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VERY SMALL GROUP ACTIONS ON R-TREES AND DEHN TWIST AUTOMORPHISMS

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0. INTRODUCTION

LENGTH functions on a group G which come from actions on **R**-trees [20, 29, 10, 3, 37] and spaces of such length functions have been of central importance in combinatorial group theory in recent years. We will be concerned with subspaces of the projective space $SLF(G)$ of translation length functions of small actions of G on **R**-trees (see [12] or Section 1 below). An element of this space will often be referred to simply as "an action" (see 1.7).

Thurston's classification of diffeomorphisms of surfaces [36] and the work of Morgan and Shalen [21–23] has led to a well-known program for studying the structure of an individual automorphism of G or of the outer automorphism group $\mathrm{Out}(G)$. An appropriate subspace X of $SLF(G)$ is taken as an analogue of Teichmuller space or its boundary, and $\mathrm{Out}(G)$ is viewed as an analogue of the mapping class group. One hopes to learn about $\mathrm{Out}(G)$ through its induced action on the space of actions X [13, 9, 19, 18, 24] and to analyze an individual automorphism by finding fixed points and studying the dynamics of its induced action on X [8, 19]. For the free group of rank n, an important underpinning, the contractibility of the spaces of actions in question, has been proven [34, 35]. See [1, 12, 30, 31] for more complete references and history.

We contribute to this program in several ways. We start by introducing (Section 2) the space $VSL(G)$—the projective space of translation length functions of very small actions of G on **R**-trees. A *very small action* of the group G on an **R**-tree $\mathscr{T}$ is a small action such that for each non-trivial $g \in G$ the fixed subtree $\mathrm{Fix}(g)$ (a) is equal to $\mathrm{Fix}(g^n)$ whenever $g^n \neq 1$ and (b) does not contain a triod (cone on three points).

Notice that if $Free(G)$ denotes the space of free actions of G on **R**-trees and $SLF(G)$ denotes the space of small actions [12] then

$$Free(G) \subset VSL(G) \subset SLF(G).$$

THEOREM I. *Let G be a finitely generated group. Then*

(a) *$VSL(G)$ is compact.*

(b) *If $G = F_n$, the free group of rank n ($n \geq 2$), then the subspace of simplicial actions in $SLF(G)$ which do not belong to $VSL(G)$ is infinite dimensional.*

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Theorem I is proved in Theorems 2.3 and 9.8. This work started with the conjecture that $SLF(F_n)$ is precisely the closure of the Culler–Vogtmann space CV_n of actions of F_n on simplicial $\mathbf{R}$-trees. (This space is referred to as X_n in [13] and as "outer space" in [14].) The conjecture might be motivated by the fact [32] that, for fundamental groups of closed surfaces of genus greater than one, $SLF(G)$ is the closure of $Free(G)$. Theorem I shows that this conjecture is far from true in general. One conclusion we draw from the results of this paper is that it is $VSL(G)$ or one of its subspaces, rather than $SLF(G)$, which plays the crucial role in these matters.

The question of whether $VSL(G)$ is in fact equal to $\overline{Free(G)}$ and whether, when $G = F_n$, $VSL(F_n) = \overline{CV_n}$ has the following partial answer.

THEOREM II. *Suppose that G is a finitely generated group.*

(a) *If G acts freely on an $\mathbf{R}$-tree (i.e. $Free(G) \neq \emptyset$) and G does not contain a copy of $\mathbb{Z}\oplus\mathbb{Z}$ then a simplicial action of G on an $\mathbf{R}$-tree lies in $VSL(G)$ if and only if it is a limit of free actions on $\mathbf{R}$-trees.*

(b) *If $G = F_n$ $(n \geq 2)$ then a simplicial action of G on an $\mathbf{R}$-tree lies in $VSL(G)$ if and only if it is a limit of free* simplicial *actions (i.e. it lies in the closure of Culler–Vogtmann space).*

Notes. (1) Rips [27] has proved the conjecture of Shalen that a finitely generated group which acts freely on an $\mathbf{R}$-tree is necessarily a free product of surface groups and free abelian groups.

(2) Using Theorem II, Bestvina and Feighn [7] have now proved that indeed $VSL(F_n) = \overline{CV_n} = \overline{Free(F_n)}$.

The proof of Theorem II and the heart of this paper involves a study of the dynamics of the induced homeomorphisms $\varphi^*: SLF(G) \to SLF(G)$, where $\varphi \in \mathrm{Out}(G)$ is *represented by a multiple Dehn twist automorphism D* (introduced in Section 6). These automorphisms are defined for the fundamental group of a graph of groups $\mathcal{G}$, and carried over via an isomorphism $h: G \to \pi_1\mathcal{G}$ to define the outer automorphism φ. Roughly speaking, D is given by replacing stable letters t_e (where e is an edge of the graph of $\mathcal{G}$) in every word of $\pi_1(\mathcal{G}, P)$ by $t_e z_e$, where z_e is in (the image of) the edge group G_e (see Definition 6.5). When G is a surface group, multiple Dehn twist automorphisms have the automorphisms induced by classical Dehn twists as special cases. They are the most immediate analogues of Thurston's reducible surface automorphisms. For free groups, elementary Nielsen transformations are simple examples of automorphisms represented by Dehn twist automorphisms. Of particular interest are automorphisms $\varphi \in \mathrm{Out}(G)$ which are representable by *proper* Dehn twists $D: \pi_1(\mathcal{G}, P) \to \pi_1(\mathcal{G}, P)$ (Definition 13.1). This requires, in addition, (a) that the graph of groups $\mathcal{G}$ is *very small* (i.e. the action of $\pi_1(\mathcal{G}, P)$ on the Bass–Serre tree $\mathcal{T}_{\mathcal{G}}$ is very small, see Section 10), and (b) that the z_e satisfy a condition which makes the twists on distinct edges not cancel each other out. We prove in [11] that, for $G = F_n$, every automorphism of F_n which can be represented as a Dehn twist automorphism actually can be represented as a proper Dehn twist.

For the free group F_n our general results allow us to draw the following picture concerning the dynamics of the homeomorphism of $SLF(F_n)$ induced by a proper Dehn twist (see Section 13).

THEOREM III. (a) *If φ is an automorphism of F_n which is representable as a proper Dehn twist automorphism, then for any action $A \in Free(F_n)$ $(n \geq 2)$—in particular for any*

$A \in CV_n$—the orbit of A is parabolic: there exists a very small simplicial action A_∞ such that

$$\lim_{n\to\infty} (\varphi^*)^n(A) = \lim_{n\to\infty} (\varphi^*)^{-n}(A) = A_\infty.$$

(b) *Every elementary Nielsen transformation φ is representable as a proper Dehn twist. In addition to having the parabolic orbits in* (a) *it also satisfies*:

(i) *$Fix(\varphi^*)$ contains a $(3n-7)$-simplex in ∂CV_n (a 1-simplex if $n = 2$) and an infinite dimensional subcomplex of simplicial actions in $SLF(F_n)$.*

(ii) *For every integer $q > 0$ there is a simplicial action $A_q \in SLF(F_n)$ such that the φ^*-orbit of A_q consists of exactly q points.*

(iii) *There is an element $A_0 \in SLF(F_n)$ such that the φ^*-orbit of A_0 is infinite and dense in itself (i.e. each point of the orbit is a limit point of the orbit.)*

Theorem III(a) is a special case of a more general and more precise result stated as Theorem 13.2 in Section 13. Parabolic orbits are still obtained under the following circumstances:

(1) The group G in question does not need to be free, but it suffices if G is an α-group ("almost locally free group"): G is a finitely generated and torsion free group such that, if g and h are non-trivial elements of G then either $\langle g, h\rangle$ is infinite cyclic or there exist powers g^n, h^m such that $\langle g^n, h^m\rangle$ is a free group of rank 2. We introduce α-groups in Section 3 and derive some of their elementary properties. Examples are free groups, torsion free word hyperbolic groups (Lemma 3.3) and finitely generated groups which act freely on **R**-trees and do not contain copies of $\mathbb{Z}\oplus\mathbb{Z}$ (Lemma 3.2). They are the finitely generated, torsion free groups among the RG groups investigated independently by Fine and Rosenberger [15].

(2) The assumption that the action $A \in SLF(G)$ be free is not necessary in order to obtain a parabolic orbit $\{(\varphi^*)^n(A) \mid n \in \mathbb{Z}\}$. What is required is only that the *twistors* z_e in the definition of the multiple Dehn twist ϕ have $\|z_e\|_A \neq 0$. (See Theorem 13.2.)

(3) The limit point A_∞ can be described more precisely in terms of the automorphism $\phi \in \mathrm{Out}(G)$. Suppose φ is represented by a proper Dehn twist of the graph of groups $\mathcal{G}$ (say for simplicity $G = \pi_1(\mathcal{G}, P)$). Then A_∞ is contained in the simplex $\sigma(\mathcal{G}) \subset VSL(G)$, where $\sigma(\mathcal{G})$ consists precisely of those (simplicial) G-actions on **R**-trees which are obtained via Bass–Serre theory from $\mathcal{G}$ by varying the lengths of the edges of $\mathcal{G}$ (for more details see Sections 3 and 9). If $\mathcal{G}$ has only one edge, as in the case (Example 6.7(a)) where φ is an elementary Nielsen transformation, then $\sigma(\mathcal{G})$ consists of a single point only, and hence the limit point A_∞ will be independent of the particular orbit $\{(\varphi^*)^n(A) \mid n \in \mathbb{Z}\}$ in consideration.

We get the following algebraic consequence of these results for α-groups.

THEOREM IV [Uniqueness of proper Dehn twist representations (13.4)]. *Suppose that G is an α-group which admits a free action on some* **R**-*tree. Suppose that $\phi: G \to G$ is an isomorphism which can be represented as a proper Dehn twist, with non-trivial twists on all edges, via an isomorphism $h_1: G \to \pi_1(\mathcal{G}_1, P_1)$ and also via an isomorphism $h_2: G \to \pi_1(\mathcal{G}_2, P_2)$. Then the graph of groups decompositions $(\mathcal{G}_1, h_1)$ and $(\mathcal{G}_2, h_2)$ of G are equivalent in the sense that the corresponding Bass–Serre actions on simplicial trees $\pi_1(\mathcal{G}_i, P_i) \times \mathcal{T}_i \to \mathcal{T}_i$ ($i = 1, 2$) are $(h_2 \circ h_1^{-1})$-equivalently isomorphic.*

Note. See [4] for an interpretation of this result in the category of graphs of groups. The authors have used Theorem IV to solve the conjugacy problem for Dehn twist automorphisms of free groups [11].

Tools developed in this paper

In the course of proving the theorems here we were led to develop a number of tools and concepts which we believe will have wider applicability.

The *very small actions* are introduced and the compactness of $VSL(G)$ is proved in Section 2.

The *α-groups* (Section 3) have already been mentioned. The significant results of [15] on how these groups and RG groups in general behave under free product with amalgamation and HNN extension should be noted. In some cases their results combined with Lemma 10.2 can be interpreted as saying that if $\mathcal{G}$ is a *very small graph of groups* (introduced in Section 10) and if the vertex groups are α-groups then $\pi_1(\mathcal{G}, P)$ is an α-group.

In Section 4 we prove the Skyscraper Lemma, which, given an action on an **R**-tree, gives the translation length of a word $\|w\|$, when $w = b_0 x_1^{n_1} b_1 \ldots x_q^{n_q} b_q$, as being asymptotically equal to $\sum n_i \| x_i \|$ under some obviously necessary hypothesis. We believe that analogues of this lemma will play a key role when generalizing the theory of spaces of length functions of actions on **R**-trees to the theory of spaces of length functions coming from appropriate actions of G on negatively curved spaces.

In Sections 5–10 we develop the theory of *metric graphs of groups*, combining the Bass–Serre theory with the theory of **R**-trees. We first (Sections 5–7) reformulate the Bass–Serre theory for our purposes and introduce Dehn twist automorphisms with particular attention to their relationship to quotient graphs of groups.

We then (Section 8) metrize graphs of groups and give a Combination Lemma which uses these as a template for constructing **R**-tree actions. (The reader should note Skora's broader treatment of this subject [33]; our hypotheses differ from his because of our particular use of this Combination Lemma in Section 12.)

In Section 9 the simplex $\sigma(\mathcal{G})$ is introduced and it is proved that, under appropriate hypotheses ("no invisible vertices; not a self-swallowing cycle"), $\sigma(\mathcal{G})$ is an $(r-1)$-dimensional simplex in $PLF(G)$, where r is the number of edges in $\mathcal{G}$. This is the foundation of our infinite dimensionality results in Theorems I and III.

Finally, in Section 10 we introduced *the very small graphs of groups*, combinatorially capturing (Proposition 10.3) the notion of a very small action on an **R**-tree. An interesting and non-obvious algebraic application of Proposition 10.3 and the fact that $VSL(G)$ is compact is that (Corollary 10.4) a quotient graph of groups of a very small graph of groups $\mathcal{G}$ is very small when $G = \pi_1(\mathcal{G}, P)$ is an α-group. The key combinatorial property which very small graphs of groups have is given by the Choice Lemma (Lemma 10.6), whereby elements of edge groups can be chosen (to be used in defining useful Dehn twist automorphisms) so that the hypothesis of the Skyscraper Lemma can be satisfied.

All of the preceding work is combined in Section 11 to prove the Fundamental Convergence Lemma 11.5 which gives our fundamental calculation concerning the dynamics of the induced action on $SLF(G)$ of Dehn twist automorphisms. This is then applied in Sections 12 and 13 to prove Theorems II, III, IV above.

PART I. GROUP ACTIONS ON **R**-TREES

1. NOTATION AND BACKGROUND ON **R**-TREES

G will denote a finitely generated group unless explicitly stated otherwise.

$\mathbb{N}$ denotes the set of *positive* integers.

F_n denotes the free group of rank n.

$F(S)$ denotes the free group with basis S.

We use the terminology of [12] in discussing **R**-trees. We recall the following definitions and facts.

1.1. An **R**-tree $\mathcal{T}$ is a metric space in which any two points $x, y \in \mathcal{T}$ are connected by a unique arc $[x, y]$ (*called a geodesic*), and this arc is isometric to a line segment in **R** of length $\mathrm{dist}(x, y)$.

From now on we assume that G acts on $\mathcal{T}$ by isometries (from the left).

1.2. For any $g \in G$ there is a *translation length*

$$\| g \| = \inf\{\mathrm{dist}(x, g(x)) \mid x \in \mathcal{T}\}$$

and a non-empty characteristic set

$$C_g = \{x \in \mathcal{T} \mid \mathrm{dist}(x, g(x)) = \| g \|\}.$$

These satisfy:

(1) $\| 1 \| = 0$; $\| g \| = \| g^{-1} \|$; $\| ghg^{-1} \| = \| h \|$.

(2) If $\| g \| = 0$ then $C_g = \mathrm{Fix}(g)$. If $\| g \| > 0$ then C_g is a real line ("axis") on which g acts by translation by $\| g \|$ units.

1.3. The action of G on $\mathcal{T}$ is *trivial* if $\| g \| = 0$ for all $g \in G$.

1.4. The action of G on $\mathcal{T}$ is *minimal* if $\mathcal{T}$ contains no non-empty proper subtree which is invariant under the action of the entire group G. A non-trivial action of G on $\mathcal{T}$ always restricts to a minimal non-trivial action on some **R**-tree $\mathcal{T}' \subset \mathcal{T}$ with the same translation length function. Indeed $\mathcal{T}' = \bigcup\{C_g \mid \| g \| > 0\}$. (See [12, 3.1].)

1.5. The action of G on $\mathcal{T}$ is *abelian* if any one of the following equivalent (see [12], 2.4) conditions is satisfied:

(a) The action fixes an end of $\mathcal{T}$

(b) There exists a homomorphism $\rho: G \to \mathbf{R}$ such that $\| g \| = |\rho(g)|$ for all $g \in G$. (In particular ρ defines an action of G on the real line by translations with the same length function.)

(c) $\| ghg^{-1}h^{-1} \| = 0$ for all $g, h \in G$.

Note. (b) or (c) implies that a length function $\| \cdot \|$ cannot come from both an abelian action and a non-abelian action.

1.6. We denote:

$\Omega(G)$	= the set of conjugacy classes of elements of G.
$LF(G)$	= the subspace of $\mathbf{R}^{\Omega(G)}$ of points $(\| g \|)_{[g] \in \Omega(G)}$ such that $\| \ \|$ is the translation length function of a non-trivial action of G on an **R**-tree.
$PLF(G)$	= the image of $LF(G)$ under the canonical map to projective space $\mathbf{R}^{\Omega(G)} - \{0\} \to P\mathbf{R}^{\Omega(G)}$ This is a compact space [12, (4.5)].

1.7. Given a set of non-trivial actions of G on **R**-trees we *define* the corresponding *space of actions* to be the subspace of $PLF(G)$ determined by the set of *translation* length functions of the given actions. If $\|\cdot\| \in \mathbf{R}^{\Omega}$ is such a length function then $[\|\cdot\|]$ will denote its image in $PLF(G)$. We sometimes simply write $\|\cdot\| \in PLF(G)$ instead of $[\|\cdot\|]$.

THEOREM (Culler and Morgan [12, (3.7)] and Alperin and Bass [2, (7.13)]). *If two minimal non-abelian actions of the finitely generated group G determine the same translation length function then the actions are equivalent (via an equivariant isometry).*

1.8. Using 1.7 we define the following subspaces of $PLF(G)$:

$SLF(G)$ = the space of small actions (see 1.9) of G on **R**-trees. This is a compact space [12, (5.3)].
$Free(G)$ = the space of free actions of G on **R**-trees.
CV_n = the space of free simplicial actions of F_n on **R**-trees (= Culler–Vogtmann space; simplicial actions in general are defined in Section 8 below).

1.9. A group is *small* if it does not contain a free group of rank 2. An action of a group G on an **R**-tree is *small* if the stabilizer of every non-degenerate arc is small. (An *arc* is a homeomorph of a compact interval or point in **R**. The *stabilizer* of a set A is the subgroup of G consisting of elements which fix A pointwise.)

1.10. We shall also have need of *based length functions*. If $P \in \mathcal{T}$ then the based length function $L_P: G \to \mathbf{R}$ is defined by

$$L_P(g) = \operatorname{dist}(P, gP).$$

It is related to the translation length function $\|\cdot\|$ by

(1) $\|g\| = L_P(g^2) - L_P(g) = \lim_{n\to\infty} L_P(g^n)/n$,
(2) $L_P(g) = \|g\| + 2\operatorname{dist}(P, C_g)$.

In general, if $L: G \to \mathbf{R}$ is a non-negative function and $g, h \in G$ then we define the Lyndon–Gromov "common part of g and h" by

$$c(g,h) = \tfrac{1}{2}(L(g) + L(h) - L(g^{-1}h)).$$

According to [10], [3], a non-negative length function $L: G \to \mathbf{R}$ is the based length function $L = L_P$ of some action of G on an **R**-tree $\mathcal{T}$ if and only if

1.11.

(i) $L(1) = 0$,
(ii) $L(g) = L(g^{-1})$,
(iii) $c(g,h) \geq \min(c(g,k), c(k,h))$ for all $g, h, k \in G$.

When $L = L_P$ one sees directly that $c(g,h)$ = length of $[P, gP] \cap [P, hP]$.

1.12. If $h: G_1 \to G_2$ is a homomorphism then any action of G_2 on an **R**-trees $\mathcal{T}$ pulls back to an action of G_1 on $\mathcal{T}$ defined by $g_1(x) = (h(g_1))(x)$, for all $g \in G_1$. If h is an isomorphism we then get an associated homeomorphism h^* (which we also call h when no confusion can occur). Thus, $h^*: PLF(G_2) \to PLF(G_1)$ is induced from the coordinatewise homeomorphism $\mathbf{R}^{\Omega(G_2)} \to \mathbf{R}^{\Omega(G_1)}$ given by $(h^*(\|\cdot\|))(g_1) = \|h(g_1)\|$ for all $g_1 \in G_1$.

2. THE COMPACTNESS OF THE SPACE $VSL(G)$ OF VERY SMALL GROUP ACTIONS

Definition 2.1. An action of a group G on an **R**-tree $\mathcal{T}$ is *very small* if:

(a) It is small.

(b) (*no obtrusive powers*) If $g \in G$ and $g^n \neq 1$ then $\mathrm{Fix}(g) = \mathrm{Fix}(g^n)$. (If $g^n \neq 1$ and $\mathrm{Fix}(g) \neq \mathrm{Fix}(g^n)$ then g^n is called an *obtrusive power*.)

(c) (*no fixed triods*) If $1 \neq g \in G$ then $\mathrm{Fix}(g)$ does not contain a triod (= cone on three points) (i.e. the subtree $\mathrm{Fix}(g)$ has no branch points; it is isometric to a closed connected subset of **R**).

We define $VSL(G) \subset PLF(G)$ to be the space of all very small actions of G on **R**-trees (using the conventions of 1.7).

