solutions of (2.1) with the aid of the operator \( A \).

Definition 2.10. Let

\[ \mathcal{H} = \{ u : u(x, t) = Af(x, t), f \in L^2(\mathbb{R}) \} \]
Define a norm on $\mathcal{H}$ by

$$
|u| = \frac{\sqrt{2}}{\|J_0\|_2} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(y) J_0(2yx)|^2 \sqrt{|y|/|x|} \, dx \, dy \right)^{1/2},
$$

where

$$
\|J_0\|_2 = \left( \int_{-\infty}^{\infty} |J_0(2yx)|^2 |x|^{-1/2} \, dx \right)^{1/2}.
$$

Note. The factor of $|x|^{-1/2}$ in the integrals is to guarantee convergence at infinity.

From this definition it is easy to prove the next few results.

**Lemma 2.11.** $A$ is an isometric operator from $L^2(\mathbb{R})$ to $\mathcal{H}$.

**Proof.** The motivation for defining the norm on $\mathcal{H}$ in this somewhat unusual way is the Plancherel theorem which tells us that for each fixed $x$ we have

$$
\int_{-\infty}^{\infty} |u(x, t)|^2 \, dt = \int_{-\infty}^{\infty} |f(y) J_0(2yx)|^2 \, dy.
$$

We include the factor $\sqrt{|y|/|x|}$ in the integral essentially to ensure convergence and guarantee that we have an isometry. The fact that $A$ is an isometry is now an easy calculation. Let

$$
c = \frac{2}{\|J_0\|_2^2},
$$

then

$$
\|Af\|^2 = c \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(y) J_0(2yx)|^2 \sqrt{|y|/|x|} \, dx \, dy
$$

$$
= c \int_{-\infty}^{\infty} \sqrt{|y|} \int_{-\infty}^{\infty} |f(y)|^2 \, dy \int_{-\infty}^{\infty} |J_0(2yx)|^2 |x|^{-1/2} \, dx
$$

$$
= c \int_{-\infty}^{\infty} \sqrt{|y|} \int_{-\infty}^{\infty} |f(y)|^2 \, dy \int_{-\infty}^{\infty} |J_0(u)|^2 \left(|u|/|4y|\right)^{1/2} \, du
$$

$$
= \int_{-\infty}^{\infty} |f(y)|^2 \, dy
$$

$$
= \|f\|_2^2.
$$
Lemma 2.12. \((\mathcal{H}, \| \cdot \|)\) is a Hilbert space.

Proof. The fact that \(\| \cdot \|\) defines an inner product on \(\mathcal{H}\) by the polarisation identity is an easy exercise. The completeness of \(\mathcal{H}\) follows from that of \(L^2\) in the following manner. Pick a Cauchy sequence \(f_n \in L^2\). The fact that \(A\) is an isometry then says that \(Af_n\) is a Cauchy sequence in \(\mathcal{H}\). We clearly must have \(Af_n \to Af \in \mathcal{H}\) where \(f_n \to f \in L^2\). Conversely, given a Cauchy sequence \(u_n \in \mathcal{H}\) there exists a Cauchy sequence \(f_n \in L^2\) such that \(Af_n = u_n\). If \(f_n \to f \in L^2\); let \(Af = u\), then by the fact that \(A\) is an isometry we must have \(u_n \to u\). Hence \(\mathcal{H}\) is complete. 

We can now prove the equivalence of the Lie representation \(\tilde{\tau}\) with \(\tau\).

First a lemma, which may be found in Gelfand and Shilov [6, p. 185].

Lemma 2.13. Let \(J_0\) be the Bessel function of the first kind of order zero, and let

\[
(1 - x^2)^{-1/2} = \begin{cases} 
(1 - x^2)^{-1/2} & \text{for } |x| \leq 1 \\
0 & \text{for } |x| > 1
\end{cases}
\]

Then

\[
\int_{-1}^{1} (1 - x^2)^{-1/2} e^{2ix\xi} dx = J_0(2\xi). \quad (2.21)
\]

For \(f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})\), we may, by the convolution theorem, write the operator \(Af\) as a convolution of the Fourier transform of \(f\) and the function \((x^2 - t^2)^{-1/2}\), where

\[
(x^2 - t^2)^{-1/2} = \begin{cases} 
(x^2 - t^2)^{-1/2} & \text{for } |t| \leq |x| \\
0 & \text{for } |t| > |x|\end{cases} \quad (2.22)
\]

This function is of considerable importance in the study of the axially symmetric wave equation as the next result suggests.

Theorem 2.14.

\[
E(x, t) = (x^2 - t^2)^{-1/2} \quad (2.23)
\]

is a fundamental solution of the axially symmetric wave equation.

The proof of this theorem may be pieced together from the preceding results and is left to the reader.

