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A PROPERTY OF THE COLORED COMPLETE GRAPH

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If the lines of the complete graph K_n are colored so that no point is on more than $\frac{1}{7}(n-1)$ lines of the same color or so that each point lies on more than $\frac{1}{7}(5n+8)$ lines of different colors, then K_n contains a cycle of length n with adjacent lines having different colors.

Introduction

Let the lines of a graph G be colored. Let an AC_m (of G) be a cycle of length m with adjacent lines having different colors.

Suppose K_n (the complete graph on n points) is colored so that no point lies on more than λ lines of the same color. Daykin asked in [2] whether K_n must then contain an AC_n for n sufficiently large. He showed that it must for $\lambda=2$ and $n \geqslant 6$. Bollobas and Erdös showed in [1] that there must be an AC_n for $n > 69\lambda$. In this paper we improve this result by reducing the bound here to $n > 7\lambda$.

Bollobas and Erdös also showed that if K_n is colored so every point lies on lines of $7n/8$ different colors then K_n must contain an AC_n. We improve this result by showing that if K_n is colored so that every point lies on lines of more than $(5n+8)/7$ different colors, then K_n must contain an AC_n.

We also give a conjecture concerning a more general problem of this type.

Theorem 1. *If $n \geqslant 7\lambda+1$ and the lines of the complete graph on n points K_n are colored so that no point lies on more than λ lines of the same color, then K_n contains an AC_n.*

We will prove this by showing first that K_n must contain an AC_m with m "large" and then that an arbitrary point may always be added to any AC_m to produce an AC_{m+1} when m is sufficiently large.

In what follows we will denote the color of the line connecting two points P and Q by (P, Q).

Lemma 1. *Under the hypothesis of Theorem 1, K_n contains an AC_m for some $m \geqslant n-\lambda$.*

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Proof. We will show how to build up a path which can always be closed into an AC_m with at most λ points left out.

Select distinct points $P_1, \ldots, P_{\lambda+2}$ so that for $3 \leqslant j \leqslant \lambda+2$

$$(P_j, P_{j-1}) \neq (P_{j-1}, P_{j-2}), \qquad (P_j, P_{j-1}) \neq (P_{j-1}, P_1), \qquad (P_j, P_1) \neq (P_2, P_1).$$

This is clearly possible as each of the conditions on P_j can exclude at most $\lambda - 1$ points. Next add the remaining points so that for $j \geqslant \lambda+3$, $(P_j, P_{j-1}) \neq (P_{j-1}, P_{j-2})$ until no more can be added. Let P_{n-k} be the last point added and let $Q_1, Q_2 \cdots Q_k$ be the remaining points. Note $k \leqslant \lambda - 1$. Then we must have $(P_{n-k}, P_{n-k-1}) = (P_{n-k}, Q_i)$, $1 \leqslant i \leqslant k$. Pick h so that $2 \leqslant h \leqslant \lambda - k + 1$ and $(P_{n-k}, P_h) \neq (P_{n-k}, P_{n-k-1})$. We have $\lambda - k$ possibilities and at most $\lambda - (k+1)$ can be excluded (as at most λ lines containing P_{n-k} have color (P_{n-k}, P_{n-k-1})) so this is possible. If $(P_{n-k}, P_h) \neq (P_h, P_{h+1})$, then $(P_h, P_{h+1}, \ldots, P_{n-k})$ is an AC_m with at most $h - 1 + k \leqslant \lambda$ points left over. If $(P_{n-k}, P_h) \neq (P_h, P_{h-1})$, then $(P_h P_{h-1} \cdots P_2 P_1 P_{h+1} P_{h+2} \cdots P_{n-k})$ is an AC_{n-k}. As $(P_{h-1}, P_h) \neq (P_h, P_{h+1})$ one of the above cases must occur, in either case giving an AC_m with $m \geqslant n - \lambda$.

Lemma 2. *Let the hypothesis of Theorem* 1 *hold. Suppose K_n contains an AC_m where $6\lambda < m < n$. Then K_n contains an AC_{m+1}.*

Proof. The idea behind this proof is that as m increases the number of ways a point may be added to an AC_m grows faster than the number of ways the addition may fail to yield an AC_{m+1}.

Let the points of the AC_m be $1, 2, \ldots, m$ in cyclic order. Let P be any remaining point. Let $L = \{j \mid (P, j) \neq (j, j-1)\}$, $R = \{j \mid (P, j) \neq (j, j+1)\}$. Since $(j-1, j) \neq (j, j+1)$ $|L| + |R| \geqslant m$. Hence we may assume without loss of generality that $|L| \geqslant m/2$. Consider pairs of points $i, j \in L$ such that $(P, i) \neq (P, j)$ and $j \neq i \pm 1$. There are at least $\frac{1}{2}|L|(|L| - \lambda - 2)$ such pairs. If $(i+1, j+1) \neq (j+1, j+2)$ and $(i+1, j+1) \neq (i+1, i+2)$ then $(i+1, i+2, \ldots, j, P, i, i-1, \ldots, j+2, j+1)$ is an AC_{m+1} (see Fig. 1). But each line $(j+1, j+2) j \in L$ can eliminate at most $\lambda - 1$ pairs. Hence if $\frac{1}{2}|L|(|L| - \lambda - 2) > |L|(\lambda - 1)$ an AC_{m+1} must exist. But this condition reduces to $|L| > 3\lambda$ which is satisfied if $m > 6\lambda$ since $L \geqslant m/2$.