Notice that if an action of G on $\mathcal{T}$ is very small then the action of G on the minimal invariant subtree $\mathcal{T}' \subset \mathcal{T}$ is also very small.

EXAMPLE 2.2. (a) Free actions are very small. (b) Length functions of small abelian actions belong to $VSL(G)$.

Proof of (b). If $G \times \mathcal{T} \to \mathcal{T}$ is a small abelian action with translation length function $\|\cdot\|$, notice first of all [12, 5.1] that G contains no free subgroup of rank 2: for, the action being abelian, if x, y generated a free subgroup of G of rank 2 then (1.5(a)) the free group $\langle xyx^{-1}y^{-1}, x^2yx^{-2}y^{-1}\rangle$ would fix an infinite ray, contradicting the assumption that the action is small.

An action by translation $G \times \mathbb{R} \to \mathbb{R}$ with the same length function exists (1.5(b)). It is certainly small, since G contains no free group of rank 2. Clearly there are no fixed triods and, since the action is by translations, $\mathrm{Fix}(g) = \mathrm{Fix}(g^n)$ if $n \neq 0$. Thus $[\|\cdot\|] \in VSL(G)$. □

The main goal of this section is to prove the following theorem.†

THEOREM 2.3. *For every finitely generated group G the space $VSL(G)$ is compact.*

COROLLARY 2.4. *$Closure(Free(G)) \subset VSL(G)$. In particular: $Closure(CV_n) \subset VSL(F_n)$.*

The reduction of Theorem 2.3 to length function inequalities

The compactness of $VSL(G)$ will be proved using the following two lemmas. Note that the length function of a minimal and non-abelian action is the length function of a unique such action (1.7) and is not the length function of any abelian action (by 1.5).

LEMMA 2.5. *Let $\|\cdot\|$ be the length function of a minimal and non-abelian action of G on an **R**-tree $\mathcal{T}$. Then the action realizing $\|\cdot\|$ has an obtrusive power (i.e. Definition 2.1(b) is false) $\Leftrightarrow$ there exist elements $g, h \in G$ and an integer $P \geq 2$ such that $g^P \neq 1 \neq g^{P-1}$ and*

(a) $\|gh\| > \|g\| + \|h\|$

(b) $\max(\|g^P h\|, \|g^P h^{-1}\|) < \|g^{P-1} h\|$.

†We prove this using the length function topology since this is the genre in which the rest of the paper is written. A shorter proof using the Gromov topology on the space of actions has been given by Paulin [25].

LEMMA 2.6. *Let $\|\cdot\|$ be the length function of a minimal and non-abelian action of a group G on an* **R***-tree $\mathscr{T}$. Then the action of G on $\mathscr{T}$* realizing *$\|\cdot\|$ has a fixed triod (i.e. Definition 2.1(c) is false) $\Leftrightarrow$ there exist elements $a, b, c, g \in G$ such that, for any ordering (r, s, t) of $\{a, b, c\}$, one has*

$$(*) \quad \operatorname{dist}(C_r, C_s) + \operatorname{dist}(C_s, C_t) > \operatorname{dist}(C_r, C_t) + 2\operatorname{dist}(C_s, C_g).$$

Proof of Theorem 2.3 *assuming Lemmas* 2.5 *and* 2.6. We show that $VSL(G)$ is a closed subspace of the compact space $SLF(G)$.

The space $\widetilde{NA}$ of length functions of minimal non-abelian actions is open in $LF(G) \subset \mathbf{R}^{\Omega}$ by (1.5(c)). The spaces $\tilde{U}$, $\tilde{V}$ of length functions of non-trivial minimal actions which have obtrusive powers or fixed triods (violating Definition 2.1(b) or (c)), respectively, have intersections $\widetilde{NA} \cap \tilde{U}$ and $\widetilde{NA} \cap \tilde{V}$ which are open subsets of $\mathbf{R}^{\Omega}$ by Lemmas 2.5 and 2.6. They are unions of sets given by inequalities of continuous functions of the variable point $\|\cdot\|$. To get this from Lemma 2.6 note [12, (1.5), (1.9)] that if $x, y \in G$, then

$$\operatorname{dist}(C_x, C_y) = \max(0, \tfrac{1}{2}(\|xy\| - \|x\| - \|y\|)).$$

Clearly $\widetilde{NA}$, $\tilde{U}$, $\tilde{V}$ are closed under multiplication by positive scalars, so the open sets $\widetilde{NA}$, $\widetilde{NA} \cap \tilde{U}$ and $\widetilde{NA} \cap \tilde{V}$ project to open subspaces NA, $NA \cap U$ and $NA \cap V$ of $PLF(G)$. By Example 2.2(b), and the fact (1.5(c)) that a length function cannot be realized by both an abelian action and a non-abelian action, we have

$$SLF(G) - VSL(G) \subset SLF(G) \cap NA.$$

Now, by the uniqueness of the action corresponding to an element $\|\cdot\| \in \widetilde{NA}$, an element of $NA \cap U$ (or of $NA \cap V$) cannot also be represented by a non-abelian action without obtrusive powers (or without fixed triods). (In contrast there is an abelian length function [12, (3.9)] which can be represented by a minimal action with fixed triods and by a minimal action on $\mathbb{R}$ without fixed triods.) Hence, by Definition 2.1 of $VSL(G)$

$$SLF(G) - VSL(G) = SLF(G) \cap [(NA \cap U) \cup (NA \cap V)].$$

Thus, $SLF(G) - VSL(G)$ is an open subset of $SLF(G)$. □

To prepare for the proofs of Lemmas 2.5 and 2.6, let us first recall the following elementary facts.

2.7 (Culler and Morgan [12], 1.5, 1.9). Let G (finitely generated as usual) act on the tree $\mathscr{T}$, and let $g, h \in G$.

(a) $C_g \cap C_h$ is empty or consists of a single point if and only if

$$\|gh\| = \|g\| + \|h\| + 2\operatorname{dist}(C_g, C_h) = \|gh^{-1}\|.$$

(b) In any case, $\max(\|gh\|, \|gh^{-1}\|) \leq \|g\| + \|h\| + 2\operatorname{dist}(C_g, C_h)$, and $\operatorname{dist}(C_g, C_h) = 0$ if and only if $C_g \cap C_h \neq \emptyset$.

LEMMA 2.8. *Assume a minimal and non-abelian action of a group G on an* **R***-tree $\mathscr{T}$ is given. Suppose that $Q \in \mathscr{T}$ and that $\mathscr{T}'$ is a component of $\mathscr{T} - \{Q\}$. Then there is an element $h \in G$ with $C_h \subset \mathscr{T}'$.*

Proof. Let P be any point of $\mathscr{T}'$. Because the action is minimal, P lies (1.4) on an axis C_h for some $h \in G$ with $\|h\| > 0$. If $Q \notin C_h$ we are done. Otherwise denote $C_h^+ = C_h \cap \mathscr{T}'$ and

$C_h^- = C_h - C_h^+$. Note that $Q \in C_h^-$. Replacing h by h^{-1} if necessary, we may assume that the translation by h along C_h yields $h(Q) \in C_h^+$.

It cannot occur that $C_g \cap C_h^-$ is non-compact for all $g \in G$. For this would give an end fixed by the group G, contradicting (see 1.5(a)) the fact that the action is abelian. Therefore, we may choose $g \in G$ such that $C_g \cap C_h^-$ is empty or is a non-empty set of finite length.

If $C_g \cap C_h^-$ is a non-empty set of finite length (an arc or a point) then h^q translates this connected set into C_h^+ for some sufficiently large q. Hence $h^q(C_g) = C_{h^q g h^{-q}} \subset \mathscr{T}'$ and we are done.

If $C_g \cap C_h^- = \emptyset$ and $C_g \cap C_h^+ = \emptyset$ then $C_g \subset \mathscr{T}'$ and we are done.

Finally if $C_g \cap C_h^- = \emptyset$ and $C_g \cap C_h^+ = \emptyset$, so that $C_g \cap C_h = \emptyset$ then $C_{gh} \cap C_h$ is a closed interval of finite length $\|h\|$ ([2], 8.1b) and one of the two preceding paragraphs applies. □

Proof of Lemma 2.5. ($\Leftarrow$) We are given g, h and $P \geq 2$ with $g^P \neq 1 \neq g^{P-1}$ such that:

(a) $\|gh\| > \|g\| + \|h\|$.
(b) $\max(\|g^P h\|, \|g^P h^{-1}\|) < \|g^{P-1} h\|$.

We claim that either $C_g \neq C_{g^P}$ or $C_g \neq C_{g^{P-1}}$, so that g has an obtrusive power. For otherwise, from 2.7 and Lemma 2.5(a)

$$C_{g^P} \cap C_h = C_{g^{P-1}} \cap C_h = C_g \cap C_h = \emptyset$$

and thus

$$\begin{aligned} \|g^P h^{-1}\| = \|g^P h\| &= \|g^P\| + \|h\| + 2\,\mathrm{dist}(C_{g^P}, C_h) \\ &\geq \|g^{P-1}\| + \|h\| + 2\,\mathrm{dist}(C_{g^{P-1}}, C_h) \\ &= \|g^{P-1} h\| \end{aligned}$$

contradicting Lemma 2.5(b).

($\Rightarrow$) Assume G has an obtrusive power: $g^P \neq 1$ and $\mathrm{Fix}(g) \neq \mathrm{Fix}(g^P)$. Since $\emptyset \neq \mathrm{Fix}(g) \subseteq \mathrm{Fix}(g^P)$, then $g^{P-1} \neq 1$ and, also, there is a non-degenerate arc $A = [Q, Q'] \subset \mathrm{Fix}(g^P)$ such that $A \cap \mathrm{Fix}(g) = \{Q\}$. Without loss of generality P is the smallest integer such that g^P fixes a point of $A - \{Q\}$. This implies that $\mathrm{Fix}(g^{P-1})$ (which contains Q) does not meet the component of $\mathscr{T} - \{Q\}$ which contains int(A).

Let $\mathscr{T}'$ be a component of $\mathscr{T} - \{Q'\}$ which does not contain $A - \{Q\}$. Such a component exists because, the action being minimal, Q' separates $\mathscr{T}$. By Lemma 2.8 there exists $h \in G$ with $C_h \subset \mathscr{T}'$. Since C_h and $C_{g^{P-1}} \supset C_g$ are disjoint, $\|gh\| > \|g\| + \|h\|$ and Lemma 2.5(a) is satisfied. To verify Lemma 2.5(b) we apply 2.7 and the fact that $\|g^P\| = \|g^{P-1}\| = 0$:

$$\begin{aligned} \max(\|g^P h\|, \|g^P h^{-1}\|) &\leq \|g^P\| + \|h\| + 2\,\mathrm{dist}(\mathrm{Fix}(g^P), C_h) \quad [\text{by } 2.7(\text{b})] \\ &< \|g^P\| + \|h\| + 2\,\mathrm{dist}(\mathrm{Fix}(g^{P-1}), C_h) \\ &= \|g^{P-1}\| + \|h\| + 2\,\mathrm{dist}(\mathrm{Fix}(g^{P-1}), C_h) \\ &= \|g^{P-1} h\| \quad [\text{by } 2.7(\text{a})]. \end{aligned}$$

□

Proof of Lemma 2.6. ($\Rightarrow$) Suppose that there exists a non-trivial $g \in G$ such that $\mathrm{Fix}(g)$ contains a triod. We shall find $h_1, h_2, h_3 \in G$ such that for any ordering (r, s, t) of $\{h_1, h_2, h_3\}$ one has

$$\mathrm{dist}(C_r, C_s) + \mathrm{dist}(C_s, C_t) > \mathrm{dist}(C_r, C_t) + 2\,\mathrm{dist}(C_s, C_g).$$

Let Q be the vertex of a triod in $\mathcal{T}$. Let $\mathcal{T}_j$ $(j = 1, 2, 3)$ be the three components of $\mathcal{T} - \{Q\}$ containing the three spokes of the triod (omitting Q).

By 2.8, there exists $h_j \in G$ such that $C_{h_j} \subset \mathcal{T}_j$, $j = 1, 2, 3$. Let A_j be the shortest arc from C_{h_j} to Q (non-degenerate since $Q \notin \mathcal{T}_j$). Then A_j must meet C_g in a non-degenerate subinterval B_j $(j = 1, 2, 3)$ such that $B_1 \cup B_2 \cup B_3$ is a triod in C_g. Writing $A_j = B_j' \cup B_j$, where $B_j' \cap B_j$ is a point, we get the situation in Fig. 1 as part of the tree $\mathcal{T}$

From Fig. 1 it is clear that

$$\begin{aligned}\operatorname{dist}(C_{h_i}, C_{h_j}) + \operatorname{dist}(C_{h_j}, C_{h_k}) &= \operatorname{dist}(C_{h_i}, C_{h_k}) + 2 \operatorname{length} A_j \\ &> \operatorname{dist}(C_{h_i}, C_{h_k}) + 2 \operatorname{dist}(C_{h_j}, C_g)\end{aligned}$$

as desired.

$(\Leftarrow)$ Finally, we assume that there is no fixed triod in $\mathcal{T}$ and we prove that for any elements $a, b, c, g \in G$ there is an ordering (r, s, t) of $\{a, b, c\}$ such that

$$\operatorname{dist}(C_r, C_s) + \operatorname{dist}(C_s, C_t) \leq \operatorname{dist}(C_r, C_t) + 2 \operatorname{dist}(C_s, C_g).$$

We start by assuming, without loss of generality, that

(i) $\operatorname{dist}(C_a, C_c) \geq \operatorname{dist}(C_a, C_b)$,

(ii) $\operatorname{dist}(C_a, C_c) \geq \operatorname{dist}(C_b, C_c)$.

If $\operatorname{dist}(C_a, C_c) \geq \operatorname{dist}(C_a, C_b) + \operatorname{dist}(C_b, C_c)$ we are done, setting $r = a$, $s = b$, $t = c$. Suppose, on the other hand, that

(iii) $\operatorname{dist}(C_a, C_b) + \operatorname{dist}(C_b, C_c) > \operatorname{dist}(C_a, C_c)$.

Then (i)–(iii) imply that all the distances are positive and the sum of any two is greater than the third. Let A_{ab} be the shortest arc from C_a to C_b. Similarly define A_{bc} and A_{ac}. Define $A_a = A_{ab} \cap A_{ac}$. Similarly define A_b and A_c. Since the sum of the lengths of any two of A_{ab}, A_{ac}, and A_{bc} is greater than the third, one checks easily that $A_{ab} \cup A_{ac} \cup A_{bc}$ is a triod with spokes A_a, A_b, A_c emanating from a vertex Q. At the end of the spoke A_a is the tree C_a; similarly for C_b, C_c.

By hypothesis, C_g does not contain any triod. So, for some $s \in \{a, b, c\}$, C_g is disjoint from the component of $\mathcal{T} - \{Q\}$ containing the spoke $A_s - \{Q\}$. Then $\operatorname{dist}(C_s, Q)$

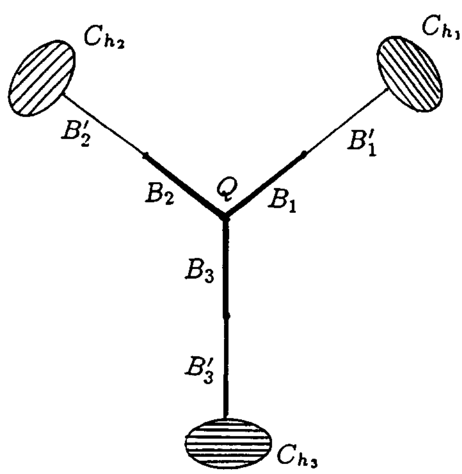

Fig. 1.

$\leq \operatorname{dist}(C_s, C_g)$, and we have, for r, t the other two elements of $\{a, b, c\}$,

$$\begin{aligned}\operatorname{dist}(C_r, C_s) + \operatorname{dist}(C_s, C_t) &= \operatorname{dist}(C_r, C_t) + 2\operatorname{dist}(C_s, Q)\\ &\leq \operatorname{dist}(C_r, C_t) + 2\operatorname{dist}(C_s, C_g)\end{aligned}$$

as desired. □

3. α-GROUPS

Recall from the Introduction that an α-group (= *almost locally free group*) is a finitely generated torsion free group such that, if $1 \neq g, h \in g$ then either $< g, h > \cong \mathbb{Z}$ or $\langle g^n, h^m \rangle$ is a free group of rank 2 for some integers m, n.

Notes 3.1. (a) The condition that α-groups be finitely generated is put in for convenience, but is usually not used. With our definition, a subgroup of an α-group is an α-group precisely when the subgroup is finitely generated.

(b) This entire paper was originally written with the more general definition, which does not assume that "$\langle g, h \rangle \cong \mathbb{Z}$ or ...", but rather that "there exist non-zero integers m, n such that $\langle g^m, h^n \rangle$ generates a free group of rank one or of rank two". However, this would require in Theorem II (= 12.1) that edge stabilizer be assumed to be finitely generated and it would complicate the proofs of elementary algebraic facts. So we have adopted the current definition.

Examples

LEMMA 3.2. *If G is a finitely generated group which acts freely on some* **R**-*tree $\mathscr{T}$ and which has no subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ then G is an α-group.*

Proof. Let g, h be non-trivial elements of G. Since the action of G on $\mathscr{T}$ is free, there are axes C_g, C_h, in $\mathscr{T}$. If $C_g \cap C_h$ is compact then, for suitable $m, n > 0$, we have $\operatorname{diam}(C_{g^n} \cap C_{h^m}) < \min\{\|g^n\|, \|h^m\|\}$. Hence, by [12, 2.6], the subgroup $\langle g^n, h^m \rangle$ is free of rank 2. If $C_g \cap C_h$ is not compact this intersection contains an infinite ray and $ghg^{-1}h^{-1}$ fixes points far out on this ray. Since the action of G is free, $gh = hg$. Thus $\langle g, h \rangle$ is an abelian group and the fact that G contains no $\mathbb{Z} \oplus \mathbb{Z}$ implies that $\langle g, h \rangle$ is infinite cyclic. Thus G is an α-group. □

Essentially the same proof in a technically more difficult setting gives the following lemma.

LEMMA 3.3. *If G is a torsion free work hyperbolic group then G is an α-group.*

Proof. See the proof of Theorem 30, p. 148 in [17]. □

Remark. Not every α-group is a word hyperbolic group. For example, Potyagailo [26] gives a finitely generated group $N \subset SO(1, 4)$ which is a (normal) subgroup of a torsion free word hyperbolic group, but which is itself not a word hyperbolic group. By Note 3.1(a) and 3.3, N is an α-group.

Elementary algebraic facts

The following notion will be useful in finding twisters z, z' such that the Dehn twists D_z and $D_{z'}$ do not cancel each other out.

Definition 3.4. Two elements g, h of a group G are *positively bonded* (or *negatively bonded*) if there exist $r \in G$ and integers m, n with $mn > 0$ ($mn < 0$) such that $g^m = rh^n r^{-1}$.

LEMMA 3.5. *Suppose that G is an α-group with non-trivial elements g and h.*

(a) *If $hg^n h^{-1} = g^m$ for some $m, n \neq 0$ then $h^p = g^q$ for some $p, q \neq 0$.*
(b) *If m, n are distinct integers then g^m is not conjugate to g^n.*
(c) *g and h cannot be both positively bonded and negatively bonded.*

Proof. (a) For any k and ℓ, h^k and g^ℓ satisfy the non-trivial relation

$$h^k (g^\ell)^{n^k} h^{-k} = (g^\ell)^{m^k}.$$

So $\langle h^k, g^l \rangle$ is not a free group of rank two for any integers k and ℓ. Thus $\langle h, g \rangle \cong \mathbb{Z}$, and the results follows.

(b) If $rg^m r^{-1} = g^n$ then $\langle r, g \rangle \cong \mathbb{Z}$ by the argument in (a). Thus $g^m = g^n$ with $m \neq n$, contradicting the fact that G is torsion free.

(c) If $g^m = rh^n r^{-1}$ and $g^p = sh^{-q} s^{-1}$ where $m, n, p, q > 0$ then g^{pn} is conjugate to g^{-qm} contradicting (b). □

Small subgroups and small actions

PROPOSITION 3.6. (a) *Every small subgroup of an α-group is abelian.*
(b) *Every non-trivial small α-group G is infinite cyclic.*

Proof. (a) Two elements g, h of the subgroup cannot have powers which generate a free group of rank two. Thus $\langle g, h \rangle \cong \mathbb{Z}$. Hence any two elements commute.