We are now in a position to state and prove the main result of this section.
Theorem 2.15. Let $G = SL(2, \mathbb{R}) \times \mathbb{R}$ and let $\{ \mathcal{F}, L^2(\mathbb{R}) \}$ be as in Definition 2.7. Let $\tau$ be the Lie representation of $G$. Then for all $g \in G$ and for all $f \in L^2(\mathbb{R})$ we have

$$(\tau(g) Af)(x, t) = (A \mathcal{F}(g) f)(x, t)$$ (2.24)

Before proceeding to the proof of the theorem, let us make some comments. As noted above, in previous work [4] we showed that the Lie point symmetries of the Schrödinger, heat and Fokker-Planck equations are equivalent to a representation of the underlying symmetry group via an operator which is essentially convolution with a fundamental solution of the equation and the Fourier transform. This is precisely the result we have here for the axially symmetric wave equation. The significance is that the Lie prolongation algorithm for computing point symmetries is in fact constructing representations of a Lie group on the solution space of the given equation. Moreover, these representations are typically irreducible (see below). This suggests that one may employ results from representation theory to study solutions of differential equations by using the equivalence of the group symmetries and some standard analytic representation of the group. A natural question to ask is under what circumstances is it true that Lie’s algorithm will produce a representation equivalent to a “standard” representation (in the case above, a principal series representation). In the case of the Schrödinger equation, the Segal-Shale-Weil representation, also called the oscillator representation by some authors)? At this point a definitive answer is not yet known, though the unitary case may be dealt with to some degree. We will discuss this question in a forthcoming publication.

One may also reverse the process. Consider the situation where one has a mapping $A$ from some function space, say $L^2$ for concreteness, and a PDE

$$\mathcal{L}(x, D^*) u = 0, \quad (2.25)$$

such that if $f \in L^2$, then $u = Af(x)$, solves our PDE. Then we may do the following. Given a group $G$ and a representation $\pi$ of $G$ on $L^2$, we may define symmetries of the PDE by the construction: Set $\rho$ to be a symmetry of (2.25) by

$$\rho(g) u(x) = (A \pi(g) f)(x)$$ (2.26)

$\rho$ will, subject to topological constraints, inherit the properties of the representation $\{ \pi, L^2 \}$. The natural question to ask then is under what circumstances is $\rho$ the representation corresponding to a group of Lie point symmetries? It turns out that it is not always the case that (2.26) defines point symmetries. There are also generalised symmetries which arise when...
one allows the underlying vector fields (1.2) to be functions of the derivatives of the dependent variables. (See the book by Olver for an extensive discussion of generalised symmetries of differential equations). Sometimes a symmetry constructed via (2.26) will be a point symmetry and sometimes it will be a generalised symmetry.

For example let us apply the relation (2.26) to the linearised KdV equation

\[ u_t = u_{xxx}. \]  

(2.27)

The Lie algebra of point symmetries is four dimensional. A basis is

\[ v_1 = \frac{\partial}{\partial x}, \quad v_2 = \frac{\partial}{\partial t}, \quad v_3 = x \frac{\partial}{\partial x} + 3t \frac{\partial}{\partial t}, \quad v_4 = u \frac{\partial}{\partial u}. \]

We use the fact that if \( f \in L^2(\mathbb{R}) \) then

\[ u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) e^{-iy^3 + ixy} dy \]

defines a solution of the linearised KdV equation. (One needs to be careful about the convergence of the integral, which must be taken in the \( L^2 \) sense).

Now we define a symmetry by allowing the reals to act on \( L^2(\mathbb{R}) \) by translation, viz:

\[ \sigma(\epsilon) u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y - \epsilon) e^{-i(y^3 + ixy)} dy \]  

(2.28)

Perhaps surprisingly this simple group action does not produce a point symmetry. A simple application of the convolution theorem shows

\[ \sigma(\epsilon) u(x, t) = e^{ix^3 - i\epsilon t} \int_{-\infty}^{\infty} u(x - 3\epsilon^2 t - y, t) K_{3\epsilon t}(y) dy. \]

This clearly does not come from one of the point symmetries listed above. Indeed if we differentiate (2.28) with respect to \( \epsilon \), then set \( \epsilon = 0 \) we obtain the generalised vector field

\[ v = i \left( xu + 3t \frac{\partial^2 u}{\partial x^2} \right) \frac{\partial}{\partial u}. \]  

(2.29)

Hence (2.28) is actually a generalised symmetry. The interested reader may readily construct many other examples.