(b) By (a) G is a finitely generated torsion free abelian group; $G \cong \mathbb{Z}^k$. Since no two elements generate a free abelian group of rank two, $G \cong \mathbb{Z}$. □

COROLLARY 3.7. *If G is an α-group and has a small action on an* **R**-*tree then the stabilizer G_e of every non-degenerate arc is abelian. If it is finitely generated then $G_e = \{1\}$ or $G_e \cong \mathbb{Z}$.*

LEMMA 3. 8. *Suppose that H is a small subgroup of the α-group G such that, if $x \in G$ and $x^p \in H$ ($p \neq 0$), then $x \in H$. Then*

$$(rHr^{-1}) \cap H = \{1\} \quad \textit{for all } r \notin H,\ r \in G.$$

(*When this conclusion holds some people say that H is malnormal in G.*)

Proof. Suppose on the contrary there exist non-trivial elements $g, h \in H$ and $r \notin H$ with $rgr^{-1} = h$. Since H is a small subgroup, $\langle g, h \rangle \cong \mathbb{Z}$ and $g^m = h^n$ for some $m, n \neq 0$. Thus

$$rg^n r^{-1} = h^n = g^m.$$

Hence by Lemma 3.5(a), $r^p = g^q \in H$ for some $p \neq 0$. Thus by hypotheses $r \in H$, contradicting our choice of r. □

LEMMA 3.9. *Suppose that G is an α-group acting on an* **R**-*tree $\mathcal{T}$ so that the action is small and there are no obtrusive powers (i.e.* $\mathrm{Fix}(g) = \mathrm{Fix}(g^p)$ *if $g \in G$ and $p \neq 0$; see* (2.1)). *If g and h are non-trivial elements of G and A is an arc in* $\mathrm{Fix}(g)$ *with $h(A) \subset$* $\mathrm{Fix}(g)$ *then $h|A = \mathrm{id}_A$.*

Proof. Elements g and $h^{-1}gh$ lie in the small group stab(A). Thus $g^m = (h^{-1}gh)^n = h^{-1}g^n h$ for some $m, n \neq 0$. So by Lemma 3.5(a), $g^p = h^q$ for some non-zero integers p, q. Hence

$$A \subset \mathrm{Fix}(g) = \mathrm{Fix}(g^p) = \mathrm{Fix}(h^q) = \mathrm{Fix}(h).$$

Therefore $h|A = \mathrm{id}_A$. □

4. THE SKYSCRAPER LEMMA

In this section we prove a theorem on word length for groups acting on **R**-trees. We use Lyndon's cancellation lemma [20, (6.1)] to stably calculate the based length and translation length of a word of the form

$$w = b_0 x_1^{n_1} b_1 x_2^{n_2} \dots x_q^{n_q} b_q \quad (\| x_i \| > 0 \text{ for all } i)$$

as the $n_i \to \infty$. The terms $x_i^{n_i}$ $(1 \leq i \leq q)$ are "skyscrapers" separated by the b_i's, and the question is whether the b_i's can prevent cancellation between the skyscrapers. We show that the most obvious necessary hypothesis is sufficient to prevent essential cancellation. This result will be the cornerstone in Section 11 of our calculation of the limit of a sequence of actions on **R**-trees.

THE SKYSCRAPER LEMMA 4.1. *Suppose that an action of an α-group G on an* **R**-*tree $\mathscr{T}$ with small edge stabilizers is given. Let $b_0, x_1, b_1, \dots, x_q, b_q \in G$ be elements such that*

(i) $\| x_i \| > 0, \qquad 1 \leq i \leq q,$

(ii) $x_i^{-n} \neq b_i x_{i+1}^m b_i^{-1}$ *if* $1 \leq i \leq q-1,\ n > 0,\ m > 0.$

Let $w = w(n_1, n_2, \dots, n_q) = b_0 x_1^{n_1} b_1 x_2^{n_2} \dots x_q^{n_q} b_q$.

(a) *For every point $P \in \mathscr{T}$ with corresponding based length function L_P (see* 1.10*) there exist constants N_P, K_P, such that, for all integers $n_1, n_2, \dots, n_q \geq N_P$,*

$$L_P(w) = \sum_{i=1}^{q} n_i \| x_i \| + K_P.$$

(b) *If furthermore, the following condition* (iii) *holds:*

(iii) $x_q^{-n} \neq b_q b_0 x_1^m (b_q b_0)^{-1}$ *for all* $m > 0,\ n > 0$

then there exist constants N, K such that, for all $n_1, \dots, n_q \geq N$,

$$\| w \| = \sum_{i=1}^{q} n_i \| x_i \| + K.$$

The proof of the Skyscraper Lemma depends on showing that the common parts $c(\,,)$ (as defined in 1.10) of certain growing words eventually stabilize.

LEMMA 4.2. *Suppose that an action of an α-group G on an* **R**-*tree $\mathscr{T}$ with the small arc stabilizers is given. Let $P \in \mathscr{T}$ with corresponding based length function L_P. Suppose that $x, y \in G$ with*

(a) $\| x \| > 0,\ \| y \| > 0,$
(b) $x^{-n} \neq y^m$ *for any positive integers m, n.*

Then there exist constants $N = N_P(x,y)$ and $K = K_P(x,y)$ such that $c(x^{-n}, y^m) = K$ for all $m, n > N$.

Proof. We have $c(x^{-n}, y^m) = \text{length}([P, x^{-n}P] \cap [P, y^m P])$. Write the arc $\alpha(n) \equiv [P, x^{-n}P]$ as the concatenation of three arcs

$$\alpha(n) = \alpha_0 \cup \alpha_1(n) \cup \alpha_2(n)$$

where $\alpha_1(n) = \alpha(n) \cap C_x$ is the arc of length $n\|x\|$ pictured in Fig. 2. Similarly write

$$\beta(m) = [P, y^m P] = \beta_0 \cup \beta_1(m) \cup \beta_2(m).$$

Case 1. Assume that $C_x \cap C_y$ is bounded. Let K be the length of the bounded (possibly degenerate) interval

$$\gamma \equiv \left(\alpha_0 \cup \bigcup_{n>0} \alpha_1(n)\right) \cap \left(\beta_0 \cup \bigcup_{m>0} \beta_1(m)\right).$$

Choose N so large that $m, n > N$ implies $m\|x\|$ and $n\|y\|$ are both greater than length(γ). Then, if m and n are greater than N we have

$$c(x^{-n}, y^m) \equiv \text{length}(\alpha(n) \cap \beta(m)) = \text{length}(\gamma) = K.$$

Case 2. Assume that $C_x \cap C_y$ has infinite diameter.

CLAIM. *$C_x = C_y$ and x and y shift points of this common axis in the same direction.*

This is because C_x and C_y have at least an infinite ray in common. The fact that the action is small implies that $\langle x, y \rangle$ cannot contain a free group of rank 2. (Otherwise two elements of the commutator subgroup of $\langle x, y \rangle$ would generate a free group of rank 2. This is impossible because each such element fixes all points sufficiently far out on the ray, and the action would not be small. Since G is an α-group there exist $m, n > 0$ such that $x^n = y^{\varepsilon m} \neq 1$ for some $\varepsilon = \pm 1$. By the hypothesis of Lemma 4.2(b), $x^n \neq y^{-m}$. Thus $x^n = y^m$ for some $m, n > 0$ and the claim follows.

From the claim we see that $c(x^{-n}, y^m) = \text{length}(\alpha_0) = \text{length}(\beta_0)$. Lemma 4.2 follows if we let $N = 0$ and $K = \text{length}(\alpha_0)$. □

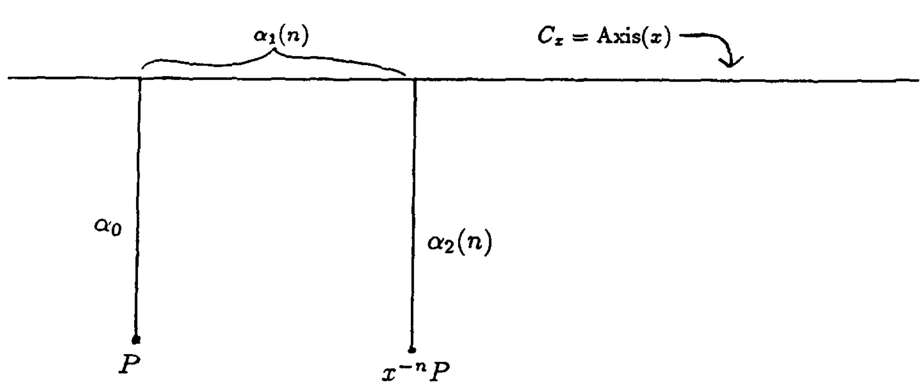

Fig. 2.

Lyndon's Cancellation Lemma

We now recall Lyndon's Cancellation Lemma (stated in our notation).

LEMMA 4.3 (Lyndon [20, (6.1)]). *If L_P is the based length function of an action of the group G on an R-tree $\mathscr{T}$ with base point P and if $w = z \dots z_{q+1}$, where*

$$c(z_{i-1}^{-1}, z_i) + c(z_i^{-1}, z_{i+1}) < L_P(z_i), \quad 2 \le i \le q$$

then

$$L_P(z_1 \cdots z_{q+1}) = \sum_{i=1}^{q+1} L_P(z_i) - 2\sum_{i=1}^{q} c(z_i^{-1}, z_{i+1}).$$

Proof. Lyndon's paper is in terms of abstract length functions, $|\cdot|$, satisfying certain axioms. We define $|z| = L_P(z)$ and note that our $c(x, y)$ becomes Lyndon's $d(x, y)$ once we change our convention to let G act on the tree on the right rather than on the left. We also note that Lyndon's proof only uses his Lemma 2.3; this assumes his axioms A2 and A4 which are trivially satisfied with these meanings of $|z|$ and $d(x, y)$. Then Lyndon's Lemma formally applies to give Lemma 4.3.

(Alternatively, one can convince oneself that Lemma 4.3 is true by drawing a picture when $q = 2$, and then doing a straightforward induction for $q > 2$. This will probably be the same induction Lyndon did.) □

Proof of the Skyscraper Lemma 4.1. Write

$$\begin{aligned} w &= b_0 x_1^{n_1} b_1 x_2^{n_2} \dots b_{q-1} x_q^{n_q} b_q \\ B_i &= b_0 \dots b_i \quad (0 \le i \le q) \\ y_i &= B_{i-1} x_i B_{i-1}^{-1} \quad (1 \le i \le q) \\ y_{q+1} &= B_q. \end{aligned}$$

Then

$$w = y_1^{n_1} \dots y_q^{n_q} y_{q+1}.$$

The hypothesis of the Skyscraper Lemma gives $\|y_i\| = \|x_i\| > 0\,(1 \le i \le q)$, and $y_i^{-n} \ne y_{i+1}^m$ for any $m, n > 0$ $(1 \le i \le q-1)$. Thus Lemma 4.2 applies to each pair y_i, y_{i+1} $(1 \le i \le q-1)$ and there exist constants K_i and N_i $(1 \le i \le q-1)$ such that for all $n_i, n_{i+1} > N_i$ we have $c(y_i^{-n_i}, y_{i+1}^{n_{i+1}}) = K_i$.

Also, if n_q is large enough that $n_q \|y_q\| > L_P(B_q)$ then $c(y_q^{-n_q}, y_{q+1}) = K_q$ is constant (the length of a constant intersection of arcs).

Choose N_P so large that

$$N_P > \max\left\{N_1, \dots, N_{q-1}, \frac{1}{\|y_q\|} L_P(B_q)\right\}$$

and that for all $n_1, \dots, n_q > N_P$ we have

$$\|y_i^{n_i}\| > K_{i-1} + K_i.$$

Then

$$L_P(y_i^{n_i}) = \|y_i^{n_i}\| + 2\,\mathrm{dist}(P, C_{y_i}) > K_{i-1} + K_i \quad (2 \le i \le q).$$

So by Lyndon's Lemma 4.3, $K_i = c(y_i^{-n_i}, y_{i+1}^{n_{i+1}})$ implies

$$\begin{aligned} L_P(w) &= \sum_{i=1}^{q+1} L_P(y_i^{n_i}) - 2\sum_{i=1}^{q} K_i \\ &= \sum_{i=1}^{q} (\| y_i^{n_i} \| + 2\,\mathrm{dist}(P, C_{y_i})) + L_P(B_q) - 2\sum_{i=1}^{q} K_i \\ &= \sum_{i=1}^{q} n_i \| x_i \| + K_P \end{aligned}$$

where

$$K_P = \sum_{i=1}^{q} 2\,\mathrm{dist}(P, C_{y_i}) + L_P(B_1) - 2\sum_{i=1}^{q} K_i$$

is independent of the n_i.

This completes the proof of part (a) of the Skyscraper Lemma.

To prove part (b), we use the fact that $\| w \| = L_P(w^2) - L_P(w)$. We let B_i, y_i be as above $(0 \le i \le q)$ but instead of $y_{q+1} = B_q$ we set

$$\begin{aligned} B_{q+j} &= b_0 \ldots b_q b_0 \ldots b_{j-1} \quad (1 \le j \le q+1) \\ y_{q+j} &= B_{q+j} x_j B_{q+j}^{-1} \quad (1 \le j \le q) \\ y_{2q+1} &= B_{2q+1} = B_q^2. \end{aligned}$$

Then

$$w^2 = y_1^{n_1} y_2^{n_2} \ldots y_q^{n_q} y_{q+1}^{n_1} \cdots y_{2q}^{n_q} y_{2q+1}.$$

Again Lemma 4.2 applies to each pair y_i, y_{i+1} $(1 \le i \le 2q)$. This is true for y_q, y_{q+1} because $y_q^{-n} \neq y_{q+1}^{m}$ $(m, n > 0)$ by hypothesis (iii) of the Skyscraper Lemma which says that $x_q^{-n} \neq b_q b_0 x_1^m (b_q b_0)^{-1}$. Then, using Lyndon's Lemma as in (a) to calculate $L_P(w^2)$ we get, for all n_i sufficiently large $(n_i > N)$, that

$$\begin{aligned} \| w \| &= L_P(w^2) - L_P(w) \\ &= 2\sum_{i=1}^{q} n_i \|x_i\| - \sum_{i=1}^{q} n_i \| x_i \| + K \\ &= \sum_{i=1}^{q} n_i \| x_i \| + K \end{aligned}$$

where K is a constant independent of the $n_i > N$. Finally K is independent of the base point P because $\| w \|$ and the $\| x_i \|$ are independent of P. □

PART II. SIMPLICIAL ACTIONS: METRIC BASS–SERRE THEORY

In Part II (Sections 5–10) we connect the Bass-Serre theory to the metric theory of groups acting on **R**-trees. We introduce a number of tools which will be needed later—most significantly Dehn twist automorphisms (Section 6) and very small graphs of groups (Section 10) — and we prove (Theorem 9.7) that the space of small simplicial actions of G on **R**-trees is often infinite dimensional.

5. GRAPHS OF GROUPS: NOTATION AND BASIC FACTS

We recast results of [29, 28, 10] with the notation modified for our purposes.

5.1. A *graph* Γ consists of a set $E = E(\Gamma)$ of *oriented edges* and a set $V = V(\Gamma)$ of *vertices* along with

(1) an involution $e \to \bar{e}$ of E with $e \neq \bar{e}$ for all $e \in E$ and

(2) two maps $\tau: E \to V$ ($\tau(e)$ is called the "terminal vertex" of e) and $\omega: E \to V$ ($\omega(e)$ is called the "initial vertex" of e) such that $\tau(e) = \omega(\bar{e})$ for all $e \in E$.

5.2. A *graph of groups* $\mathcal{G}$ consists of a connected graph $\Gamma = \Gamma(\mathcal{G})$, a family of groups $\{G_v | \, v \in V(\Gamma)\}$, a family of groups $\{G_e | \, e \in E(\Gamma)\}$ with $G_e = G_{\bar{e}}$ for all $e \in E$, and a family of injections $\{f_e: G_e \rightarrowtail G_{\tau(e)} \,|\, e \in E\}$.

$$\mathcal{G} = (\Gamma, \{G_v\}, \{G_e\}, \{f_e: G_e \rightarrowtail G_{\tau(e)}\}).$$

A graph of groups $\mathcal{G}$ is called *finite* if the underlying graph Γ is finite.

5.3. The *Bass group* $\beta(\mathcal{G})$ of a graph of groups $\mathcal{G}$ (called $F(G, Y)$ on p. 41 of [29]) is defined as

$$\beta(\mathcal{G}) = (*(G_v)_{v \in V} * F(\{t_e \,|\, e \in E\}))/R$$

where $*(G_v)_{v \in V}$ denotes the free product of the vertex groups, $F\{t_e \,|\, e \in E\}$ is the free group on $\{t_e \,|\, e \in E\}$ and R is the normal subgroup determined by the relations

(a) $t_{\bar{e}} = t_e^{-1}$ for all $e \in E$,

(b) $f_{\bar{e}}(g) = t_e f_e(g) t_e^{-1}$ for all $e \in E$, $g \in G_e$.

For notational convenience we will, in many situations, use the symbol t_i for t_{e_i} with $e_i \in E$.

For all vertices $v \in V$ the groups G_v (and hence their subgroups $f_e(G_e)$) inject via the canonical map into $\beta(\mathcal{G})$.

5.4. A *connected word in* $\beta(\mathcal{G})$ is an element $w \in \beta(\mathcal{G})$ of the form

$$w = r_0 t_1 r_1 t_2 \ldots t_q r_q$$

where $r_0 \in G_{\omega(e_1)}$, $\tau(e_i) = \omega(e_{i+1}) r_i \in G_{\tau(e_i)}$ for all $i = 1, \ldots, q-1$, $r_q \in G_{\tau(r_q)}$. We say w *is a word of type* c where c denotes the path $e_1 e_2 \cdots e_q$ in Γ.

A *reduced* word $w = r_0 t_1 r_1 t_2 \cdots t_q r_q$ is a connected word which satisfies either

(1) $q = 0$ and $r_0 \neq 1$, or

(2) $q > 0$ and, for all $i = 1, \ldots, q-1$, if $r_i \in f_{e_i}(G_{e_i})$ then $t_i \neq t_{i+1}^{-1}$.

Further w is *cyclically reduced* if w is reduced, if $\omega(e_0) = \tau(e_q)$ and if $t_q = t_0^{-1}$ *implies* $(r_q r_0) \notin f_{e_q}(G_{e_q})$.

5.5. Reduced words have the following fundamental properties:

(1) They represent non-trivial elements of $\beta(\mathcal{G})$.

(2) Every non-trivial element of $\beta(\mathcal{G})$ can be written as a reduced word.

(3) All reduced words which represent the same element of $\beta(\mathcal{G})$ have the same sequence of t_i [i.e. they are of type c for the same (not necessarily reduced) path $c = e_1 e_2 \cdots e_q$ in Γ]. In particular they have the same *combinatorial length* q.

(4) Any two reduced words which represent the same element of $\beta(\mathscr{G})$ can be transformed into each other by a finite sequence of replacements of the form $t_i \mapsto f_{\bar{e}_i}(g)^{-1} t_i f_{e_i}(g)$ with $g \in G_{e_i}$ (i.e. without changing the path c by introducing new t_i).

5.6. The *fundamental group* of a graph of groups $\mathscr{G}$ is defined and denoted as follows if $P \in V$ and T_0 is a maximal tree in the graph Γ:

(a) $\pi_1(\mathscr{G}, P) = \{w \in \beta(\mathscr{G}) \mid w$ is a word of type c for some closed loop c based at $P\}$.
(b) $\pi_1(\mathscr{G}, T_0) = \beta(\mathscr{G})/(t_e = 1$ if $e \in T_0)$.

The composition of the canonical maps $\pi_1(\mathscr{G}, P) \xrightarrow{i} \beta(\mathscr{G}) \xrightarrow{\pi} \pi_1(\mathscr{G}, T_0)$ is an isomorphism for all vertices $P \in \Gamma$ and all maximal trees $T_0 \subset \Gamma$. In particular all vertex groups G_e inject (canonically up to conjugation) into $\pi_1(\mathscr{G}, T_0)$.

The choice of any such T_0 induces a well-defined retraction $(\pi \circ i)^{-1} \circ \pi : \beta(\mathscr{G}) \to \pi_1(\mathscr{G}, P)$, which maps every reduced word w in $\beta(\mathscr{G})$ of type c to the element $bwd^{-1} \in \pi_1(\mathscr{G}, P)$ where b and d are the uniquely determined reduced words in the letters t_e, $e \in T_0$, which are read off from arcs in T_0 connecting P to the initial and terminal vertices of the path c.

5.7. A graph of groups $\mathscr{G}$ defines a simplicial tree $\mathscr{T}_{\mathscr{G}}$ as follows. Fix a vertex $P \in V$. Every vertex in $\mathscr{T}_{\mathscr{G}}$ will have a level. There is one vertex of $\mathscr{T}_{\mathscr{G}}$ of level 0, and is written as $\langle r_0 \rangle$ for every $r_0 \in G_P$ (including $r_0 = 1$). The vertices of $\mathscr{T}_{\mathscr{G}}$ of level $q > 0$ are identified with equivalence classes of reduced words

$$w = r_0 t_1 \dots t_q r_q \in \beta(\mathscr{G}) \quad (r_0 \in G_P)$$

under the equivalence relation: $\langle w \rangle = \langle w' \rangle$ iff

(a) w and w' are words of type c for the same path c in Γ, and
(b) $w^{-1} w' \in G_v$, where v is the terminal point of c.

We say that such a vertex $\langle w \rangle$ of $\mathscr{T}_{\mathscr{G}}$ *lies above* or *projects to* $v \in \Gamma$ and that $\langle r_0 \rangle$ lies above P if $r_0 \in G_P$.

The *positively oriented edges* of $\mathscr{T}_{\mathscr{G}}$ are defined to be in one–one correspondence with the ordered pairs $(\langle w_q \rangle, \langle w_{q+1} \rangle)$ of vertices of level q and $q+1$ $(q \geq 0)$ such that $w_{q+1} = w_q t_{q+1} r_{q+1}$. Such an edge is said to *lie above* or to *project to* $e_{q+1} \in E(\Gamma)$ where $t_{q+1} = t_{e_{q+1}}$. The inverse edge $(\langle w_{q+1} \rangle, \langle w_q \rangle)$ is said to lie above $\bar{e}_{q+1}$. Note that in this situation $\langle w_q t_{q+1} r_{q+1} \rangle = \langle w'_q t'_{q+1} r'_{q+1} \rangle$ if and only if $t_{q+1} = t'_{q+1}$ and $w_q^{-1} w'_q \in f_{\bar{e}_{q+1}}(G_{e_{q+1}})$. Thus a vertex of level $q+1$ is the end point of only one edge emanating from the set of vertices of level q. It follows that $\mathscr{T}_{\mathscr{G}}$ is a tree, that each reduced word $w = r_0 t_1 \dots t_q r_q$ determines an arc of q segments from $\langle 1 \rangle$ to $\langle w \rangle$, and that $\mathscr{T}_{\mathscr{G}}$ is a union of these arcs. (Compare [10].)

5.8. The fundamental group $\pi_1(\mathscr{G}, P)$ acts naturally on $\mathscr{T}_{\mathscr{G}}$ from the left (by left multiplication and subsequent reduction of the vertex labels). One readily deduces that this is a simplicial action without inversion of edges and that the quotient of $\mathscr{T}_{\mathscr{G}}$ modulo this action is the graph Γ. The stabilizer of any simplex in $T_{\mathscr{G}}$ which lies above a vertex or edge $\sigma \in \Gamma$ is isomorphic [conjugate in $\beta(\mathscr{G})$] to G_σ. Indeed, if $w \in \beta(\mathscr{G})$ is a reduced word along a path from P to v and if $g \in \pi_1(\mathscr{G}, P)$ then $g\langle w \rangle \equiv \langle gw \rangle = \langle w \rangle \Leftrightarrow g \in wG_v w^{-1}$, by 5.7(b). Thus:

5.8a. $\mathrm{stab}(\langle w \rangle) = wG_v w^{-1}$.

If $\tilde{e} = (\langle w_1 \rangle, \langle w_2 \rangle)$ is an oriented edge of $\mathcal{T}_{\mathcal{G}}$ with vertex representatives chosen so that $w_2 = w_1 t_e$ then, taking intersections of vertex stabilizers, we see that

5.8b. $\mathrm{stab}(\tilde{e}) = w_1 f_{\bar{e}}(G_e) w_1^{-1} = w_2 f_e(G_e) w_2^{-1}$.

5.9. In the opposite direction, if we start with a group G which acts without inversion on some simplicial tree $\mathcal{T}^s$ then [29, p. 55] for any vertex P of the quotient graph $\Gamma = \mathcal{T}^s/G$ we obtain an isomorphism $G = \pi_1(\mathcal{G}, P)$. Here the graph of groups $\mathcal{G}$ is built on Γ with the edge groups G_e and the vertex groups G_v of $\mathcal{G}$ isomorphic to the stabilizers of simplexes $\tilde{e}$ and $\tilde{v}$ in $\mathcal{T}^s$ which project to e and v in Γ. The injections f_e come from the inclusions $G_{\tilde{e}} \subset G_{\tau(\tilde{e})} \cong G_{\tilde{v}}$ where $\tilde{v} = \widetilde{\tau(e)}$. The tree $\mathcal{T}_{\mathcal{G}}$ constructed from $\mathcal{G}$ as in 5.7 is then $\pi_1(\mathcal{G}, P)$-equivariantly isomorphic to $\mathcal{T}^s$.

Remark. We shall need the Bass–Serre theory in the form outlined above. After writing this section we became aware of Bass' paper [4]. Section 1 of that paper gives a careful exposition, with complete proofs, of this form of the theory.

6. MULTIPLE DEHN TWIST AUTOMORPHISMS

Definition 6.1. Let $\mathcal{G}$ be a graph of groups. Fix an (oriented) edge $e \in E$. Assume that z is an element of the center of the group $G_e = G_{\bar{e}}$. Define an automorphism $D = D_z$ of the free product $*(G_v)_{v \in V} * F(\{t_{e'} \mid e' \in E\})$ by the condition that

(a) $D(t_e) = t_e f_e(z)$, $D(t_{\bar{e}}) = t_{\bar{e}} f_{\bar{e}}(z)^{-1}$, and
(b) $D(r) = r$ if $r \in G_v$ $(v \in V(\Gamma))$, $D(t_{e'}) = t_{e'}$ if $e' \in E(\Gamma) - \{e, \bar{e}\}$.

LEMMA 6.2. (1) *This automorphism induces an automorphism (also called $D = D_z$) on the Bass group $\beta(\mathcal{G})$.*

(2) *For all $P \in V$ the automorphism $D: \beta(\mathcal{G}) \to \beta(\mathcal{G})$ restricts to an automorphism*

$$D = D_z : \pi_1(\mathcal{G}, P) \to \pi_1(\mathcal{G}, P).$$

Notation 6.3. We call $D_z : \pi_1(\mathcal{G}, P) \to \pi_1(\mathcal{G}, P)$ a *single Dehn twist of $\pi_1(\mathcal{G}, P)$, based on $\mathcal{G}$.* The element $z \in \mathrm{Center}(G_e)$ will sometimes be called the *twistor* and we say that we have *twisted on the edge e.* Note that the n-fold composition $(D_z)^n = D_{z^n}$.

Proof of 6.2. (1) We have to show that both sides of the equations (a) and (b) of 5.3 are mapped by D to the same element in $\beta(\mathcal{G})$. This is obviously true for all edges other than e. It is readily checked for e if one notes that $D(t_e)^{-1} = D(t_{\bar{e}})$ and that $z \in \mathrm{Center}(G_e)$ implies $f_e(g) = f_e(zgz^{-1})$.

(2) If w is a word of type c for some loop based at P then so are $D(w)$ and $D^{-1}(w)$. Thus D takes $\pi_1(\mathcal{G}, P)$ onto itself. □

The following is a direct consequence of Definition 6.1.

LEMMA 6.4. *The elements of any sequence of single Dehn twists $D_{z_1}, \ldots, D_{z_p}$ based on a given graph of groups $\mathcal{G}$ commute.* □

Definition 6.5. (1) If $\{e_1, \ldots, e_p\} \subset E(\Gamma)$ and $e_i \notin \{e_j, \bar{e}_j\}$ for all $i \neq j$ then $D = D_{z_1} \circ \cdots \circ D_{z_p}$ is called a *multiple Dehn twist* (also called simply a *Dehn twist*) with twistors z_i.

(2) An automorphism φ of a group G *is representable as a multiple Dehn twist t automorphism* if there exist a graph of groups $\mathcal{G}$, an isomorphism $h: G \to \pi_1(\mathcal{G}, P)$ and a multiple Dehn twist D of $\pi_1(\mathcal{G}, P)$ such that $\varphi = h^{-1} \circ D \circ h$. We also say that φ *is represented by the multiple Dehn twist D based on* $\mathcal{G}$.

Remark 6.6. As is true for all (outer) automorphisms of a group G, any Dehn twist automorphism D of $G = \pi_1(\mathcal{G}, P)$ induces a homeomorphism D^* (sometimes simply denoted D) on each of the spaces $LF(G)$, $PLF(G)$, $SLF(G)$ and $VSL(G)$, by permuting their coordinates (see 1.12).

Example 6.7. (a) *Nielsen automorphisms of* $F(x_1, \ldots, x_n)$. Let $\mathcal{G}$ be the graph of groups in Fig. 3. Let D be the Dehn twist of $\pi_1(\mathcal{G}, P)$ based on $\mathcal{G}$ given by

$[D(t) = tf_e(z) = ta_j]$.

Let φ and ψ be the Nielsen automorphisms of $F(x_1, \ldots, x_n)$ given by $\varphi(x_i) = \psi(x_i) = x_i$ $(i \neq j)$, $\varphi(x_j) = x_k^{\varepsilon} x_j$ and $\psi(x_j) = x_j x_k^{\varepsilon}$. Then φ is represented by the Dehn twist D via the isomorphism $h_1 : F(x_1, \ldots, x_n) \to \pi_1(\mathcal{G}, P)$ with $h_1(x_i) = a_i$ $(i \neq j)$ and $h_1(x_j) = t$ while ψ is represented by the Dehn twist D^{-1} via the isomorphism h_2 with $h_2(x_i) = a_i$ $(i \neq j)$ and $h_2(x_j) = t^{-1}$.

(b) It is easy to construct automorphisms of F_n which are representable as Dehn twist automorphisms but for which this fact is not obvious. As an example, consider

$$\varphi : F_5 \to F_5 \text{ by} \quad \begin{aligned} a &\mapsto a^2 e \\ b &\mapsto bae \\ c &\mapsto cedae^2 bdebda^2 e \\ d &\mapsto e^{-1} a^{-1} dae \\ e &\mapsto e^{-1} a^{-1} e. \end{aligned}$$

Remark 6.8. For every automorphism $\varphi \in \mathrm{Out}(G)$ which can be represented by a multiple Dehn twist automorphism, the growth function $GF_\varphi(\cdot)$ is *of linear type*:

$$GF_\varphi(m) \equiv \limsup \left\{ \frac{|\varphi^m[w]|_A}{|[w]|_A} \,\middle|\, w \in G \right\} \leq K \cdot m$$

for some $K \in \mathbf{R}$ where $|[w]|_A$ denotes the word length of the shortest representative of the conjugacy class $[w]$ with respect to any fixed generating system A of G.

The authors do not know for α-groups in general whether this linear growth characterizes Dehn twists as outer automorphisms (up to replacing φ by some power). For free groups, however, this can be shown to be true using relative train track representatives as

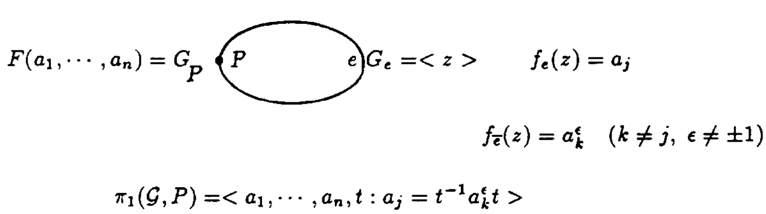

Fig. 3.

introduced by Bestvina and Handel in [8]. It is proved in [11] that every Dehn twist automorphism has a relative train track representative such that the transition matrix of every stratum is the identity matrix. The converse is only partially true: no train track map with at least one exponentially growing stratum represents a Dehn twist automorphism (by the growth inequality above). However, there are relative train track maps with every stratum a permutation matrix such that the growth of the automorphism represented is not linear, but polynomial of higher degree. The algorithmic question as to which ones are linear (and hence represent roots of Dehn twist automorphisms) and which are not is solved in [11].

7. THE QUOTIENT GRAPH OF GROUPS

7.1. Let $\mathcal{G}$ be any graph of groups, and let $\Gamma_+ \subset \Gamma$ be a not necessarily connected subgraph which contains all vertices of Γ. For every connected component Γ_α of Γ_+ we choose a base point P_α (possibly $\Gamma_\alpha = P_\alpha$) and a maximal tree T_α. Every such component defines a *subgraph of groups* $\mathcal{G}_\alpha$ of $\mathcal{G}$. The Bass groups $\beta(\mathcal{G}_\alpha)$ are canonically embedded in $\beta(\mathcal{G})$.

We define a *quotient graph of groups* $\mathcal{G}/\{\mathcal{G}_\alpha\}$ by contracting each component Γ_α of Γ_+ to a vertex. The edges E_j of the quotient graph $\Gamma/\{\Gamma_\alpha\}$ are in one-to-one correspondence with the edges e_j of $\Gamma - \Gamma_+$. The incidence maps τ and ω for $\Gamma/\{\Gamma_\alpha\}$ are inherited from Γ in the obvious way. The vertex groups of $\mathcal{G}/\{\mathcal{G}_\alpha\}$ are defined by $G_\alpha = \pi_1(\mathcal{G}_\alpha, P_\alpha)$, and the edge groups by $G_{E_j} = G_{e_j}$. For any edge E_j with $\tau(e_j) \in \Gamma_\alpha$ the injection of $f_{E_j}: G_{E_j} \to G_\alpha$ is defined as composition

$$G_{E_j} \xrightarrow{\text{id}} G_{e_j} \xrightarrow{f_{e_j}} G_{\tau(e_j)} \subset \beta(\mathcal{G}_\alpha) \to \pi_1(\mathcal{G}_\alpha, P_\alpha)$$

where the last map is the retraction given by the maximal tree T_α (see 5.6).

LEMMA 7.2. *Let $\mathcal{G}$ be a graph of groups as in 7.1, with a specified maximal tree T_α in every component Γ_α. Let $P \in \Gamma$ be a vertex of Γ, and let P_0 be its image in $\Gamma/\{\Gamma_\alpha\}$. Then there is a canonical isomorphism*

$$f_{\{T_\alpha\}}: \pi_1(\mathcal{G}, P) \to \pi_1(\mathcal{G}/\{\mathcal{G}_\alpha\}, P_0)$$

which maps a word $w = r_0 t_1 r_1 t_2 \ldots t_q r_q$ of type c (a loop in Γ based at P) to the word $W = \hat{R}_0 \tau_1 \hat{R}_1 \tau_2 \cdots \tau_p \hat{R}_p$. Here each R_i is maximal subword of w which is read off along a subpath c_i as c passes through a component Γ_{α_i} of Γ_+, and $\hat{R}_i$ denotes the image of R_i under the retraction $\beta(\mathcal{G}_{\alpha_i}) \to \pi_1(\mathcal{G}_{\alpha_i}, P_{\alpha_i})$ determined by T_{α_i} as in 5.6. The symbols τ_i in the word W stand for t_{E_i}. Thus $R_i = b_i R_i d_i^{-1}$ where the b_i and d_i are words in the t's which belong to T_{α_i}. (See Fig. 4.)

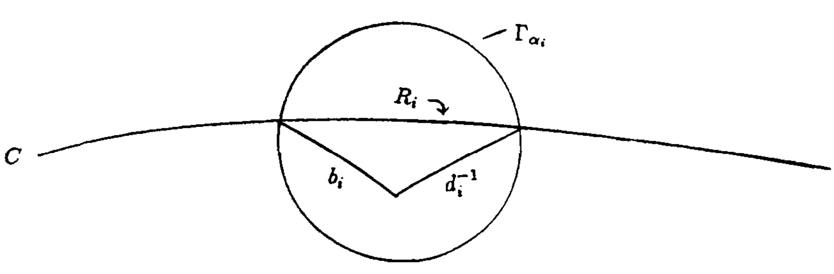

Fig. 4.

Proof. For notational simplicity we will consider only the special case $G_{E_i} = \{1\}$ for all edges E_j of $\Gamma/\{\Gamma_\alpha\}$. The general case (which will not be used in the sequel) follows the same scheme.

In the special case there is a canonical isomorphism

$$\beta(\mathscr{G}) \to \mathop{*}_{\alpha}(\beta(\mathscr{G}_\alpha)) * F(\{t_{E_j} \mid E_j \in E(\Gamma/\{\Gamma_\alpha\})\})/t_{\bar{E}_j} = t_{E_j}^{-1}$$

which maps t_{e_j} to t_{E_j} for all $e_j \in (\Gamma - \Gamma_+)$. We choose a subset T_e of $E(\Gamma - \Gamma_+)$ so that $T = \bigcup_\alpha T_\alpha \cup T_e$ forms a maximal tree in Γ. Observe that the corresponding subset $T_E \subset E(\Gamma/\{\Gamma_\alpha\})$ defines a maximal tree of $\Gamma/\{\Gamma_\alpha\}$. The above isomorphism induces an isomorphism

$$\begin{aligned}\pi_1(\mathscr{G}, T) &= \beta(\mathscr{G})/\langle\{t_e \mid e \in T\}\rangle \\ &\to F(\{t_{E_j} \mid E_j \in E(\Gamma/\{\Gamma_\alpha\}) - T_E\}) \mathop{*}_{\alpha} \{\beta(\mathscr{G}_\alpha)\}/\langle\{t_e \mid e \in T_\alpha\}\rangle \\ &= \pi_1(\mathscr{G}/\{\mathscr{G}_\alpha\}, T_E).\end{aligned}$$

That this isomorphism coincides with the map $f_{\{T_\alpha\}}$ follows directly from the definition in 5.6 of the isomorphisms $\pi_1(\mathscr{G}, T) \cong \pi_1(\mathscr{G}, P), \pi_1(\mathscr{G}/\{\mathscr{G}_\alpha\}, T_E) \cong \pi_1(\mathscr{G}/\{\mathscr{G}_\alpha\}, P_0)$ and $\beta(\mathscr{G}_\alpha)/\langle t_e \mid e \in T_\alpha\rangle \cong \pi_1(\mathscr{G}_\alpha, \mathscr{T}_\alpha) \cong \pi_1(\mathscr{G}_\alpha, P_\alpha)$. □

Remark 7.3. It is crucial for the Fundamental Convergence Lemma in Section 11 to understand how a Dehn twist D defined on a graph of groups $\mathscr{G}$ acts on the fundamental group of any quotient $\mathscr{G}/\{\mathscr{G}_\alpha\}$ via the isomorphism $f_{\{T_\alpha\}}$ given in the preceding lemma. From the definition of Dehn twist it follows (using the notation of Lemma 7.2) that if

$$w = R_0\tau_1R_1\tau_2\ldots\tau_pR_p \quad \text{with} \quad G_{E_j} = \{1\}, \tau_j = T_{E_j}$$

then

$$D(w) = D(R_0)\tau_1D(R_1)\cdots\tau_pD(R_p)$$

and

$$f_{\{T_\alpha\}}D(w) = \widehat{D(R_0)}\tau_1\widehat{D(R_1)}\cdots\tau_p\widehat{D(R_p)}.$$

Note! $\widehat{D(R_i)} = b_iD(R_i)d_i^{-1}$ will in general be different from

$$D(\hat{R}_i) = D(b_i)D(R_i)D(d_i^{-1}).$$

8. METRIC GRAPHS OF GROUPS AND A COMBINATION LEMMA FOR R-TREES

Definition 8.1. *A metric graph of groups* $\mathscr{G}(L)$ consists of a graph of groups $\mathscr{G}$ and an edge-length function $L{:}E \to \mathbf{R}$ such that $L(e) = L(\bar{e}) \geq 0$ for all $e \in E$, with L not identically zero. For the sake of explicit computation when $\mathscr{G}$ is finite we fix an ordering of the oriented edges $e_1, e_2, \ldots, e_r$ and we write

$$\mathscr{G}(L) = \mathscr{G}(L(e_1), \ldots, L(e_r)).$$

Metric graphs of groups correspond to "simplicial actions on **R**-trees" according to the following discussion.

8.2. A *simplicial structure* on an **R**-tree $\mathcal{T}$ is a family of *edges* (embedded non-degenerate arcs) $\sigma_\alpha^1 = [a_\alpha, b_\alpha] \subset \mathcal{T}$ with *vertices* a_α, b_α such that:

(i) $\mathcal{T} = \bigcup_\alpha \sigma_\alpha^1$.
(ii) If $\alpha \neq \beta$ then $\sigma_\alpha^1 \cap \sigma_\beta^1$ is empty or is a common vertex of σ_α^1 and σ_β^1.
(iii) $\mathring{\sigma}_\alpha^1 = (a_\alpha, b_\alpha)$ is open in $\mathcal{T}$.
(iv) The set of vertices of $\mathcal{T}$ has no accumulation points in $\mathcal{T}$.

A *simplicial* **R**-*tree* $\mathcal{T}$ is an **R**-tree with a given simplicial structure.

Note that:

(1) $\mathcal{T}$ usually does not have the weak topology induced by the simplicial structure.
(2) From (iii) it follows that every branch point of $\mathcal{T}$ is a vertex.
(3) A simplicial structure on $\mathcal{T}$ determines a (combinatorial) simplicial tree which is connected; i.e. any two vertices may be connected by a finite sequence of edges.

If a group G acts without inversion of edges by simplicial isometries on the simplicial **R**-tree $\mathcal{T}$ then the graph of groups $\mathcal{G} = \mathcal{T}/G$ of 5.9 with $\pi_1\mathcal{G} \cong G$ becomes a metric graph with positive edge-length function if we assign to each edge e the length $L(e)$ of some lift of e in $\mathcal{T}$. Conversely, given a metric graph of groups $\mathcal{G} = \mathcal{G}(L)$ with positive edge-lengths, the associated simplicial tree $\mathcal{T}_\mathcal{G}$ of 5.7 becomes a simplicial **R**-tree $\mathcal{T}_{\mathcal{G}(L)}$ if we assign to each edge of $\mathcal{T}_\mathcal{G}$ the length of its projection and then give $\mathcal{T}_\mathcal{G}$ the shortest-path metric. The based length function $L_\mathcal{G}$ and the translation length function $\|\cdot\|_\mathcal{G}$ of the associated action of $\pi_1(\mathcal{G}, P)$ by left multiplication may by read off from the construction of $\mathcal{T}_\mathcal{G}$ in 5.7.