This idea is discussed further in Craddock and Dooley [5].
Proof. In order to establish the theorem we will prove the equivalence for the basis vectors of the Lie algebra. That is, we will show that
\[
(\tau(\exp v)A\mathbf{f})(x, t) = (A\mathcal{F}(\exp vX_i)\mathbf{f})(x, t)
\]
for \(i = 1, \ldots, 4\). (We are using the correspondence between \(v_i\) and \(X_i\) as elements of the Lie algebra). Then the fact that any element of \(\text{SL}(2, \mathbb{R})\) may be written as a product of the form \(\exp v_1 X_1 \cdots \exp v_n X_n\) will give the result. We begin with \(X_1\). We will prove results for \(f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})\). That fact that \(L^1 \cap L^2\) is dense in \(L^2\) allows us to extend the results to the whole of \(L^2\).

(1) \[
(\tau(\exp v_1)A\mathbf{f})(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(y) J_0(2y|x|) e^{2iy(t-x)} dy.
\]

Now
\[
(A\mathcal{F}(\exp v_1)\mathbf{f})(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(y) e^{-2iy} J_0(2y|x|) e^{2iyt} dy.
\]

\[
= (\tau(\exp v_1)A\mathbf{f})(x, t).
\]

(2) \[
(\tau(\exp v_2)A\mathbf{f})(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-2f(y)} J_0(2ye^{-t}) e^{2iye^{-t}} dy
\]

and by Lemma 2.8 we have
\[
(A\mathcal{F}(\exp v_2)\mathbf{f})(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{1/2} f(e^y) J_0(2y|x|) e^{2iyt} dy
\]

\[
= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-1/2} f(u) J_0(2ue^{-t}) e^{2iue^{-t}} du
\]

\[
= (\tau(\exp v_2)A\mathbf{f})(x, t).
\]

(3) The \(X_4\) case is trivial:
\[
(\tau(\exp v_4)A\mathbf{f})(x, t) = e^A\mathbf{f}(x, t)
\]

\[
= A(\mathcal{F}(\exp vX_4)\mathbf{f})(x, t).
\]
Finally we have to deal with the $X_\lambda$ case. This involves quite a complex calculation. Observe first that Lemma 2.8 gives
\[
\langle \mathcal{D}(\exp \varepsilon X_\lambda) f(x, t) \rangle
\]
\[
= \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{|1 + \varepsilon t|} f(y) e^{-2\varepsilon y} J_0(2\varepsilon x) e^{2\varepsilon y \varepsilon} \, ds \, dy
\]
\[
= \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{|\varepsilon t + 1|} f(y) e^{\varepsilon (r(1 + \varepsilon t) - 1)/|\varepsilon t + 1|} J_0(2\varepsilon x) e^{2\varepsilon y \varepsilon} \, dr \, dy
\]
(where we have made the change of variables $r = \frac{1}{1 + \varepsilon t}$)
\[
= \frac{1}{\pi^{3/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{|\varepsilon t + 1|} f(u) e^{2\varepsilon u} e^{-2\varepsilon y(r(1 + \varepsilon t) - 1)/|\varepsilon t + 1|} J_0(2\varepsilon x) e^{2\varepsilon y \varepsilon} \, du \, dr \, dy
\]
\[
\times J_0(2\varepsilon x) \, du \, dy \, dr.
\]
We now employ Fubini's theorem and Lemma 2.14 to obtain
\[
\frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{|\varepsilon t + 1|} f(u) e^{2\varepsilon u} \left( x^2 - \left( \frac{r(1 + \varepsilon t) - t}{\varepsilon t + 1} \right)^2 \right) \left( x^2 - \frac{r^2((1 + \varepsilon t)^2 - \varepsilon^2 x^2)}{\varepsilon t + 1} \right) \left( x^2 - \frac{2t((1 + \varepsilon t)^2 - \varepsilon^2 x^2)}{\varepsilon t + 1} \right) \left( x^2 - \frac{2t((1 + \varepsilon t)^2 - \varepsilon^2 x^2)}{\varepsilon t + 1} \right) \, du \, dr \, dy.
\]
where
\[
G(\varepsilon, x, t) = \left( \frac{x^2}{(1 + \varepsilon t)^2 - \varepsilon^2 x^2} - \frac{t - \varepsilon((1 + \varepsilon t)^2 - \varepsilon^2 x^2)}{(1 + \varepsilon t)^2 - \varepsilon^2 x^2} \right)^{-1/2}.
\]
So we may observe that (2.30)
\[
= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{(1 + \varepsilon t)^2 - \varepsilon^2 x^2}} f(u) J_0 \left( \frac{2ux}{(1 + \varepsilon t)^2 - \varepsilon^2 x^2} \right) \, du.
\]
Next we see that

\[
(\tau(\exp \epsilon X_3) A f)(x, t) = \frac{1}{\sqrt{(1 + \epsilon t)^2 - \epsilon^2 x^2}} \left( Af \right) \left( \frac{x}{(1 + \epsilon t)^2 - \epsilon^2 x^2}, \frac{t + \epsilon(t^2 - x^2)}{(1 + \epsilon t)^2 - \epsilon^2 x^2} \right)
\]

\[
= \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} f(u) e^{2\epsilon \epsilon (t + \epsilon(t^2 - x^2)) / ((1 + \epsilon t)^2 - \epsilon^2 x^2)} J_0 \left( \frac{2\epsilon u}{(1 + \epsilon t)^2 - \epsilon^2 x^2} \right) du
\]

\[
= (A \tilde{\mathcal{H}}(\exp \epsilon X_3) f)(x, t).
\]

Extending to the whole of \( L^2(\mathbb{R}) \) and \( SL(2, \mathbb{R}) \) completes the proof. 