PROPOSITION 8.3. *Let $\mathcal{G}(L) = \mathcal{G}(L(e_1), \dots, L(e_r))$ be a metric graph of groups with positive edge-lengths and with base point P. Then there is an action of $\pi_1(\mathcal{G}, P)$ on the* **R**-*tree $\mathcal{T}_{\mathcal{G}(L)}$, such that for any reduced word $w = r_0 t_1 r_1 t_2 \dots r_{q-1} t_q r_q \in \pi_1(\mathcal{G}, P)$ one has*

$$L_{\mathcal{G}(L)}(w) = \sum_{i=1}^{q} L(e_i) \quad \textit{and} \quad \|w\|_{\mathcal{G}(L)} = L_{\mathcal{G}(L)}(w^2) - L_{\mathcal{G}(L)}(w).$$

If w cyclically reduces in $\beta(\mathcal{G})$ to $\hat{w} = s_j t_j r_{j+1} \dots r_{p-1} t_p$ $(0 \leq j < p \leq q)$ then

$$\|w\|_{\mathcal{G}(L)} = \sum_{i=j}^{p} L(e_i). \qquad \square$$

Remark 8.4. We note that 5.9 is valid in this metric context. If the metric graph of groups $\mathcal{G}(L)$ is obtained as $\mathcal{T}/G$ from an action without inversion of $G = \pi_1(\mathcal{G}, P)$ on some simplicial **R**-tree $\mathcal{T}$ with translation length function $\|\cdot\|_\mathcal{T}$, then the tree $\mathcal{T}_{\mathcal{G}(L)}$ is G-equivariantly isometric to $\mathcal{T}$. In particular one has $\|\cdot\|_\mathcal{T} = \|\ \|_{\mathcal{G}(L)}$.

From Proposition 8.3 it will now follow that if $\mathcal{G}$ is a finite metric graph of groups with positive edge-length function L then a quotient graph of groups $\mathcal{G} = \mathcal{G}'/\{\mathcal{G}_\alpha\}$ (as in 7.1) may be interpreted in $PLF(G)$ as the result of letting the lengths of the edges of $\Gamma_+ = \bigcup_\alpha \Gamma_\alpha \subset \Gamma$ all become zero. To be precise, let Γ_+ be a proper subgraph of Γ which contains all vertices of Γ. Let $e_1, \dots, e_s$ and $E_1, \dots, E_s$ $(s \geq 1)$ be edges of $\Gamma - \Gamma_+$ and the corresponding edges of $\Gamma/\{\Gamma_\alpha\}$. Let $e_{s+1}, \dots, e_{s+r}$ be the edges of Γ_+. Identify $\pi_1\mathcal{G} \cong \pi_1\mathcal{G}'$ via Lemma 7.2 using a family of trees $T_\alpha \subset \Gamma_\alpha$. Then, from the normal forms of corresponding words

$$w = r_0 t_1 \dots t_q r_q \mapsto R_0 t_1 \dots t_p R_p$$

we have the following.

8.5 (Corollary to 8.3). *Let $\mathcal{G}' = \mathcal{G}/\{\mathcal{G}_\alpha\}$ and define $L'(E_j) = L(e_j)$ $(1 \le j \le s)$. Then under the identification $\pi_1\mathcal{G} \cong \pi_1\mathcal{G}'$*

$$\text{(a)}\quad L_{\mathcal{G}'}(R_0 t_1 \dots t_p R_p) = \sum_{k=1}^{p} L'(E_k)$$

$$= \text{“}L_{\mathcal{G}}(w) \text{ with } L(e_i) = 0 \text{ if } e_i \in \Gamma_+\text{”}$$

(in particular $L_{\mathcal{G}'}$ is independent of the choice of trees T_α).

(b) *The tree $\mathcal{T}_{\mathcal{G}'}$ is obtained from the tree $\mathcal{T}_{\mathcal{G}}$ by squeezing arcs above simplexes of Γ_+ to a point.*

$$\text{(c)}\quad \|\cdot\|_{\mathcal{G}'(L')} = \lim_{\substack{\lambda \to 0 \\ 1 \le i \le r}} \|\cdot\|_{\mathcal{G}(L(e_1), \dots, L(e_s), \lambda_1 L(e_{s+1}), \dots, \lambda_r L(e_{s+r}))}.$$

Proof. Assertions (a) and (c) are immediate from Proposition 8.3. Assertion (b) allows from the construction of $\mathcal{T}_{\mathcal{G}}$ and $\mathcal{T}_{\mathcal{G}'}$ in 5.7. □

Generalizing Proposition 8.3, we now present a method for combining **R**-trees to build a new **R**-tree according to the pattern given by a metric graph of groups. A more general construction (which does not have the restriction (#) below) is presented in a different vein in [33], but we shall need the explicit length function given in this lemma.

COMBINATION LEMMA 8.6. *Let $\mathcal{G}(L)$ be a metric graph of groups with base point P. Assume that each of the vertex groups G_v acts on some **R**-tree $\mathcal{T}_v$ with base point P_v, and let L_v denote the based length function on G_v of this action. Assume that*

$$\textit{for all edge groups } G_e \textit{ the restriction of } L_{\tau(e)} \textit{ to } f_e(G_e) \textit{ is } 0. \tag{\#}$$

Then for reduced words $r_0 t_1 r_1 t_2 \dots r_{q-1} t_q r_q$ the definition

$$L_+(r_0 t_1 r_1 t_2 \dots r_{q-1} t_q r_q) = \sum_{j=0}^{q} L_j(r_j) + \sum_{j=1}^{q} L(e_j)$$

*(where L_j stands for L_{v_j} with $r_j \in G_{v_j}$) gives a well-defined based length function $L_+(\mathcal{G}(L), \{L_v\}) = L_+ : \pi_1(\mathcal{G}, P) \to \mathbf{R}$ which satisfies the Lyndon–Chiswell axioms (1.11). Hence it defines an action of $\pi_1(\mathcal{G}, P)$ on an **R**-tree $\mathcal{T}_+ = \mathcal{T}_+(\mathcal{G}(L), \{L_v\})$, with translation length function*

$$\|w\|_+ = L_+(w^2) - L_+(w).$$

Remark. The **R**-tree T_+ resulting from the length function L_+ is called the *$\mathcal{G}$-combination of the trees $\mathcal{T}_v$*. It may be seen from the viewpoint of [10] to be the union of arcs $[P, wP]$, as w ranges over G. If $w = r_0 t_1 \dots t_q r_q$ then the arc $[P, wP]$ is the concatenation of the arc $[P_0, r_0 P_0]$ of length $L_0(r_0)$ in T_{v_0}, followed by an arc of length $L(e_1)$ which ends at the base point P_{v_1} in a copy of T_{v_1}, followed by the arc $[P_{v_1}, r_1 P_{v_1}]$ in this copy of T_{v_1}, etc.

Proof of Lemma 8.6. The length function L_+ is well defined since

(a) every element of $\pi_1(\mathcal{G}, P)$ is represented by a unique path in Γ, and
(b) by 5.5(4) it suffices to show that for every vertex group G_v one has

$$L_v(r) = L_v(f_e(x)r) = L_v(r f_{\bar{e}}(y))$$

for all $r \in G_v$ and $x \in G_e$ with $\tau(e) = v$ or $y \in G_e$ with $w(y) = v$. The latter is a direct consequence of our assumption (#) and the axioms (1.11) for based length functions.

It remains to show that the function L_+ satisfies the Chiswell axioms (1.11). We must verify:

(i) $L_+(1) = 0$.
(ii) $L_+(w) = L_+(w^{-1})$ for all $w \in \pi_1(\mathscr{G}, P)$, and
(iii) $c(u, w) \geq \min\{c(u, v), c(v, w)\}$ for all $u, v, w \in \pi_1(\mathscr{G}, P)$, where $c(x, y) \equiv 1/2(L_+(x) + L_+(y) - L_+(x^{-1}y))$.

The first two conditions are clearly satisfied. To prove (iii) write $u = u_0 r u_1$ and $v = u_0 s v_1$, where u_0 is a reduced word ending in an edge generator t_i and $r, s \in G_{\tau(e_i)}$. Furthermore, either the reduced words u_1, v_1 differ in their first letter t_{i+1}, or $r^{-1}s$ is not contained in $f_{\bar{e}_{i+1}}(G_{e_{i+1}})$. This implies that $u_1^{-1} r^{-1} s v_1$ is a reduced word representing $u^{-1}v$. Then (iii) follows with a bit of computation from the fact that $c(u, v) = L_+(u_0) + c_i(r, s)$ for $c_i(r, s) = 1/2(L_i(r) + L_i(s) - L_i(r^{-1}s))$, and from the fact that L_i is a based length function and hence satisfies the axiom (iii); and the analogous facts hold for $c(u, w)$ and $c(v, w)$. □

COROLLARY 8.7. *If, in the Combination Lemma, the actions of the vertex groups G_v on the trees $\mathscr{T}_v$ are free and the edge groups are (consequently) trivial then the resulting action of $\pi_1(\mathscr{G}, P)$ on $\mathscr{T}_+$ is free.*

Proof. One computes directly that if $w \neq 1$ then

$$\|w\|_+ \equiv L_+(w^2) - L_+(w) > 0.$$ □

9. MINIMAL GRAPHS OF GROUPS AND THE SIMPLEX $\sigma(\mathscr{G})$ OF LENGTH FUNCTIONS

Definition 9.1. A graph of groups $\mathscr{G}$ is *minimal* if:

(a) The underlying graph Γ is finite.
(b) For every proper connected subgraph $\Gamma_0 \subset \Gamma$ with subgraph of groups $\mathscr{G}_0$ the injection $\pi_1(\mathscr{G}_0, P) \to \pi_1(\mathscr{G}, P)$ fails to be onto (P a vertex of Γ_0).

The motivation for this definition is given by the following proposition.

PROPOSITION 9.2. *Suppose that $G = \pi_1(\mathscr{G}, P)$ is finitely generated. Then the associated action of G on $\mathscr{T}_\mathscr{G}$ is minimal if and only if $\mathscr{G}$ is a minimal graph of groups.*

Proof. Exercise for the reader. □

An immediate consequence of the definition and Lemma 7.2 is Lemma 9.3.

LEMMA 9.3. *If $\mathscr{G}$ is a minimal graph of groups then any quotient graph of groups $\mathscr{G}/\{\mathscr{G}_\alpha\}$ is also minimal.*

Definition 9.4. Let $\mathscr{G}$ be a graph of groups, based on a graph Γ with r geometric edges and base point P. Let $|E| = \{e_1, \ldots, e_r\}$ consist of exactly one oriented edge from each pair $e, \bar{e} \in E$. The set of all non-negative edge-length functions $L: |E| \to \mathbf{R}$ with $\sum_{1 \leq i \leq r} L(e_i) = 1$, constitute a closed $(r-1)$-simplex $\sigma^{r-1} \subset \mathbf{R}^r$. By Proposition 8.3 and 8.5 any point of σ^{r-1} determines a based length function $L: \pi_1(\mathscr{G}, P) \to \mathbf{R}$ and hence a point of $PLF(\pi_1(\mathscr{G}, P))$. We denote

$$\sigma(\mathscr{G}) = [\text{the image of the map } \sigma^{r-1} \to PLF(\pi_1(\mathscr{G}, P))].$$

We wish to assert that this map is an embedding and that $\sigma(\mathcal{G})$ has dimension $r-1$. This will be true for a minimal graph of groups $\mathcal{G}$ provided that every vertex of $\mathcal{G}$ leads to a degree of freedom in $PLF(G)$. We must require that there be no *invisible vertices.*

Definition 9.5. A vertex v of the graph of groups $\mathcal{G}$ is *invisible* if v is a valence-two vertex—say $v = \tau(e_1) = \tau(e_2)$—and if both injections $f_{e_i}: G_{e_i} \to G_v$ $(i = 1, 2)$ are isomorphisms. A *visible* vertex is one which is not invisible.

Invisible vertices frequently occur in minimal graphs of groups. (Hint: Barycentrically subdivide an edge of your favorite minimal graph of groups.) Note that a vertex v of Γ is visible iff

(a) v has valence one in Γ or
(b) every point $\tilde{v}$ which projects to v is a branch point of $\mathcal{T}_{\mathcal{G}}$.

To see how invisible vertices distort the information given by $\sigma(\mathcal{G})$, suppose that $\tau(e_1) = \tau(e_2)$ is an invisible vertex as above, let $\mathcal{G}_{e_1}$ be the subgraph of groups carried by the closure of e_1 and let $\mathcal{G}' = \mathcal{G}/\{\mathcal{G}_{e_1}\}$. Let E_2 be the image of e_2 in $\mathcal{G}'$. If we let $L'(E_2) = L(e_1) + L(e_2)$ and leave other lengths unchanged then, under the natural identification $\pi_1\mathcal{G} \cong \pi_1\mathcal{G}'$ (see Lemma 7.2), we get an identification $\sigma(\mathcal{G}') = \sigma(\mathcal{G})$ although $\mathcal{G}$ and $\mathcal{G}'$ are based on graphs with different numbers of edges.

PROPOSITION 9.6. *Let $\mathcal{G}$ be a minimal graph of groups with r geometric edges which has no invisible vertices. Assume that $\mathcal{G}$ is non- "self-swallowing" (i.e. $\mathcal{G}$ is not a simple cycle such that in one direction all edge group injections f_e are surjective). Then the map $\sigma^{r-1} \to \sigma(\mathcal{G})$ of Definition 9.4 is a homeomorphism and $\sigma(\mathcal{G})$ has dimension $r - 1$.*

Remark. If $\mathcal{G}$ is self-swallowing with edges $e_1, e_2, \dots, e_r$ forming a simple loop and each $f_{e_i} = f_i$ an isomorphism, then any word is equivalent to a reduced word of the form $w = r_0(t_1 t_2 \dots t_r)^{m(w)}$. Whatever the lengths $L(e_i)$, we have $\|w\|_{\mathcal{G}(L)} = |m(w)| \sum_{i=1}^{r} L(e_i)$. Thus $\sigma(\mathcal{G})$ consists of a single point of $PLF(G)$. Self-swallowing cycles were first pointed out by Bestvina and Feighn [5].

Proof of Proposition 9.6. (1) For every edge $e \in E(\Gamma)$ there is a reduced path $e = e_0, e_1, e_2, \dots, e_p$ $(p \geq 0)$ "with a new group element at the end of the path". This means that either

(1a) there exists an element $g \in G_{\tau(e_p)}$ which is not contained in $f_{e_p}(G_{e_p})$ or
(1b) there is a loop $e_{p+1}, \dots, e_s$, based at $\tau(e_p)$ such that $e_{p+1} \neq \bar{e}_p$ and $e_s \neq e_p$.

Proof of (1). If there is no reduced path such that (1a) holds then, for every reduced path $e, e_1, \dots, e_p$, the map f_{e_p} is surjective, and minimality implies that $\tau(e_p)$ does not have valence 1. By the finiteness of Γ every such path must eventually come back and touch itself. Starting with e, build such a path $e = e_0, e_1, \dots, e_q$ $(q \geq 0)$ such that $e_1, \dots, e_{q-1}$ is an arc and $\tau(e_q) = w(e_{s+1})$, $s \leq q - 1$. If $e_{s+1} \neq e_0$, then $e_{s+1}, \dots, e_q$ is a loop giving a group element at the end of the path $e_0, e_1, \dots, e_s$. If $e_{s+1} = e_0$, so that $e_0, e_1, \dots, e_s$ is a circle C, then, since $\mathcal{G}$ is not a self-swallowing loop, there is a branch point of C at some vertex $\tau(e_k)$. Out of this branch point either an arc grows which ends in a loop, or which comes back and touches C at a branch point. Thus one of the possibilities in Fig. 5 occurs in our graph, and (1) is proved.

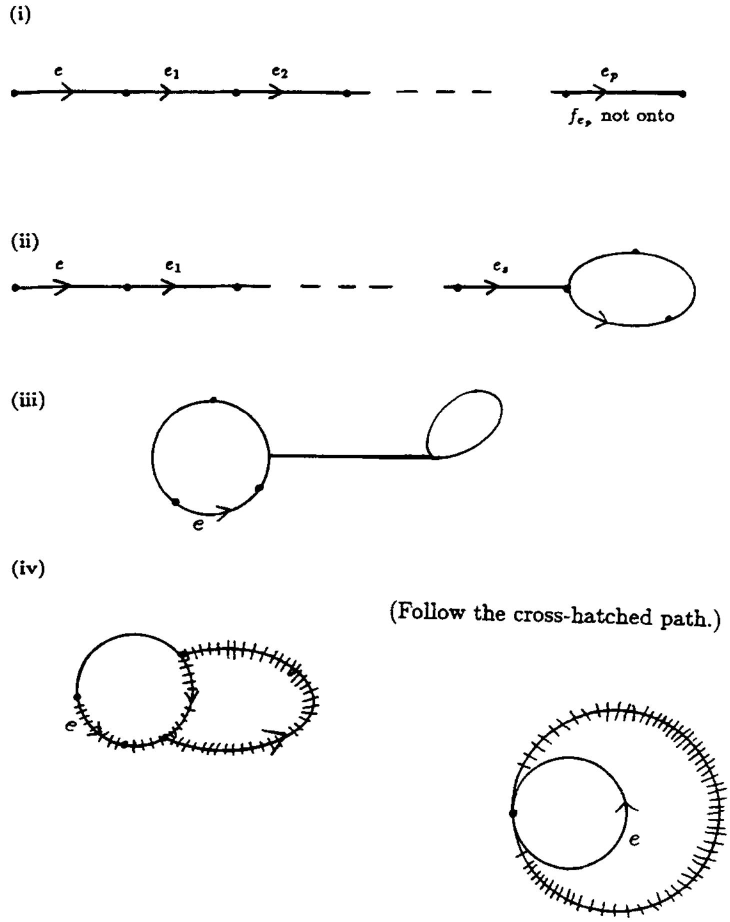

Fig. 5.

(2) For each edge $e \in E(\Gamma)$ with, say, $P = \tau(e)$ we now define a reduced word $\gamma_e \in \pi_1(\mathscr{G}, P)$:

(a) If f_e is not surjective, let $\gamma_e \in G_P - f_e(G_e)$.

(b) If f_e is surjective and if $\exists$ a path $e, e_1, \dots, e_p$ with a new group element g at the end of the path such that $f_{\bar{e}_1}$ is not surjective, set

$$\gamma_e = t_1 \dots t_r g t_r^{-1} \dots t_1^{-1} h \in \pi_1(\mathscr{G}, P)$$

where $h \in G_p - f_{\bar{e}_1}(G_{\bar{e}_1})$.

(c) If every edge e' with $\tau(e') = P$ has $f_{e'}(G_{e'}) = G_P$ then, since P is not an invisible vertex, it is a branch point. Hence, there are reduced paths $e, e_1, \dots, e_p$ and $e, e'_1, \dots, e'_s$ with $e_1 \neq e'_1$, with new group elements h and h' at the end of these paths. In this case we set

$$\gamma_e = t_1 \dots t_p h t_p^{-1} \dots t_1^{-1} t'_1 \dots t'_s h' (t'_s)^{-1} \dots (t'_1)^{-1}$$

and

$$\delta_e = \begin{cases} \gamma_e & \text{in case (a)} \\ \gamma_e^2 & \text{in cases (b) and (c).} \end{cases}$$

Note that γ_e and δ_e are cyclically reduced.

Suppose now that an edge-length function $L: |E| \to \mathbf{R}$ determines the translation length $\|\cdot\|_L$ as in Section 8. Then for any edge e we consider the words

$$w_1(e) = t\gamma_e t^{-1}\gamma_{\bar{e}} \quad (t = t_e)$$
$$w_2(e) = t\delta_e t^{-1}\delta_{\bar{e}}.$$

These are cyclically reduced, by the definition of γ_e and δ_e. Thus (Proposition 8.3), $\|w_i\|_L$ $(i = 1, 2)$ is the sum of the lengths of the edges of the paths traversed. It follows that

$$L(e) = \|w_1(e)\| = \tfrac{1}{2}\|w_2(e)\|.$$

Therefore, if $\|\cdot\|_L = \|\cdot\|_{L'}$ for edge-length functions L and L' then $L(e) = L'(e)$ for every edge e, and hence $L = L'$. Finally if $[\|\cdot\|_L] = [\|\cdot\|_{L'}]$ for some $L, L' \in \sigma^{r-1}$ then $\|\cdot\|_L = C\|\cdot\|_{L'} = \|\cdot\|_{C\cdot L'}$. Thus $L = C \cdot L'$ and, since the sums of the coordinates of L and of L' each equals one, we deduce that $C = 1$. Thus $\sigma^{r-1} \to \sigma(\mathcal{G}) \subset PLF(G)$ is injective and is a homeomorphism, completing the proof of Proposition 9.6. □

As an application of Proposition 9.6 we give the promised infinite dimensionality result.

THEOREM 9.7. *If $G = A * B$ is a free product of non-trivial groups, one of which contains an element of infinite order, then the subspace of $PLF(G)$ determined by small simplicial actions is infinite dimensional.*

Proof. Suppose that $b \in B$ has infinite order. We show that this space contains a k-simplex for every $k > 0$. Indeed, by Proposition 9.6, the minimal, small graph of groups decomposition $\mathcal{G}$ pictured in Fig. 6 determines a k-simplex $\sigma(\mathcal{G}) \subset PLF(G)$. Here $\langle x_i \rangle$ and $\langle y_i \rangle$ are infinite cyclic groups with generators x_i, y_i $(1 \le i \le k)$. We define $f_{\bar{e}_i}(y_i) = x_i$ $(1 \le i \le k)$, $f_{e_i}(y_i) = x_{i+1}^2$ $(1 \le i \le k-1)$ and $f_{e_k}(y_k) = b^2$. □

10. VERY SMALL GRAPHS OF GROUPS

Definition 10.1. A graph of groups $\mathcal{G}$ is called *very small*, if

(a) every edge group G_e is small (i.e. it does not contain a free subgroup of rank 2),
(b) for every edge e and every $r \in G_{\tau(e)}$, if $1 \ne r^p \in f_e(G_e)$, then $r \in f_e(G_e)$, and
(c) (*no conjugate triples*) for any three distinct edges e_1, e_2 and e_3 with common terminal vertex v, and for any non-trivial elements $z_1 \in G_{e_1}$, $z_2 \in G_{e_2}$ and $z_3 \in G_{e_3}$ the images $f_{e_1}(z_1)$, $f_{e_2}(z_2)$ and $f_{e_3}(z_3)$ are not all conjugates of each other in G_v.

LEMMA 10.2. *If $\mathcal{G}$ is a very small graph of groups and if $\pi_1(\mathcal{G}, P)$ is an α-group then the following strengthenings of Definition* 10.1 (a) *and* (b) *hold*:

(a*) *If a non-trivial edge group G_e is finitely generated then it is infinite cyclic. In any case, each edge group G_e is an abelian group.*

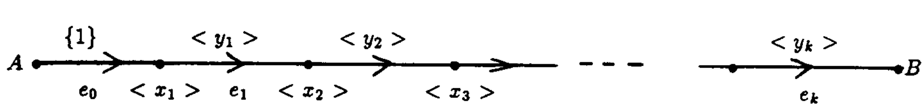

Fig. 6.

(b*) *If* $r \in G_{\tau(e)}$ *and* $r \notin f_e(G_e)$ *then*

$$r f_e(G_e) r^{-1} \cap f_e(G_e) = \{1\}.$$

Moreover (b*) $\Rightarrow$ *Definition* 10.1(b).

Proof. (a*) follows from Proposition 3.6 and (b*) follows from Lemma 3.8. Also (b*) $\Rightarrow$ Definition 10.1(b) since $1 \neq r^p \in f_e(G_e)$ implies that $r r^p r^{-1} \in f_e(G_e)$. Hence $r \in f_e(G_e)$ by (b*). □

PROPOSITION 10.3. *Let $\mathcal{G}$ be a graph of groups with base point P.*

(1) *If the action of $G = \pi_1(\mathcal{G}, P)$ on the tree $\mathcal{T}_{\mathcal{G}}$ is very small, then $\mathcal{G}$ is very small.*
(2) *If $\mathcal{G}$ is very small, and if G is an α-group, then its action on $\mathcal{T}_{\mathcal{G}}$ is very small.*

Remark. In (2) one can replace the hypothesis that the group G is an α-group by the hypothesis that the graph of groups $\mathcal{G}$ satisfies (b*) of Lemma 10.2.

Proof. We use the facts and notation developed in 5.7 and 5.8.

(1) If $e \in E(\mathcal{G})$ and $\tilde{e}$ is any edge in $\mathcal{T}_{\mathcal{G}}$ which projects to e then $\operatorname{stab}(\tilde{e}) \cong G_e$. Since the action on T_s is small, G_e is a small group and Definition 10.1(a) is satisfied.

If $r \in G_{\tau(e)}$ and $1 \neq r^p \in f_e(G_e)$, let $\langle w \rangle = \tau(\tilde{e})$. Then (5.8a, 5.8b) we have

$$\operatorname{stab}(\langle w \rangle) = w G_{\tau(e)} w^{-1} \quad \text{and} \quad \operatorname{stab}(\tilde{e}) = w f_e(G_e) w^{-1}.$$

So $(wrw^{-1})^p \in \operatorname{stab}\langle \tilde{e} \rangle$ and, the action on $\mathcal{T}_{\mathcal{G}}$ being very small, we conclude that $wrw^{-1} \in \operatorname{stab} \tilde{e}$. Thus $r \in f_e(G_e)$, and Definition 10.1(b) is satisfied.

Similarly, given the data of Definition 10.1(c), let $\langle w \rangle = \tilde{v}$ be a vertex in $\mathcal{T}_{\mathcal{G}}$ above v. Let v_1, v_2, v_3 be the initial points of e_1, e_2, e_3. For any $g_i \in G_v$ $(i = 1, 2, 3)$ the element of $\pi_1(\mathcal{G}, P)$

$$w(g_i f_{e_i}(z_i) g_i^{-1}) w^{-1} = (w g_i t_i^{-1}) f_{\bar{e}_i}(z_i)(w g_i t_i^{-1})^{-1}$$

fixes the edge $\tilde{e}_i$ spanned by $\tilde{v} = \langle w \rangle$ and $\tilde{v}_i = \langle w g_i t_i^{-1} \text{ reduced} \rangle$. Thus if there are elements $r, g_1, g_2, g_3 \in G_v$ with $r = g_i f_{e_i}(z_i) g_i^{-1}$ $(i = 1, 2, 3)$ then wrw^{-1} fixes three edges $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ emanating from $\tilde{v}$. Since no non-trivial element of $\pi_1(\mathcal{G}, P)$ fixes a triod, we thus cannot have the images $f_{e_i}(z_i)$ all conjugate in G_v; so Definition 10.1(c) is satisfied.

(2) We assume now that $\mathcal{G}$ is a very small graph of groups with G an α-group and show that the (simplicial) action of G on $\mathcal{T}_{\mathcal{G}}$ is very small (Definition 2.1).

If A is a non-degenerate arc in $\mathcal{T}_{\mathcal{G}}$ then A contains an interior point x of some edge $\tilde{e}$. Since G acts without inversion, $G_e \cong \operatorname{stab}(\tilde{e}) = \operatorname{stab}\{x\} \supset \operatorname{stab} A$. Since G_e is a small group, so is $\operatorname{stab}(A)$. Thus the action of G and $\mathcal{T}_{\mathcal{G}}$ is small.

If there were a non-trivial element g with an obtrusive power—$\operatorname{Fix}(g^p) \neq \operatorname{Fix}(g)$, $g^p \neq 1$—then $\operatorname{Fix}(g)$ would be a proper subtree of $\operatorname{Fix}(g^p)$. There would be an edge $\tilde{e} = (\langle w_1 \rangle, \langle w_2 \rangle)$ with $w_2 = w_1 t$ in $\operatorname{Fix}(g^p)$ which meets $\operatorname{Fix}(g)$ only in $\tau(\tilde{e}) = \langle w_1 \rangle$ (notation of 5.7). Then $g = w_1 r w_1^{-1}$ for some $r \in G_{\tau(e)}$, $r \notin f_e(G_e)$ and $g^p = w_1 x w_1^{-1}$ for some $x \in f_e(G_e)$. But then $r^p = x \in f_e(G_e)$, contradicting Definition 10.1(b). Thus the action has no obtrusive powers.

Finally we show that $\operatorname{Fix}(g)$ never contains a triod if $g \neq 1$. If $\operatorname{Fix}(g)$ contains a triod it contains three edges $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ emanating from a vertex $\tilde{v} = \langle w \rangle$. Since G is an α-group and we have already proved that the action on $\mathcal{T}_{\mathcal{G}}$ is small and has no obtrusive powers, Lemma 3.9 implies directly (or Lemma 10.2(b*) implies with a bit of calculation) that $\tilde{e}_1, \tilde{e}_2$ and

$\tilde{e}_3$ project to distinct edges e_1, e_2, e_3 of Γ emanating from vertex v. But g fixes $\tilde{e}_i$, so that $g = wx_iw^{-1}$ for some non-trivial $x_i \in f_{\bar{e}_i}(G_{e_i})$ $(i = 1, 2, 3)$. Therefore x_1, x_2, x_3 are mutually conjugate elements of G_v coming from distinct edges. This contradicts Definition 10.1(c). Thus $\mathrm{Fix}(g)$ does not contain a triod. □

Corollary 10.4. *Let $\mathcal{G}$ be a very small graph of groups such that $G = \pi_1(\mathcal{G}, P)$ is an α-group. Then every quotient graph of groups $\mathcal{G}' = \mathcal{G}/\{\mathcal{G}_\alpha\}$ is also very small.*

Proof. If we assign length 1 to all edges of $\mathcal{G}$ and to all the edges of $\mathcal{G}'$ then, from 8.5(c), the action of G on $\mathcal{T}_{\mathcal{G}'}$ is the limit of actions on $\mathcal{T}_{\mathcal{G}}$ as the length of the edges in the $\mathcal{G}_\alpha$ go to zero. By Proposition 10.3(2) and Theorem I [= Theorem 2.3], the action of G on $\mathcal{T}_{\mathcal{G}'}$ is a limit of very small actions and thus is very small. By Proposition 10.3(1) it follows that $\mathcal{G}'$ is very small. □

Recall (Definition 3.4) that two elements g, h of a group G are *positively* (*or negatively*) *bonded* if there exist $r \in G$, $m, n \neq 0$ such that $g^m = rh^nr^{-1}$ with $mn > 0$ (or with $mn < 0$).

Definition 10.5. (1) Let e_1 and e_2 be distinct edges of $\mathcal{G}$ with common terminal vertex $\tau(e_1) = \tau(e_2) = v$. Let $1 \neq z_i \in \mathrm{Center}\, G_{e_i}$ and denote $x_i = f_{e_i}(z_i)$ $(i = 1, 2)$. We say that *e_1 and e_2 are positively* (*or negatively*) *bonded via z_1 and z_2* if x_1 and x_2 are positively (or negatively) bonded in G_v.

(2) If $\tau(e_1) = \tau(e_2)$ we say that e_1 and e_2 are *positively* bonded (negatively bonded) if there exist $z_1 \in G_{e_1}$ and $z_2 \in G_{e_2}$ as in (1). Finally e_1 and e_2 are *bonded* if they are either positively or negatively bonded.

Lemma 10.6. *Let $\mathcal{G}$ be a finite graph of groups with base point P.*

(a) *If $\mathcal{G}$ is very small, then for every edge e of $\mathcal{G}$ there exists at most one edge $e' \neq e$ which is bonded to e.*

(b) *If $\pi_1(\mathcal{G}, P)$ is an α-group, then no two edges of $\mathcal{G}$ can be both positively and negatively bonded via the same pair of elements.*

Proof. (a) If there are two edges e' and e'', both bonded to e, we would have

$$x_e^m = rx_{e'}^n r^{-1} \quad \text{and} \quad x_e^p = sx_{e''}^q s^{-1}$$

for some $r, s \in G_{\tau(e)}$ and $n, m, p, q \neq 0$. Hence x_e^{pm}, $x_{e'}^{pn}$ and $x_{e''}^{qn}$ are conjugate in $G_{\tau(e)}$. By Definition 10.1(c) this contradicts the assumption that $\mathcal{G}$ is very small.

(b) Apply Lemma 3.5(c). □

The crucial use of the hypothesis that $\mathcal{G}$ is a very small graph of groups is to prove the following Choice Lemma. It is this lemma that we use in Section 11 to show that we may apply the Skyscraper Lemma of Section 4.

Choice Lemma 10.7. *Let $\mathcal{G}$ be a finite very small graph of groups with based point P. Suppose that $G = \pi_1(\mathcal{G}, P)$ is an α-group. Then for every non-trivial edge group G_e we may choose a non-trivial element $z_e \in G_e$ such that, denoting $x_e = f_e(z_e)$, we have:*

(a) $z_{\bar{e}} = z_e^{-1}$.

(b) *If $\tau(e) = \tau(e') = v$, $r \in G_v$ and if $t_e r t_{\bar{e}'}$ is a reduced word (i.e. either $e \neq e'$ or $r \notin f_e(G_e)$) then for all $m, n > 0$*

$$x_e^n \neq rx_{e'}^m r^{-1}.$$

(c) *Suppose such z_e have been chosen satisfying* (a) *and* (b). *Let $w = t_1 r_1 \dots r_{q-1} t_q$ be a reduced word in $\beta(\mathcal{G})$ and let i, j be indices with $1 \le i < j \le q$ and with both G_{e_i} and G_{e_j} non-trivial. Then, for all $m, n > 0$,*

$$x_{e_i}^n \ne (r_i t_{i+1} \dots r_{j-1} t_j) x_{e_j}^{-m} (r_i t_{i+1} \dots t_j)^{-1}.$$

Proof. We prove first that (c) follows from (a) and (b). Notice that from (a) we get

$$r_{j-1} t_j x_{e_j}^{-m} t_j^{-1} r_{j-1}^{-1} = r_{j-1} x_{\bar{e}_j}^m r_{j-1}^{-1}.$$

If $j = i + 1$ then the fact that $t_{j-1} r_{j-1} t_j$ is reduced implies by (b) that $x_{e_i}^n \ne r_{j-1} x_{\bar{e}_j}^m r_{j-1}^{-1}$ and (c) is proved.

Proceeding inductively, if $j > i + 1$ and (c) is false then we have

$$x_{e_i}^n = r_i t_{i+1} \dots t_{j-1} (r_{j-1} x_{\bar{e}_j}^m r_{j-1}^{-1}) t_{j-1}^{-1} \dots r_i^{-1}$$

for some $m, n > 0$. But since w was reduced, and the right-hand side collapses to an element of a vertex group, we see that $r_{j-1} x_{\bar{e}_j}^m r_{j-1}^{-1} \in f_{e_{j-1}}(G_{e_{j-1}})$. Therefore $G_{e_{j-1}}$ is non-trivial, and we consider the non-trivial element $x_{e_{j-1}} \in f_{e_{j-1}}(G_{e_{j-1}})$ satisfying (a) and (b). Since G is an α-group and $G_{e_{j-1}}$ is small, there are powers $m_j > 0$ and $n_j \ne 0$ such that

$$r_{j-1} x_{\bar{e}_j}^{mm_j} r_{j-1}^{-1} = x_{e_{j-1}}^{n_j}.$$

Since $t_{j-1} r_{j-1} t_j$ is reduced, we see from part (b) that $n_j < 0$. But then

$$x_{e_i}^{nm_j} = (r_i t_{i+1} \dots t_{j-1}) x_{e_{j-1}}^{-|n_j|} (r_i \dots t_{j-1})^{-1}$$

contradicting the induction hypothesis. Thus (c) follows from (a) and (b).

To prove (a) and (b), first observe from the malnormality of $f_e(G_e)$ in $G_{\tau(e)}$ given by Lemma 10.2 that if $e = e'$ then an arbitrary choice of the $z_e \in G_e$ satisfies statement (b). Thus (b) will be satisfied if we can choose the z_e so that: *if $e \ne e'$ and $\tau(e) = \tau(e')$ then e and e' are not positively bonded via z_e and $z_{e'}$.*

We start with an arbitrary choice of the non-trivial elements $z_e \in G_e$ for all edges e with $G_e \ne \{1\}$. After possibly switching some $z_{\bar{e}}$ to z_e^{-1}, we may assume that condition (a) is satisfied.

We then consider all maximal connected subgraphs Γ_i of Γ with the property that any two edges e, e' of Γ_i with $\tau(e) = \tau(e')$ are bonded via z_e and $z_{e'}$. If two such subgraphs are distinct then they have no edges in common by Lemma 10.6(a). (They may however have vertices in common.) Clearly (b) is satisfied for edges e and e' contained in distinct subgraphs Γ_i.

By Lemma 10.6(a) and the fact that $\mathcal{G}$ is finite, Γ_i is either homeomorphic to an interval or to S^1. In the first case, suppose the arc is $e_1, e_2, \dots, e_q$ with $\tau(e_i) = \tau(\bar{e}_{i+1})$. We may, for $i \ge 2$, inductively change z_{e_i} and $z_{\bar{e}_1}$ to $z_{e_i}^{-1}$ and $z_{\bar{e}_i}^{-1}$ if necessary so that e_{i-1} and $\bar{e}_i$ are negatively bonded. Then they will not be positively bonded, by Lemma 10.6(b). Thus (b) will also be satisfied for any two edges contained in such a subgraph Γ_i.

Finally, applying the same "switching technique" to a subgraph Γ_i which is homeomorphic to S^1, we obtain a cycle of edges in Γ, $e_1, e_2, \dots, e_q$ with $\tau(e_i) = \tau(\bar{e}_{i+1})$ ($i \bmod q$), where (1) all $e_i, \bar{e}_{i+1}$ are negatively bonded for $i = 1, \dots, q$, and (2) $e_1, \bar{e}_q$ are either positively or negatively bonded. If negatively bonded, (b) is again satisfied for any two edges contained in the subgraph Γ_i. We show that e_1 and $\bar{e}_q$ cannot be positively bonded by the chosen z's: for $1 \le i \le q - 1$: $x_{e_i}^{m_i} = r_i x_{\bar{e}_{i+1}}^{-n_{i+1}} r_i^{-1}$ for some $m_i, n_{i+1} > 0, r_i \in G_{\tau(e_i)}$. Suppose $x_{e_q}^{m_q} = r_q x_{\bar{e}_1}^{n_1} r_q^{-1}$ for some $m_q, n_1 > 0$, $r_q \in G_{\tau(e_q)}$. This would imply

$$
\begin{aligned}
x_{e_1}^{m_1 m_2 \dots m_q} &= r_1 x_{\bar{e}_2}^{-n_2 m_2 \dots m_q} r_1^{-1} = r_1 t_2 x_{e_2}^{n_2 m_2 \dots m_q} t_2^{-1} r_1^{-1} \\
&= \cdots \\
&= (r_1 t_2 r_2 \dots r_{q-1} t_q) x_{e_q}^{n_2 n_3 \dots n_q m_q} (r_1 t_2 r_2 \dots r_{q-1} t_q)^{-1} \\
&= (r_1 t_2 r_2 \dots r_{q-1} t_q r_q) x_{\bar{e}_1}^{n_1 n_2 \dots n_q} (r_1 t_2 r_2 \dots r_{q-1} t_q r_q)^{-1} \\
&= (r_1 t_2 r_2 \dots r_{q-1} t_q r_q t_1) x_{e_1}^{-n_1 n_2 \dots n_q} (r_1 t_2 r_2 \dots r_{q-1} t_q r_q t_1)^{-1}
\end{aligned}
$$

which by Lemma 3.5 contradicts the assumption that G is an α-group.

PART III. THE DYNAMICS OF MULTIPLE DEHN TWIST AUTOMORPHISMS

11. THE FUNDAMENTAL CONVERGENCE LEMMA

Assumption.

$\mathcal{G}$ is a minimal very small graph of groups such that $G = \pi_1(\mathcal{G}, P)$ is an α-group. (11.1)

The purpose of this section is to prove Fundamental Convergence Lemma 11.5 which gives our fundamental calculation concerning the dynamics of the induced actions on $SLF(G)$ of multiple Dehn twist automorphisms of G. For certain sequences $\{D_k\}$ of such automorphisms of G and certain sequences $\{\|\cdot\|_k\}$ of elements of $SLF(G)$ we calculate $\lim_k D_k(\|\cdot\|_k)$. This limit exists and is an explicitly named element of $\sigma(\mathcal{G})$.

The setup

We partition the set of edges E of $\Gamma(\mathcal{G})$ into two sets, each closed under the operation, $e \mapsto \bar{e}$,

$$E = \{e_1, e_2, \dots, e_A\} \cup \{E_1, E_2, \dots, E_B\}$$

such that $G_{E_b} = \{1\}$, $1 \le b \le B$. (Either set may be empty; some of the G_{e_a} may also be trivial.) Denote

$$\Gamma_+ = \Gamma - \{E_1, E_2, \dots, E_B\}$$

$$\{\Gamma_\alpha\}_{1 \le \alpha \le S} = \text{the set of components of } \Gamma_+$$

$$G_\alpha = \pi_1(\Gamma_\alpha, P_\alpha).$$

Note that each G_α is finitely generated. (This follows from the fact that G is finitely generated, using the form of reduced words.)

Assumption.