We have thus constructed a new model for the symmetries of the axially symmetric wave equation. Some results immediately follow.

**Corollary 2.16.** \{\tau, \mathcal{H}\} is an irreducible representation of \( G \).

**Corollary 2.17.** \( SL(2, \mathbb{R}) \times \mathbb{R} \) is a global group of symmetries of (2.1).

**Proof.** This follows by the equivalence of \( \tau \) and \( \tilde{\mathcal{H}} \). Since the latter is defined for the whole of \( G \), we can define \( \tau \) for all elements of \( G \) by the equivalence established in Theorem 2.15.

**Corollary 2.18.** The restriction of \( \tau \) to \( SL(2, \mathbb{R}) \) is an irreducible, unitary representation of \( SL(2, \mathbb{R}) \).

The next result follows from the irreducibility of \( \tau \).

**Corollary 2.19.** Pick \( u \in \mathcal{H}, u \neq 0 \). Define

\[
S = \left\{ \sum_{i=1}^{n} \tau(g_i) u, n \in \mathbb{N}; \{ g_i \} \text{ ranges over all finite subsets of } G \right\}.
\]

Then \( \overline{S} = \mathcal{H} \), where \( \overline{S} \) is the closure of \( S \).

Since we have a correspondence between \( L^2(\mathbb{R}) \) and \( \mathcal{H} \) we may use information about \( L^2(\mathbb{R}) \) and the representation \( \mathcal{H} \) to deduce information about \( \mathcal{H} \).
3. THE TWO DIMENSIONAL LAPLACE EQUATION

Laplace’s equation in two dimensions is

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \] (3.1)

Any attempt to analyse the symmetries of (3.1) as we did the axially symmetric wave equation runs into an immediate problem. The algebra of Lie point symmetries, is infinite dimensional, and the full symmetry group is the group of analytic functions on \( \mathbb{C} \). More explicitly, if we let

\[ v = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} \tag{3.2} \]

then \( v \) generates a symmetry of (3.1) if and only if \( \xi \) and \( \eta \) satisfy the Cauchy–Riemann equations. Consequently there are infinitely many nontrivial point symmetries of (3.1).

It is worth noting that according to Miller \[8\], the phenomenon of a linear PDE in \( n \) variables possessing an infinite dimensional Lie algebra of symmetries only occurs in the case \( n = 2 \). If the PDE is nonlinear this is no longer true. There are results on the maximum size of a Lie group of point symmetries admitted by a Linear PDE. See for example \[8\] and \[9\].

Consequently analysing the symmetries of the Laplace equation will be a more difficult task than in the previous cases we have considered. There is also a further difficulty. The previous examples we treated were all evolution equations, where it is natural to consider initial value problems. The Laplace equation is not an equation of evolution and the natural problems involve boundary values.

The reason why this is an important difference is that the intertwining operators we have constructed previously all arose in a natural way out of the solution of an initial value problem by Fourier transform. In the case of the Laplace equation one has to specify boundary conditions, making the task of deriving any intertwining operator more difficult.

Nevertheless the phenomena we have been describing still holds true, at least to some extent. We may consider particular subgroups of the full group of symmetries, and perform an analysis that parallels that for (2.1) in many important aspects.

Let us begin with the following lemma. We omit the proof, which is an easy application of Lie’s theory.

**Lemma 3.1.** Let the Lie algebra of (3.1) be \( \mathfrak{L} \). Then the following vector fields span a finite dimensional subalgebra of \( \mathfrak{L} \).
This is the Lie algebra for the Mobius group.

We next observe that:

**Proposition 3.2.** \( \{v_1, ..., v_7\} \) generate a copy of \( \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{R} \) as a Lie algebra over \( \mathbb{R} \).

**Proof.** We pick a basis for \( \mathfrak{sl}_2(\mathbb{C}) \) as a Lie algebra over \( \mathbb{R} \).

\[
X_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
X_4 = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad X_5 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad X_6 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}
\]

and let \( X_7 \) a basis for the Lie algebra of \( \mathbb{R} \). Calculating the commutator table for \( \{v_i\} \) and \( \{X_i\} \) shows that the map which sends \( v_i \to X_i \) is a Lie algebra isomorphism.