A small action of G_α $(1 \le \alpha \le S)$ on an $\mathbf{R}$-tree $\hat{\mathcal{T}}_\alpha$ with based length function L_α and translation length $\|\cdot\|_\alpha$ is given. (11.2)

The G-trees $\mathcal{T}_k$

We construct a sequence $\mathcal{T}_k$ $(k = 1, 2, \dots)$ of G-trees as follows. Let L be a positive edge-length function on E so that $\mathcal{G} = \mathcal{G}(L) = \mathcal{G}(L(e_1), \dots, L(e_A), L(E_1), \dots, L(E_B))$ is a metric graph of groups (Section 8) which gives rise to the metric simplicial tree $\mathcal{T}_{\mathcal{G}}$ with translation length function $\|\cdot\|_{\mathcal{G}}$. The G-tree $\mathcal{T}_k$ ($k \in \mathbb{N}$) is defined, combining the constructions of 7.1 and Combination Lemma 8.6, as the $\mathcal{G}/\{\mathcal{G}_\alpha\}$ combination of the trees $(1/k)\hat{\mathcal{T}}_\alpha$. To be explicit, if L_k and $\|\cdot\|_k = L_k^2 - L_k$ denote the based and translation length functions for the action of G on $\mathcal{T}_k$ and if $w = r_0 t_1 r_1 \ldots t_q r_q \in \pi_1(\mathcal{G}, P)$ is a reduced word then, according to Lemma 7.2 and Combination Lemma 8.6, we find $L_k(w)$ as follows.

Write w as a product of subwords $w = R_0 T_1 R_1 \ldots T_p R_p$ where each $T_i = t_{E_b}$ for some $b = b(i)$ ($1 \leq b \leq B$) and each R_i is a word in the Bass group $\beta(\mathcal{G}_\alpha)$ of one of the components $\Gamma_\alpha = \Gamma_{\alpha(i)}$ of Γ_+. Choosing a maximal tree in each component Γ_α of Γ_+ we get (5.6) a retraction $R \mapsto \hat{R}$ of $\beta(\mathcal{G}_\alpha)$ onto $\pi_1(\mathcal{G}_\alpha, P_\alpha)$, where $\hat{R} = b(t) R \bar{d}(t)$. (Here $b(t)$ and $d(t)$ are the products of the t_e corresponding to the unique paths b and d in the maximal tree from P_α to the initial and terminal points of the path followed by R.) Then $\mathcal{T}_k$ is the **R**-tree with small G-action determined by the based length function

$$\begin{aligned} L_k(w) &= L_k(R_0 T_1 \ldots T_p R_p) \\ &= \sum_{i=0}^{p} \frac{1}{k} L_{\alpha(i)}(\hat{R}_i) + \sum_{j=1}^{p} L(E_{b(j)}). \end{aligned} \tag{11.3}$$

The Dehn twists

We consider sequences $\{D_k\}$ of Dehn twists of $\pi_1(\mathcal{G}, P)$ with the following properties. We choose for each $e \in \{e_1, \ldots, e_A\}$ with $G_e \neq \{1\}$ a non-trivial element $z_e \in G_E$ (with image $x_e = f_e(z_e)$) so that the conclusion of the Choice Lemma 10.7 holds. (Note (Lemma 10.2) that G_e is abelian, so $z_e \in \text{Center}\, G_e$.) For each such e, let $\{n_k(e)\}_{k \in \mathbb{N}}$ be any sequence of non-negative integers satisfying $n_k(e) = n_k(\bar{e})$ and

$$n_k(e) = 0 \quad \text{for all } k \tag{11.4a}$$

or

$$n_k(e) > 0 \text{ for all } k \text{ and } \lim_k \left(\frac{n_k(e)}{k}\right) \equiv \ell_a \text{ exists with } \ell_a > 0 \text{ (where } e = e_a). \tag{11.4b}$$

Edges in (b) are called *twisted edges* with twistors z_e. Edges e_a with $G_{e_a} = \{1\}$ or which satisfy (a) are called *non-twisted.* Let D_k denote the multiple Dehn twist given by

$$D_k(t_e) = t_e x_e^{n_k(e)} \quad \text{for all } e \text{ with } G_e \neq \{1\}$$

and also denote $D_k = D_k^* : PLF(G) \to PLF(G)$.

With this setup we denote $\Gamma_{\alpha(a)} =$ the component of Γ_+ containing e_a ($1 \leq a \leq A$). The group $G_{\tau(e_a)}$ is identified in the usual way (5.6) with a subgroup of $\pi_1(\mathcal{G}_{\alpha(a)}, P_\alpha(a))$ via paths in the chosen maximal tree in $\Gamma_{\alpha(a)}$. Thus $G_{\tau(e_a)}$ acts on $\mathcal{T}_{\alpha(a)}$. We assume the edges $e_1, \ldots, e_A$ have been indexed so that the e_a are twisted edges if $1 \leq a \leq C$ and non-twisted if $C + 1 \leq a \leq A$. We can now state the following lemma.

Fundamental Convergence Lemma 11.5. *Given* (11.1)–(11.4), *suppose that* $\|x_{e_a}\|_{\alpha(a)} > 0$ *precisely when* $1 \leq a \leq C$. *Then*

$$\lim_k D_k(\|\cdot\|_k) = \|\cdot\|_{\mathcal{H}} \in \sigma(\mathcal{G})$$

where $\mathcal{H}$ *is the metric graph of groups*

$$\mathcal{H} = \mathcal{G}(\ell_1 \cdot \|x_{e_1}\|_{\alpha(1)}, \ldots, \ell_C \cdot \|x_{e_C}\|_{\alpha(C)}, 0, \ldots, 0, L(E_1), \ldots, L(E_B)).$$

Proof. Consider an arbitrary element $g \in \pi_1(\mathscr{G}, P)$. Since

$$(D_k \|\cdot\|_k)(g) = \|D_k(g)\|_k = L_k(D_k(g)^2) - L_k(D_k(g))$$
$$= L_k(D_k(g^2)) - L_k(D_k(g))$$

it suffices by Proposition 8.3 to prove that, for all $w \in \pi_1(\mathscr{G}, P)$, if $w = r_0 t_1, \dots, t_q r_q$ (reduced) then

$$\lim_k (L_k D_k(w)) = L_{\mathscr{H}}(w) = \sum_i L_{\mathscr{H}}(\hat{e}_i) \tag{11.6}$$

where, in the last sum, $\hat{e}_i \in \{e_1, \dots, e_A, E_1, \dots, E_B\}$ and $t_i = t_{\hat{e}_i}$. By definition of $\mathscr{H}$ the terms of the right-hand side are

$$\begin{aligned} L_{\mathscr{H}}(\hat{e}_i) &= \ell_a \cdot \|x_{e_a}\|_{\alpha(a)} && \text{if } \hat{e}_i = e_a \ (1 \le a \le C) \\ L_{\mathscr{H}}(\hat{e}_i) &= L(E_b) && \text{if } \hat{e}_i = E_0 \ (1 \le b \le B) \\ L_{\mathscr{H}}(\hat{e}_i) &= 0 && \text{if } \hat{e}_i = e_a \ (C \le a \le A). \end{aligned} \tag{11.7}$$

For the left-hand side of (11.6), (11.3) and the important note in Remark 7.3 give

$$\begin{aligned} L_k(D_k(w)) &= \sum_{i=0}^{p} \frac{1}{k} L_{\alpha(i)}(\widehat{D_k(R_i)}) + \sum_{j=1}^{p} L(E_{b(j)}) \\ &= \sum_{i=0}^{p} \frac{1}{k} L_{\alpha(i)}(\widehat{D_k(R_i)}) + \sum_{j=1}^{p} L_{\mathscr{H}}(E_{b(j)}). \end{aligned} \tag{11.8}$$

To calculate the limit of $(1/k) L_{\alpha(i)}(\widehat{D_k(R_i)})$, we write the connected subword R_i as a word in the Bass group $\beta(\mathscr{G}_{\alpha(i)})$:

$$R_i = r_{m-1} t_m \dots t_n r_n \quad \text{(reduced in } \beta(\mathscr{G})\text{)}.$$

Suppose that $t_{m_1}, t_{m_2}, \dots, t_{m_s}$ are the t's in R_i which belong to twisted edges. Letting w_j be the subword of R_i between t_{m_j} and $t_{m_{j+1}}$, we have

$$R_i = w_0 t_{m_1} w_1 t_{m_2} w_2 \dots t_{m_s} w_s. \tag{11.9a}$$

Let $e(j)$ be the twisted edge corresponding to t_{m_j} [i.e. $e(j) = e_{a(m_j)}$]. Then

$$D_k(R_i) = w_0 (t_{m_1} x_{m_1}^{n_k(e(1))}) w_1 (t_{m_2} x_{m_2}^{n_k(e(2))}) \dots (t_{m_s} x_{m_s}^{n_k(e(s))}) w_s. \tag{11.9b}$$

If $c(t)$ and $d(t)$ are the t-words following paths in the chosen subtree of $\Gamma_{\alpha(i)}$ from the base point to the initial and end points of R_i we then have

$$\widehat{D_k(R_i)} = b_0 x_{m_1}^{n_k(e(1))} b_1 x_{m_2}^{n_k(e(2))} \dots b_{s-1} x_{m_s}^{n_k(e(s))} b_s \quad \text{where } \begin{aligned} b_0 &= c(t) w_0 t_{m_1} \\ b_s &= w_s d(t)^{-1}. \end{aligned} \tag{11.9c}$$

Since $w = r_0 t_1 \dots t_q r_q$ is a reduced word in $\beta(\mathscr{G})$, its subword $t_{m_1} r_{m_1} \dots r_{m_s - 1} t_{m_s}$ is reduced. Thus since the z_e were chosen to satisfy the Choice Lemma 10.7, we see from 10.7(c) that in (11.9c)

$$x_{m_j}^{n_k(e(j))} \ne b_j x_{m_{j+1}}^{-n_k(e(j+1))} b_j^{-1} \quad (1 \le j \le s-1). \tag{11.10}$$

The left- and right-hand sides of this equation lie in the same vertex group and remain unequal when this vertex group is injected into $\pi_1(G_{\alpha(i)}, P_{\alpha(i)})$ under the retraction of $\beta(\mathscr{G}_{\alpha(i)})$. For simplicity we use the same notation for the retracted terms, writing x_{m_j}, b_j, etc. instead of $\hat{x}_{m_j}$, $\hat{b}_j$, etc.

Now we may apply part (a) of the Skyscraper Lemma 4.1 to the action of $\mathscr{G}_{\alpha(i)}$ on $\mathscr{T}_{\alpha(i)}$. By assumption the $n_k(e(j))$ go to infinity. Therefore there is a constant K_i such that for all $n_k(e(j))$ sufficiently large we have

$$L_{\alpha(i)}(\widehat{D_k(R_i)}) = \sum_{\nu=1}^{s} n_k(e(\nu))\,\|x_{m_\nu}\|_{\alpha(i)} + K_i \tag{11.11}$$

$$\lim_{k\to\infty}\left[\frac{1}{k}L_{\alpha(i)}(\widehat{D_k(R_i)})\right] = \lim_{k\to\infty}\sum_{\nu=1}^{s}\frac{n_k(e(\nu))}{k}\,\|x_{m_\nu}\|_{\alpha(i)}$$

$$= \sum_{\nu=1}^{s}\ell_{a(m_\nu)}\,\|x_{m_\nu}\|_{\alpha(i)}. \tag{11.12}$$

Equation (11.12) says that in the subword R_i each twisted edge t_e contributes $\ell_{a(e)}\|x_e\|_{\alpha(e)}$ each time it occurs in R. Thus, by (11.8), we see that $\lim_k(D_k(w))$ is a sum in which each $t_i = t_{e_a}$ contributes $\ell_a\|x_{e_a}\|_{\alpha(a)}$ if $1 \le a \le C$ and contributes 0 if e_a is a non-twisted edge $(C+1 \le a \le A)$. On the other hand if $t_i = t_{E_b}$ for some edge E_b with trivial stabilizer then t_i contributes the length of E_b in $\mathscr{G}$. Thus (11.6) is verified and the Fundamental Convergence Lemma is proved. □

12. VERY SMALL SIMPLICIAL ACTIONS ARE LIMITS OF FREE ACTIONS

THEOREM 12.1. *Suppose that G is an α-group which acts freely on some* **R***-tree. Then every very small simplicial action has a projective length function which is a limit of projective length functions of free actions. In other words,*

$$\textit{If } Free(G) \neq \emptyset \textit{ then } Simpl(G) \cap VSL(G) \subset \overline{Free(G)}.$$

Proof. The projective length function of a very small simplicial action of G on an **R**-tree may, by 5.9, Propositions 9.2 and 10.3, be realized as $[\|\cdot\|_{\mathscr{G}}]$ where $\mathscr{G}$ is a minimal very small graph of groups with positive edge-length function L and with corresponding simplicial very small action of $\pi_1(\mathscr{G}, P)$ on $\mathscr{T}_{\mathscr{G}}$. We use the notation of Section 11 and apply the Fundamental Convergence Lemma.

By hypothesis there is a free action of G on an **R**-tree $\mathscr{T}$. Each subgraph of groups $\pi_1(\mathscr{G}_\alpha, P_\alpha)$ determined by a component Γ_α of $\Gamma - \{e \mid G_e = \{1\}\}$ then acts freely with length function $\|\cdot\|_\alpha$ on $\mathscr{T}$. We choose twistors z_e according to Choice Lemma 10.7 and we twist on every edge e with $G_e \neq \{1\}$. (This is possible: G_e is abelian (10.2), so that $\mathrm{Center}(G_e) \neq \{1\}$.) We choose the sequences $n_k(e)$ of Section 11 so that

$$\ell_\alpha \equiv \lim_k\left(\frac{n_k(e)}{k}\right) = \frac{L(e)}{\|x_e\|_\alpha}$$

when $e \in \Gamma_\alpha$. Then, by Corollary 8.7, each action $\|\cdot\|_k$ is free. From the Fundamental Convergence Lemma 11.5, $\lim_k D_k(\|\cdot\|_k) = \|\cdot\|_{\mathscr{H}} = \|\cdot\|_{\mathscr{G}}$. □

ADDENDUM 12.2. *Every very small simplicial action of the free group F_n on an* **R***-tree $(n \ge 2)$ lies in the boundary of the Culler–Vogtmann space CV_n—i.e. is a limit of free simplicial actions.*

Proof. In the preceding proof, $\pi_1(\mathscr{G}_\alpha, P_\alpha)$ is a free group of finite rank. It thus acts in the usual way as the group of covering transformations of a simplicial tree $\mathscr{T}_\alpha$ with quotient space a finite wedge of circles. The resulting actions given by the Combination Lemma in the construction of Section 11 are simplicial actions on **R**-trees $\mathcal{T}_k$ with quotients graphs Γ'_k, each homeomorphic to (Γ with each Γ_α shrunk to a point and a wedge of circles attached to this point).

Thus, using the trees $\mathcal{T}_\alpha$ in the preceding proof, rather than the tree $\mathcal{T}$, as the tree on which $\pi_1(\mathcal{G}_\alpha, P_\alpha)$ acts, we get the fact that the given very small simplicial action is a limit of free *simplicial* actions. □

13. THE INDUCED DYNAMICS OF DEHN TWIST AUTOMORPHISMS ON $SLF(G)$

Definition 13.1. If $\mathcal{G}$ is a graph of groups and $D = D_{z_1} \circ D_{z_2} \circ \cdots \circ D_{z_p}$ is a multiple Dehn twist automorphism of $\pi_1(\mathcal{G}, P)$ (see Definition 6.5(1)) then D is *properly based* on $\mathcal{G}$ if

(1) $\mathcal{G}$ is minimal,

(2) $\mathcal{G}$ is very small,

(3) the non-trivial elements $z_a (= z_{e_a}) \in \mathrm{Center}(G_{e_a})$ are chosen so that the conclusion of Choice Lemma 10.7 is satisfied (denoting $x_e = f_e(z_e)$):

(a) $z_{\bar{e}} = z_e^{-1}$.

(b) If $\tau(e) = \tau(e') = v$, $r \in G_v$ and $t_e r t_{\bar{e}'}$ is a reduced word in $\beta(\mathcal{G})$ then, for all $m, n > 0$, $x_e^n \neq r x_{e'}^m r^{-1}$.

THEOREM 13.2 (Parabolic orbits). *Let* $G = \pi_1(\mathcal{G}, P)$ *be an* α*-group and let* $D = D_{z_1} \circ \cdots \circ D_{z_p}$ *be a multiple Dehn twist property based on* $\mathcal{G}$. *Let* $[\|\cdot\|] \in SLF(G)$ *with* $\|z_i\| > 0$ $(i = 1, 2, \ldots, p)$. *Then*

(a) $$\lim_k D^k([\|\cdot\|]) = \lim_k D^{-k}([\|\cdot\|]) = [\|\cdot\|_{\mathcal{H}}] \in \sigma(\mathcal{G})$$

where

$$\mathcal{H} = \mathcal{G}(\|z_1\|, \ldots, \|z_p\|, 0, 0, 0, 0, \ldots, 0)$$

$= \mathcal{G}$ *with each twisted edge* e_i *assigned length* $\|z_i\|$ *and each untwisted edge assigned length* 0;

(b) *the limit point is an interior point of* $\sigma(\mathcal{G})$ *if* D *twists on every edge. Otherwise, if* $\mathcal{G}$ *has no invisible vertices then the limit point is an interior point of the face of* $\sigma(\mathcal{G})$ *which is determined by the quotient graph of groups* $\mathcal{G}$ *obtained by squeezing every untwisted edge of* $\mathcal{G}$ *to a point.*

Proof. $D^k = D_{z_1^k} \circ D_{z_2^k} \circ \cdots \circ D_{z_p^k}$, since the D_{z_i} commute. We apply the Fundamental Convergence Lemma. Set $E = \{e_1, e_2, \ldots, e_A\}$, so that $\Gamma = \Gamma_+$ and there is only one component $\Gamma_\alpha = \Gamma$ in the setup of Section 11. By hypothesis we are given an action of G on a tree $\mathcal{T}$ which determines $[\|\cdot\|] \in SLF(G)$. Then $\mathcal{T}_k$ of Section 11 is just $(1/k)\mathcal{T}$. We let D^k play the role of D_k of Section 11. Since $D^k(t_e) = t_e x_e^k$ we let $n_k(e) = k$ for all twisted edges and we have $\lim_k (n_k(e)/k) = 1$. Thus $\lim_k D^k([\|\cdot\|]) = [\|\cdot\|]_{\mathcal{H}}$ by the Fundamental Convergence Lemma.

Further, $\lim_k D^{-k}([\|\cdot\|]) = \lim_k (D^{-1})^k([\|\cdot\|]) = [\|\cdot\|]$. For the argument above applies to $D^{-1} = D_{z_1^{-1}} \circ \cdots \circ D_{z_p^{-1}}$ and the limit point is the same because $\|z_i\| = \|z_i^{-1}\|$.

Assertion (b) follows from Proposition 9.6. We simply note that $\mathcal{G}$ cannot be a self-swallowing cycle since G is an α-group (so that a non-trivial edge group must be $\mathbb{Z}$) and $\mathcal{G}$ satisfies Definition 10.1(b). If $\mathcal{G}$ were a self-swallowing cycle then all vertices would be invisible. □

COROLLARY 13.3 (Application to free actions). *Suppose that G is an α-group which acts freely on some* **R**-*tree [i.e.* $Free(G) \neq \emptyset$*]. If* $\varphi: G \xrightarrow{\cong} G$ *can be represented by a proper Dehn twist automorphism,*

$$h: G \xrightarrow{\cong} \pi_1(\mathscr{G}, P)$$

$$\varphi = h \circ D \circ h^{-1}$$

then every element of $Free(G)$ *has a parabolic orbit with limit point in* $h^* \sigma(\mathscr{G})$*. If only one edge is twisted then all orbits have the same limit point in* $h^* \sigma(\mathscr{G})$.

Proof. Under the homeomorphism $h^*: PLF(\pi_1(\mathscr{G}, P)) \to PLF(G)$ the orbits of D map to the orbits of φ. □

13.4 (Uniqueness of proper Dehn twist representations). *Suppose that G is an α-group which admits a free action on some* **R**-*tree. Suppose that* $\varphi: G \to G$ *is an isomorphism which can be represented as a proper Dehn twist, with non-trivial twists on all edges, via an isomorphism* $h_1: G \to G_1 = \pi_1(\mathscr{G}_1, P_1)$ *and also via an isomorphism* $h_2: G \to G_2 = \pi_1(\mathscr{G}_2, P_2)$*. Then the graph of groups decompositions* $(\mathscr{G}_1, h_1)$ *and* $(\mathscr{G}_2, h_2)$ *of G are equivalent in the sense that the corresponding Bass–Serre actions on simplicial trees* $G_i + \mathscr{T}_i \to \mathscr{T}_i$ $(i = 1, 2)$ *are* $(h_2 \circ h_1^{-1})$-*equivariantly isomorphic.*

Note. If some of the edges are not twisted upon then the proof gives the fact that the graph of groups decompositions $(\hat{\mathscr{G}}_1, \hat{h}_1)$ and $(\hat{\mathscr{G}}_2, \hat{h}_2)$ are equivalent, where $\hat{\mathscr{G}}_i$ is the quotient graph of groups which results when the non-twisted edges are collapsed to points and $\hat{h}_i$ is the composition of h_i with the canonical isomorphism $\pi_1(\mathscr{G}_i) \to \pi_1(\hat{\mathscr{G}}_i)$.