We obviously now need to know the representation theory of \( SL(2, \mathbb{C}) \). This has been completely described and once more we take as our reference Knapp [7, p. 33].
Theorem 3.3. The full non unitary principal series of $SL(2, \mathbb{C})$ is given by

$$R_k, w = \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) = | -bz + d |^{-2} - w \left( \frac{-bz + d}{-bz + d} \right)^{-k} f \left( \frac{az - c}{-bz + d} \right)$$

(3.3)

$$(k, w) \in \mathbb{Z} \times \mathbb{C}.$$ The Hilbert space is $L^2(\mathbb{C}, (1 + |z|^2)^{Re w} \, dx \, dy)$. With this inner product, $R_k, w$ is unitary if and only $Re w = 0$.

Our immediate interest is in the group homomorphism (3.3). The Lie point symmetries generated by the vector fields of Lemma 3.1, generate a group homomorphism of $SL(2, \mathbb{C})$ which is identical with (3.3). To see this we perform a simple comparison. First we need to determine the action of the point symmetries of Lemma 3.1.

Proposition 3.4. Let $\{v_1, ..., v_7\}$ be as in Lemma 3.1, and let the action on solutions of (3.1) they generate be denoted by $\sigma$, then we have

$$\sigma(\exp v_1) u(x, y) = u(x - \epsilon, y)$$

(3.4)

$$\sigma(\exp v_2) u(x, y) = u(x, y - \epsilon)$$

(3.5)

$$\sigma(\exp v_3) u(x, y) = u(x \cos \epsilon - y \sin \epsilon, x \sin \epsilon + y \cos \epsilon)$$

(3.6)

$$\sigma(\exp v_4) u(x, y) = u(e^{-\epsilon x}, e^{-\epsilon y})$$

(3.7)

$$\sigma(\exp v_5) u(x, y) = u \left( \frac{x + \epsilon(x^2 + y^2)}{1 + 2\epsilon x + \epsilon^2(x^2 + y^2)}, \frac{y}{1 + 2\epsilon x + \epsilon^2(x^2 + y^2)} \right)$$

(3.8)

$$\sigma(\exp v_6) u(x, y) = u \left( \frac{x}{1 + 2\epsilon y + \epsilon^2(x^2 + y^2)}, \frac{y + \epsilon(x^2 + y^2)}{1 + 2\epsilon y + \epsilon^2(x^2 + y^2)} \right)$$

(3.9)

$$\sigma(\exp v_7) u(x, y) = e^{\epsilon u}(x, y).$$

(3.10)

If we now take $w = -2$, and look at the representation of $SL(2, \mathbb{C}) \otimes \mathbb{R}$ given by $\mathbb{R}^h \rightarrow \mathbb{J}$, where $\mathbb{J}(r) f(z) = e^r f(z)$ is a one dimensional non unitary representation of $\mathbb{R}$, then we observe that if $f(z) = u(x, y) + iv(x, y)$ then
Comparing the two group actions we see that they involve the same transformation. What is happening is that we are getting an action of $SL(2, \mathbb{C})$ by fractional linear transformation realised on two different spaces. In the case of Proposition 3.4, the action is on the space of solutions of (3.1) and in the case of Theorem 3.3, the action is on $L^2(\mathbb{C}, (1 + |z|^2)^{-2} \, dz)$. Interesting though this fact is, it is not terribly helpful if one wishes to analyse symmetries of the Laplace equations, since $L^2(\mathbb{C}, (1 + |z|^2)^{-2} \, dz)$ contains “few” harmonic functions. Consequently it is not a useful model for solutions of (3.1). Ideally we would like to change the Hilbert space upon which the homomorphism (3.3) acts.
Let us briefly exemplify this approach. Define the fractional linear transformation

\[ T_g z = \frac{az - c}{-bz + d}, \]  

where

\[ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}). \]

Now if \( F(z) \) is analytic \( F(T_g z) \) will fail to be analytic at at most finitely many points. With this in mind we may construct a finite dimensional Hilbert space of analytic functions with finitely many poles.

**Definition 3.5.**

\[ \mathcal{H}_k = \left\{ z \to \frac{\alpha_{-k}}{(z - \beta_{-k})^k} + \frac{\alpha_{-k+1}}{(z - \beta_{-k+1})^{k-1}} + \cdots + \alpha_0 + \alpha_1 z + \cdots \alpha_k z^k; \alpha_i, \beta_i \in \mathbb{C} \right\} \]

For \( f \in \mathcal{H}_k \), define its norm by

\[ \| f \| = \left( \sum_{j=-k}^{k} |\alpha_j|^2 \right)^{1/2}. \]

\( \mathcal{H}_k \) is a finite dimensional Hilbert space of functions analytic at finitely many points and \( \mathcal{H}_k \) is a representation of \( SL(2, \mathbb{C}) \) on \( \mathcal{H}_k \). It is however nonunitary. We could of course consider other such spaces of analytic functions as Hilbert spaces for our representation. This approach seems somewhat limited.

Let us consider a different approach. The group of point symmetries for the 2 dimensional Laplace equation is infinite dimensional. We selected a particular subgroup for our analysis. We can of course pick a smaller subgroup. Take the Lie algebra spanned by \( \{ v_1, v_4, v_5 \} \). It is not difficult to show that these vector fields generate a copy of \( SL_2(\mathbb{R}) \). The principal series of \( SL(2, \mathbb{R}) \) is usually realised on \( L^2(\mathbb{R}, (1 + x^2)^{-1} dx) \) and there is a “natural” mapping from \( L^2(\mathbb{R}) \) to the solution space of the two dimensional Laplace equation arising from the fundamental solution.
Proposition 3.6. If \( f \in L^2(\mathbb{R}) \), then

\[
    u(x, y) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(\xi) e^{-2|\xi|^2 y + 2i\xi} d\xi
\]

\[
    = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) \left[ \frac{1}{2} \frac{\partial}{\partial y} \left( \log(x^2 + (y - \xi)^2) \right) \right] d\xi
\]

\[
    = \mathcal{A} f(x, y)
\]

defines a solution of (3.1).

We will now show that the element of the principal series of \( SL(2, \mathbb{R}) \) indexed by \((+, -1)\) conjugated with the Fourier transform is equivalent to the symmetries generated by exponentiating \( \{v_1, v_4, v_5\} \) under the intertwining operator \( \mathcal{A} \) of Proposition 3.6.

Definition 3.7. \( \{ \widetilde{P}, L^2(\mathbb{R}, (1 + x^2)^{-1} dx) \} \) is the representation of \( SL(2, \mathbb{R}) \) given by

\[
    \widetilde{P} \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x) = \left( \mathcal{F} \mathcal{R}^+, -1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathcal{F}^{-1} f \right)(x)
\]

for

\[
    f \in X(\mathbb{R}) = L^2(\mathbb{R}) \cap L^2(\mathbb{R}, (1 + x^2)^{-1} dx).
\]

For \( f \not\in L^2(\mathbb{R}) \) we extend to \( L^2(\mathbb{R}, (1 + x^2)^{-1} dx) \) by a straightforward density argument.

With this definition the following result is easy and left to the reader.

Lemma 3.8. \( \{ \widetilde{P}, L^2(\mathbb{R}, (1 + x^2)^{-1} dx) \} \) is a genuine representation of \( SL(2, \mathbb{R}) \).

Now we prove a result analogous to Theorem 2.15.

Theorem 3.9. Let \( \sigma \) denote the Lie representation of Proposition 3.4 restricted to the copy of \( SL(2, \mathbb{R}) \) generated by \( \{v_1, v_4, v_5\} \) and let \( \{ \widetilde{P}, L^2(\mathbb{R}, (1 + x^2)^{-1} dx) \} \) be the representation of \( SL(2, \mathbb{R}) \) given by Definition 3.7. Then for all \( g \in SL(2, \mathbb{R}) \) we have the equivalence

\[
    (\sigma(g) \mathcal{A} f)(x, y) = (\mathcal{A} \widetilde{P}(g) f)(x, y)
\]

with the intertwining operator \( \mathcal{A} \) given by Proposition 3.6 for \( y > 0 \).
Proof. The proof runs exactly the same as for Theorem 2.15, and so we give only the calculation for $v_5$. $v_5$ corresponds at the level of the Lie algebra to $X_5$ and so we aim to show that $(\mathcal{A}(\exp v_5))f(x, y) = (\mathcal{A}(\exp v_5))f(x, y)$. Clearly we have

\[
(\mathcal{A}(\exp v_5))f(x, y) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f \left( e^{-2\text{Im} \xi} (1 + \xi y) + 2\text{Re} \xi e^{\frac{s}{1 + \xi y}} \xi \right) e^{-2\text{Im} \xi} ds.
\]  

(3.16)

Hence

\[
(\mathcal{A}(\exp v_5))f(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f \left( e^{-2\text{Im} \xi} (1 + \xi y) + 2\text{Re} \xi e^{\frac{s}{1 + \xi y}} \xi \right) e^{-2\text{Im} \xi} ds
\]

(3.17)