Proof of 13.4. Choose a free action A_1 of G_1 on an **R**-tree. Then

$$(h_1 \circ h_2^{-1})^*: PLF(G_1) \to PLF(G_2)$$

pulls A_1 back to a green action $A_2 = (h_1 \circ h_2^{-1})^*(A_1)$. The limit points $A_{i,\infty} \in SLF(G_i)$ of the parabolic orbits $\{(D_i^*)^n(A_i)\}_{n=1}^{\infty}$ $(i = 1, 2)$ also correspond under $(h_1 \circ h_2^{-1})^*$. Thus by 1.7 there is an $(h_1 \circ h_2^{-1})$-equivariant isometry between the unique actions on **R**-trees determined by $A_{1,\infty}$ and $A_{2,\infty}$. But $A_{i,\infty}$ is an interior point of $\sigma(\mathscr{G}_i)$. So these actions on **R**-trees are the Bass–Serre simplicial actions determined by $\mathscr{G}_1$ and $\mathscr{G}_2$ with the extra data that the edges have specified positive lengths. Ignoring these lengths, we get an $(h_1 \circ h_2^{-1})$-equivariant simplicial isomorphism between these trees, as claimed. □

Orbits of non-free actions

Corollary 13.3 demonstrates, in the case that $G = F_n$, that points in the Culler–Vogtmann space CV_n have parabolic orbits under the action induced by multiple Dehn twist automorphisms. What are the orbits of points in ∂CV_n, of points in $SLF(F_n)$? We end this paper by exhibiting some of the orbits under Nielsen automorphisms. First we give a useful lemma.

LEMMA 13.5. *Let D be a Dehn twist based on* $\mathscr{G}$ *and let* $\hat{\mathscr{G}}$ *be any graph of groups with quotient graph of groups* $\mathscr{G}$*. Identify* $\pi_1(\hat{\mathscr{G}}, \hat{P}) = \pi_1(\mathscr{G}, P) = G$ *according to Lemma* 7.2 *and view* $\sigma(\mathscr{G})$ *as a face of* $\sigma(\hat{\mathscr{G}})$ *according to* 8.5*. Then* $D^*: PLF(G) \to PLF(G)$ *fixes every point of* $\sigma(\hat{\mathscr{G}})$.

Proof. The Dehn twist D may be viewed as a Dehn twist based on $\hat{\mathcal{G}}$. By Proposition 8.3 the based length of no word in $\beta(\hat{\mathcal{G}})$ changes. □

COROLLARY 13.6. *Let $\varphi \in Aut(F_n)$, $n \geq 2$, be an elementary Nielsen transformation of F_n, inducing $\varphi^*: PLF(F_n) \to \mathrm{PLF}(F_n)$. Then* Fix φ^* *contains an infinite dimensional simplex of $SLF(F_n)$. If $n = 2$ it contains a 1-simplex of ∂CV_n and if $n > 2$ it contains a $(3n - 7)$-simplex of ∂CV_n.*

Proof. As in Example 6.7(a) there is (up to a permutation of the generators) an isomorphism $h: F_n \to \pi_1(\mathcal{G}, P)$ with $\varphi = h^{-1} \circ D \circ h$, where $\mathcal{G}$ is the graph of groups in Fig. 7. We blow up $\mathcal{G}$ to $\hat{\mathcal{G}}$ in different ways and apply Lemma 13.5.

If we blow up the vertex using the idea of Theorem 9.7 we get the graph of groups $\mathcal{G}_k$ with $k + 2$ edges pictured in Fig. 8. Then $\sigma(\mathcal{G}_k)$ is a $(k + 1)$-simplex by Proposition 9.6. It is fixed by D^*, applying Lemma 13.5, and $\sigma(\mathcal{G}_k)$ is a face of $\sigma(\mathcal{G}_{k+1})$ since $\mathcal{G}_k = \mathcal{G}_{k+1}/(e_1 = pt)$. Thus $\bigcup_k \sigma(\mathcal{G}_k)$ is an infinite dimensional simplex in $Fix\, D^*$, and $h^*(\bigcup_k \sigma(\mathcal{G}_k))$ is an infinite dimensional simplex in Fix φ^*. When $k = 0$, $\mathcal{G}_0$ gives a 1-simplex in ∂CV_n. This is easy to see directly, or one can note that $\mathcal{G}_0$ is a very small graph of groups and apply Addendum 12.2.

Finally one can blow up the vertex of $\mathcal{G}$ differently when $n > 2$ to get a graph with $3n - 6$ edges (Fig. 9). This gives a $(3n - 7)$-simplex in ∂CV_n. □

For simplicity of notation we state and prove the final two propositions for $F_2 = F(a, b)$ rather than $F_n = F(a_1, a_2, \ldots, a_n)$.

PROPOSITION 13.7. *Let $\varphi: F(a, b) \to F(a, b)$ by $\varphi(a) = ab$, $\varphi(b) = b$. Then for each integer $k \geq 2$ there is a point $A_k = [\|\cdot\|_k] \in SLF(F(a, b))$ such that the φ^*-orbit of A_k consists of precisely k points.*

Proof. Consider the graph of groups decomposition (of Fig. 10) for $k \geq 2$. We view $\mathcal{T}_k \equiv \mathcal{T}_{\mathcal{G}_k}$ as a metric $F(a, b)$ tree with norm $\|\cdot\|_k$ via the isomorphism $h: F(a, b) \to \pi_1(\mathcal{G}_k, P)$ given by $a \mapsto a$, $b \mapsto tbt^{-1}$. Thus a cyclically reduced word $w = a^{n_1} b^{m_1} \ldots a^{n_q} b^{m_q}$ in $F(a, b)$ corresponds to an element $h(w) = a^{n_1} t b^{m_1} t^{-1} \ldots t b^{m_q} t^{-1} \in \pi_1(\mathcal{G}_k, P)$, which becomes reduced upon replacing $tb^{m_i}t^{-1}$ by x^{ℓ} if $m_i = \ell k$. Thus $\|a^{n_1} b^{m_1} \ldots b^{m_q}\|_k = 2 \cdot (\#$ of m_i not congruent to $0 \bmod k)$. We let $A_k = [\|\cdot\|_k]$.

Under the induced action (called φ rather than φ^* here) of the Dehn twist φ we get $\|\varphi^{\ell}(g)\|_k = \|g\|_k$ for all $g \in F(a, b)$ if $\ell \equiv 0(k)$; thus $\varphi^{\ell}(\|\cdot\|_k) = \|\cdot\|_k$ if $\ell \equiv 0(k)$. But

$$(\varphi^{\ell} \|\cdot\|_k)(a) \equiv \|\varphi^{\ell}(a)\|_k = \|ab^{\ell}\|_k = 2 \quad \text{if } \ell \not\equiv 0(k)$$

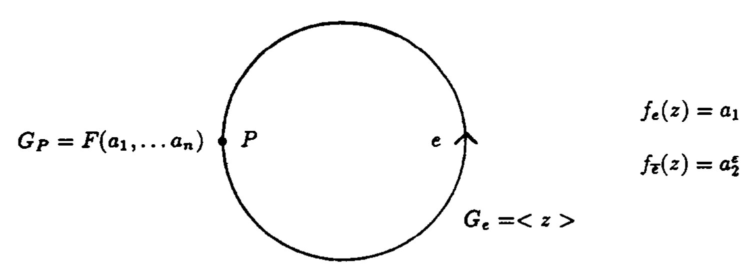

Fig. 7.

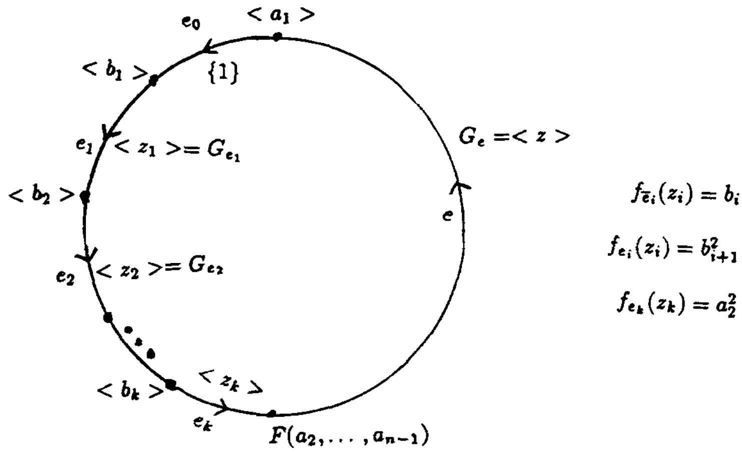

Fig. 8.

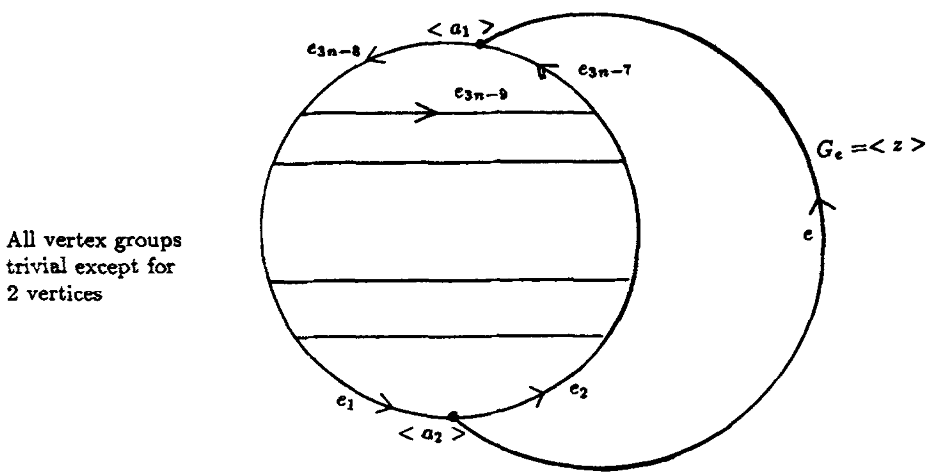

Fig. 9.

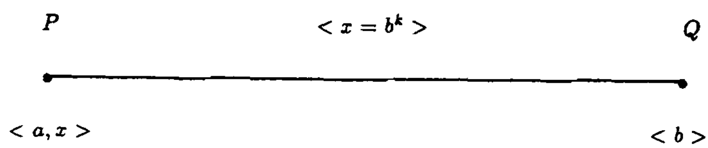

Fig. 10.

while $\|\cdot\|_k(a) = 0$; hence $\varphi^\ell A_k \neq A_k$ if $\ell \not\equiv 0(k)$. Thus the φ-orbit of A_k has exactly k points. □

PROPOSITION 13.8. *Let $\varphi: F(a, b) \to F(a, b)$ by $\varphi(a) = ab$, $\varphi(b) = b$. Then there is an action $A_\infty \in SLF(F_2)$ such that the orbit $\{\varphi^k(A^\infty) | k \in \mathbb{Z}\}$ is infinite and dense in itself.*

Proof. (1) It suffices to exhibit $A_\infty \in SLF(F_2)$ such that

(a) $\lim_{s \to \infty} \varphi^{2^s}(A_\infty) = A_\infty$,

(b) $m_1 \neq m_2 \Rightarrow \varphi^{m_1}(A_\infty) \neq \varphi^{m_2}(A_\infty)$ if $m_1, m_2 > 0$, for (b) shows that the orbit is infinite, and (a) shows that

$$\varphi^m(A_\infty) = \varphi^m \lim_{s\to\infty} (\varphi^{2^s} A_\infty), \quad m \in \mathbb{Z}$$
$$= \lim_{s\to\infty} \varphi^{2^s}(\varphi^m A_\infty)).$$

Therefore $\varphi^m(A_\infty)$ is a limit point of the infinite set $\{\varphi^{m+2^s}(A_\infty)\}_{s\in\mathbb{N}}$.

(2) Let $A_\infty = [\|\cdot\|_\infty]$ where we define

$$\|a^n\|_\infty = \|b^m\|_\infty = 0$$
$$\|a^{n_1} b^{odd\cdot 2^{i_1}} a^{n_2} b^{odd\cdot 2^{i_2}} \dots a^{n_q} b^{odd\cdot 2^{i_q}}\|_\infty = \frac{1}{2^{i_1}} + \frac{1}{2^{i_2}} + \cdots + \frac{1}{2^{i_q}}$$
$$\|w\|_\infty = \|w \text{ cyclically reduced in } F(a,b)\|_\infty.$$

(We shall show in step 3 that $A_\infty \in SLF(G)$.)

(2a) Notice that if $w = a^{n_1} b^{odd\cdot 2^{i_1}} \dots a^{n_q} b^{odd\cdot 2^{i_q}}$ then

$$\varphi^{2^s}(w) = (ab^{2^s})^{n_1} b^{odd\cdot 2^{i_1}} \dots (ab^{2^s})^{n_q} b^{odd\cdot 2^{i_q}}$$
$$= (ab^{2^s})^{n_1-1} ab^{odd\cdot 2^{i_1}+2^s} \dots (ab^{2^s})^{n_q-1} ab^{odd\cdot 2^{i_q}+2^s}.$$

But then

$$s > \max\{i_1, \dots, i_q\} \Rightarrow b^{odd\cdot 2^{i_j}+2^s} = b^{(odd+2^{s-i_j})2^{i_j}} \quad (j = 1, 2, \dots, q).$$

Thus

$$\varphi^{2^s}(\|w\|_\infty) = \|\cdot\|_\infty(\varphi^{2^s}(w)) = \sum_{j=1}^{q}\left(\frac{n_j-1}{2^s} + \frac{1}{2^{i_j}}\right)$$
$$\Rightarrow \lim_{s\to\infty} \varphi^{2^s}\|\cdot\|_\infty(w) = \frac{1}{2^{i_1}} + \frac{1}{2^{i_2}} + \cdots + \frac{1}{2^{i_q}} = \|\cdot\|_\infty(w)$$
$$\Rightarrow \lim_{s\to\infty} \varphi^{2^s} A_\infty = \lim_{s\to\infty} [\varphi^{2^s}\|\cdot\|_\infty] = [\|\cdot\|_\infty] = A_\infty.$$

(2b) We must show that $\varphi^{m_1}(A_\infty) \neq \varphi^{m_2}(A_\infty)$ if $m_1 \neq m_2$. Write $m_1 = \mathfrak{o}_1 2^{k_1}$, $m_2 = \mathfrak{o}_2 2^{k_2}$, $\mathfrak{o}_1, \mathfrak{o}_2$ odd numbers, $k_1, k_2 \geq 0$.

Case 1. If $k_1 = k_2 = k$, $\mathfrak{o}_1 < \mathfrak{o}_2$ then

$$(D^{m_i}\|\cdot\|_\infty)(a) = \|ab^{m_i}\| = \frac{1}{2^k} \quad \text{for } i = 1, 2.$$

But if $\mathfrak{o}$ is an odd number such that $\mathfrak{o}_2 + \mathfrak{o} = 2^{\ell_2}$ for some positive integer ℓ_2 we have

$$\mathfrak{o}_1 + \mathfrak{o} < \mathfrak{o}_2 + \mathfrak{o} \;\Rightarrow\; \mathfrak{o}_1 + \mathfrak{o} = odd\cdot 2^{\ell_1} \quad \text{where } \ell_1 < \ell_2.$$

Thus

$$(D^{m_i}\|\cdot\|_\infty)(ab^{\mathfrak{o}\cdot 2^k}) = \|ab^{\mathfrak{o}_i\cdot 2^k + \mathfrak{o}\cdot 2^k}\|_\infty = \|ab^{2^k(\mathfrak{o}+\mathfrak{o}_i)}\|_\infty = \frac{1}{2^{k+\ell_i}}, \quad i = 1, 2.$$

So $(D^{m_1}\|\cdot\|_\infty)(ab^{\mathfrak{o}\cdot 2^k}) \neq D^{m_2}\|ab^{\mathfrak{o}\cdot 2^k}\|_\infty$ while $(D^{m_1}\|\cdot\|_\infty)(a) = (D^{m_2}\|\cdot\|_\infty)(a)$. Therefore $[D^{m_1}\|\cdot\|_\infty] \neq [D^{m_2}\|\cdot\|_\infty]$.

Case 2. Suppose $k_1 < k_2$. Then $(\varphi^{m_i}\|\cdot\|_\infty)(a) = \|ab^{m_i}\|_\infty = 1/2^{k_i}$. Therefore

$$\frac{(\varphi^{m_1}\|\cdot\|_\infty)(a)}{(\varphi^{m_2}\|\cdot\|_\infty)(a)} = \frac{2^{k_2}}{2^{k_1}} > 1.$$

But

$$\varphi^{m_1}(ab^{2k_1}) = ab^{o_1\cdot 2^{k_1}+2^{k_1}} = ab^{2^{k_1}(o_1+1)}$$

$$\varphi^{m_2}(ab^{2k_1}) = ab^{o_2\cdot 2^{k_2}+2^{k_1}} = ab^{2^{k_1}(o_2+2^{k_2-k_1})}$$

where $(o_1 + 1)$ is even while $(o_2 + 2^{k_2-k_1})$ is odd. Therefore there is a positive number x such that

$$\frac{(\varphi^{m_1}\|\cdot\|_\infty)(ab^{2k_1})}{(\varphi^{m_2}\|\cdot\|_\infty)(ab^{2k_2})} = \frac{2^{k_1}}{2^{k_1+x}} < 1.$$

Hence

$$\varphi^{m_1}\|\cdot\|_\infty \neq \text{a multiple of } \varphi^{m_2}\|\cdot\|_\infty.$$

So

$$\varphi^{m_1}[a_\infty] \neq \varphi^{m_2}[a_\infty].$$

(3) Where does $\|\cdot\|_\infty$ come from? Why is $[\|\cdot\|_\infty]$ an element of $SLF(F_2)$? Consider the metric graph of groups $\mathscr{G}_s$ with length function $\|\cdot\|_s$ which is pictured in Fig. 11, with amalgamations $x_j = x_{j-1}^2$ $(2 \le j \le s)$ and $x_1 = b^2$. We identify $F(a, b)$ with $\pi_1(\mathscr{G}_s, P)$ via the isomorphism

$$a \mapsto a$$

$$b \mapsto (tt_s \dots t_1)b(tt_s \dots t_1)^{-1}.$$

Note that the distance from P to Q is 1.

Note that, in the Bass group $\beta(\mathscr{G}_s)$,

$$t_1 b^2 t_1^{-1} = x_1$$

$$t_2 t_1 b^4 t_1^{-1} t_2^{-1} = x_2, \quad \text{etc.}$$

Thus when $w = a^{n_1} b^{odd\cdot 2^{i_1}} \dots a^{n_q} b^{odd\cdot 2^{i_2}}$ is written as a word in $\pi_1(\mathscr{G}_s, P)$, we get

$$\|\cdot\|_s(w) = 2\left(\frac{1}{2^{y_1}} + \frac{1}{2^{y_2}} + \cdots + \frac{1}{2^{y_s}}\right) \quad \text{where } y_j = \min\{s, i_j\}.$$

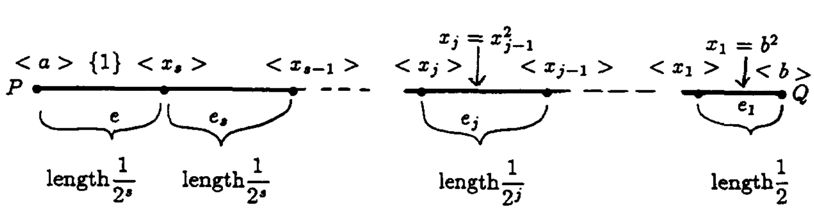

Fig. 11.

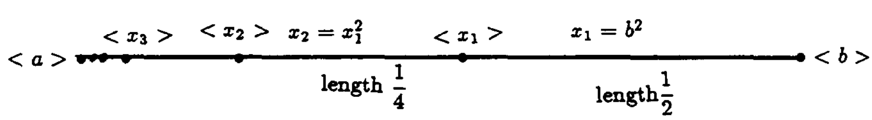

Fig. 12.

Thus (except for the factor of 2),

$$\|\cdot\|_\infty = \lim_{s\to\infty} \|\cdot\|_s.$$

Since each $[\|\cdot\|_s] \in SLF(F_2)$ it follows that $\|\cdot\|_\infty$ is a translation length function of an action on an $\mathbf{R}$-tree and $[\|\cdot\|_\infty] \in SLF(F_2)$. □

Note. To picture the $\mathbf{R}$-tree, consider the infinite "graph of groups" in Fig. 12, with $x_1 = b^2$, $x_s = x_{s-1}^2$ for all $s \geq 2$.

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