We make the change of variables $r = \frac{1}{1+\xi}$ in the above integral to obtain

\[
(\mathcal{A}(\exp v_5))f(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f \left( (1 - r) e^{-2\text{Im} \xi} (1 + \xi y) + 2\text{Re} \xi e^{\frac{s}{1 + \xi y}} \xi \right)
\]

\[
\left( e^{-2\text{Im} \xi} (1 + \xi y) + 2\text{Re} \xi e^{\frac{s}{1 + \xi y}} \xi \right) e^{-2\text{Im} \xi} ds
\]

\[
\left( e^{2\text{Im} \xi (y - u(1 - \text{Re} \xi))} d\xi + e^{2\text{Im} \xi (y - u(1 - \text{Re} \xi))} d\xi \right)
\]

\[
= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f \left( \frac{y}{y^2 + \left( x - \frac{u}{1 - \text{Re} \xi} \right)^2} \right) \left( \frac{y}{y^2 + \left( x - \frac{u}{1 - \text{Re} \xi} \right)^2} \right)
\]

\[
\left. \frac{du}{(1 - \text{Re} \xi)^2} \right|_{1}^{\infty} + \left. \frac{du}{(1 - \text{Re} \xi)^2} \right|_{-\infty}^{\infty}
\]

\[
= \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) G(x, y, u, \epsilon) e^{2\text{Re} \xi} dt du,
\]
where
\[
G(x, y, u, \varepsilon) = \frac{y}{\left(1 + 2\varepsilon x + \varepsilon^2(x^2 + y^2)\right)}
\]
\[
\left(\begin{array}{c}
1 + 2\varepsilon x + \varepsilon^2(x^2 + y^2) \\
1 + 2\varepsilon x + \varepsilon^2(x^2 + y^2)
\end{array}\right)
\]
\[
\left(\begin{array}{c}
x + \varepsilon(x^2 + y^2) \\
1 + 2\varepsilon x + \varepsilon^2(x^2 + y^2)
\end{array}\right)
\]
Now we make the change of variables
\[
k = u - \frac{x + \varepsilon(x^2 + y^2)}{1 + 2\varepsilon x + \varepsilon^2(x^2 + y^2)}
\]
and apply Fubini's theorem to reduce this to
\[
\frac{1}{\pi} \int_{-\infty}^{\infty} f(t) e^{2i\varepsilon t (x + \varepsilon(x^2 + y^2))} \left(1 + 2\varepsilon x + \varepsilon^2(x^2 + y^2)\right) dt
\]
\[
\int_{-\infty}^{\infty} \frac{ye^{2ik} dk}{\left(1 + 2\varepsilon x + \varepsilon^2(x^2 + y^2)\right)k^2 + \frac{y^2}{1 + 2\varepsilon x + \varepsilon^2(x^2 + y^2)}}
\]
which in turn
\[
= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(t) e^{2i\varepsilon t (x + \varepsilon(x^2 + y^2))} \left(1 + 2\varepsilon x + \varepsilon^2(x^2 + y^2)\right) - 2 |t| y/(1 + 2\varepsilon x + \varepsilon^2(x^2 + y^2)) dt
\]
\[
= (\sigma(\exp i\varepsilon f))(x, y)
\]
and this completes the proof.  

From this result we immediately have

**Corollary 3.10.** \( SL(2, \mathbb{R}) \) is a global group of symmetries for the Laplace equation on \( \mathbb{R}^2 \). Moreover the Lie symmetries act as an irreducible representation on the Hilbert space
\[
\mathcal{H} = \{u : Au = 0; u = \mathcal{A}f(x, y), f \in L^2(\mathbb{R}, (1 + x^2)^{-1}dx), y > 0\},
\]
where the norm on \( \mathcal{H} \) is given by
\[
\|u\|_{\mathcal{H}} = \|\mathcal{A}f\|_{\mathcal{H}} = \|f\|_{L^2(\mathbb{R}, (1 + x^2)^{-1}dx)}.
\]

We could of course also have worked with the vector fields \( \{e_2, e_4, e_6\} \) which also span a copy of \( sl_2 \). We leave the details to the interested reader.
After this partial analysis of the symmetries of the Laplace equation on \( \mathbb{R}^2 \) it is natural to turn to the Laplacian on higher dimensional spaces. We shall not do that here (the details in the case of \( \mathbb{R}^3 \) are contained in the author’s Ph.D. thesis [2]). Instead we make a few observations.

Whereas the Lie algebra of point symmetries for (3.1) is infinite dimensional, for the Laplace equation on \( \mathbb{R}^n, n > 2 \) the Lie algebra of point symmetries is finite dimensional. The general symmetry group is the group of Mobius transformations. For the specific case \( n = 3 \) the Lie algebra of point symmetries is eleven dimensional (excluding the infinite dimensional ideal arising from superposition of solutions). For details of the group symmetries see Sattinger and Weaver [10]. The group may be readily identified with \( SO(4, 1) \). An irreducible, principal series representation of \( SO(4, 1) \) generates the same group action as the point symmetries of this Laplace equation, i.e., exactly the same situation as described by Proposition 3.4 exists. (See for example Wilson [11] for the construction of the principal series of \( SO(4, 1) \).) One can pick subgroups of the full group of symmetries and perform an analogous analysis to the one given above for \( SL(2, \mathbb{R}) \) symmetries of (3.1). However at this stage we have been unable to find a suitable model for the whole group as can be done with evolution equations.

Finally we could also consider the wave equation on, for example, \( \mathbb{R}^2 \). The group of point symmetries is readily identifiable with \( SO(3, 2) \). The analysis in this case follows closely the analysis for the axially symmetric wave equation, with the intertwining operator provided by the standard Fourier transform solution. There are however some important technical problems involved and so we prefer to leave the details to a subsequent publication.

4. APPLICATIONS

Let us now consider an example to illustrate how the interplay between symmetry calculations and representation theory can yield interesting information. Let us say that a linear PDE

\[
P(x, D^x) u = g
\]

\( x \in \Omega \subseteq \mathbb{R}^n \), is integrable if there exists a vector space of functions (or distributions) \( V(\Omega), \Omega \subseteq \Omega \) connected, and a linear map \( A \) defined on \( V \), such that if \( u = Af, f \in V \), then \( u \) solves (4.1).

Clearly for an integrable equation we may define group symmetries by the construction (2.26).
It is obviously of interest to know when a PDE is integrable in the above sense. PDEs possessing fundamental solutions being the most common example of integrable equations. The following simple result is an easy example of using the interplay between group symmetries and representation theory.

**Theorem 4.1.** Let \( P(x, D^a) u = 0 \) be a linear PDE which possesses a Heisenberg group \( H_{2n+1} \) of unitary Lie symmetries, acting as multiplication by \( e^{i\xi} \) on the centre, \( (\lambda \in \mathbb{R}, \lambda \neq 0, \text{and } c \in \text{the centre of } H_{2n+1} \). Then \( P(x, D^a) u = 0 \) is integrable.

**Proof.** The proof is a straightforward application of the Stone–Von Neumann theorem. The Stone–Von Neumann theorem says that all unitary representations of the Heisenberg group which act as multiplication by the centre are unitarily equivalent. Thus let \( H_{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \), then

\[
\pi(a, b, c) f(z) = e^{i\lambda c + i(a, z - b)} f(z - b).
\]

\( f \in L^2(\mathbb{R}^n) \) is a unitary representation of the Heisenberg group. \( (a, b, c) \in H_{2n+1} \). Let \( \sigma(g) \) be the unitary symmetry of the PDE defined on a Hilbert space \( \mathcal{H} \) of solutions, where

\[
\sigma(0, 0, c) u(x) = e^{i\lambda x} u(x) \quad u \in \mathcal{H}. \tag{4.2}
\]

By the Stone–Von Neumann theorem \( \sigma \) and \( \pi \) are unitarily equivalent. Therefore there exists a unitary operator \( A : L^2(\mathbb{R}^n) \to \mathcal{H} \). Hence the PDE is integrable.

Let us sketch an example, referring the reader to [4] for the details. Consider the Schrödinger (complex heat) equation, \( u_{xx} = iu_t \). It can be shown to possess a Lie algebra of point symmetries acting by

\[
\sigma(\exp \epsilon X_1) u(x, t) = u(x - \epsilon, t) \tag{4.3}
\]

\[
\sigma(\exp \epsilon X_2) u(x, t) = e^{\epsilon} u(x, t) \tag{4.4}
\]

\[
\sigma(\exp \epsilon X_3) u(x, t) = e^{-i \epsilon x + i \epsilon^2 t} u(x - 2 \epsilon t, t), \tag{4.5}
\]

where

\[
X_1 = \frac{\partial}{\partial x}, \quad X_2 = iu \frac{\partial}{\partial u}, \quad X_3 = 2t \frac{\partial}{\partial x} - i xu \frac{\partial}{\partial u}. \tag{4.6}
\]
$X_1, X_2, X_3$ generate the Heisenberg Lie algebra. If we restrict to solutions with $L^2(\mathbb{R})$ initial data, we may define a norm by setting

$$
\|u\|^2 = \|u(x, 0)\|^2 = \int_{-\infty}^{\infty} |u(x, 0)|^2 \, dx.
$$

These symmetries may be shown to be unitary and by Theorem 4.1 there exists a mapping from $L^2(\mathbb{R})$ to the space of solutions of the Schrödinger equation. In fact this operator is convolution with the fundamental solution. Further this mapping intertwines $\sigma$ and $\pi_j$ for $\lambda = 1$. Reference [4] contains further information about this analysis. In [3] we show that this approach can ultimately be used to decompose a Hilbert space of solutions of the Schrödinger equation as a direct integral. In [5] we show that using the relationship between symmetries and representations can allow one to construct fundamental solutions of heat equations on different manifolds and on $\mathbb{R}$ with particular drift terms.

REFERENCES