If $i, j \in L$, $(P, j) \neq (P, i)$, $(i+1, j+1) \neq (j+1, j+2)$, $(i+1, j+1) \neq (i+1, i+2)$, then an AC_{m+1} exists as shown in Fig. 1.

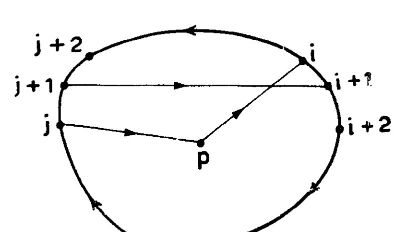

Fig. 1.

Theorem 1 now follows at once from Lemmas 1 and 2.

It is also possible to show that under the hypothesis of Theorem 1 K_n must contain an AC_m for any $m, 3 \leq m \leq n$, but we will not give the details here.

Theorem 2. *Let the lines of K_n be colored so that every point lies on lines of at least ψn different colors. Let $\psi > (5n+8)/7n$. Then K_n contains an AC_n.*

First we make some more definitions. A path $P_1 P_2 \cdots P_m$ is an AP_m if adjacent lines have different colors. P_1 is a good end if $(P_1, x) = (P_1, P_2) \Rightarrow x = P_2$. Otherwise P_1 is a bad end. A path with both ends good is a good path. We will prove the theorem by showing that K_n contains many good paths and then that this implies the existence of an AC_n.

Lemma 3. *Let the lines of K_n be colored so that every point lies on at least ψn different colors. Let $m > 2(1-\psi)n+1$. Then any m point subgraph K_m of K_n contains a good AP_2 or a good AP_3.*

Proof. Let the vertices of K_m be $P_1, P_2, \ldots, P_m$. For $i = 1, 2, \ldots, m$ let c_i be the cardinality of the largest set of edges in K_m containing P_i and having the same color. Let d_i be the number of edges in K_m containing P_i such that no other edge in K_m containing P_i has the same color. In other words d_i is the number of ways P_i can be a good end (in K_m). Let b_i be the number of edges of different colors P_i lies on. Then we have

(1) $b_i + (n-m) \geq \psi n$.

(2) $d_i + c_i + 2(b_i - d_i - 1) \leq m - 1$.

Hence

(3) $d_i \geq c_i + m - 2(1-\psi)n - 1$.

By assumption $m > 2(1-\psi)n + 1$. Hence

(4) $\sum d_i > \sum c_i$.

Consider the lines contributing to $\sum d_i$. If K_m has no good AP_2, then there are $\sum d_i$ such lines each having one good and one bad end. Consider the bad ends of these lines. If P_i is a bad end on more than c_i of these lines, then P_i must be a bad end on 2 of these lines of different colors, which gives a good AP_3. But this case must occur because of (4).

Lemma 4. *Let K_n be colored so that no point lies on more than λ lines of the same color. Suppose K_n contains K disjoint good AP's where $K > \frac{1}{2}(\lambda+2)$. Then K_n contains an AC_n.*

Proof. Let K_n contain good AP's $P_1, \ldots, P_K$. Partition the remaining points into AP's $Q_1, \ldots, Q_h$ so that no points of $Q_{j+1}, \ldots, Q_h$ can be added to either end of $Q_j, j=1,\ldots,h-1$. Then we claim we can successively insert $Q_h, Q_{h-1},\ldots,Q_2,Q_1$ between pairs of good AP's at each step reducing the number of good AP's by 1. For suppose $Q_h, Q_{h-1},\ldots,Q_{j+1}$ have already been inserted. There are $K-h+j$ good paths at this stage. Note $|Q_2|+\cdots+|Q_h|\leq\lambda-1$ ($|S|$ denotes the number of points in S) and $|Q_2|\cdots|Q_{h-1}|\geq 2$, $|Q_h|\geq 1$. Hence $2h-4+1\leq\lambda-1\Rightarrow h\leq\frac{1}{2}(\lambda+2)<K$. Now no points of $Q_{j+1},\ldots,Q_n$ can be added to either end of Q_j so at most $\lambda-2h+2j$ additional points cannot be added to either end. Now the good paths contain $2(K-h+j)$ ends so since $K>\frac{1}{2}(\lambda+2)\Rightarrow 2K-2h+2j>\lambda-2h+2j$ we can connect some P_i to one end of Q_j and since $K>\frac{1}{2}(\lambda+2)\Rightarrow 2(K-1)-2h+2j>\lambda-2h+2j$ we can connect the other end of Q_j to some one of the remaining P's. Continuing in this fashion we end with the points of K_n partitioned into $K-h$ good paths which can then be connected to form an AC_n.

The theorem can now be proved as follows. Since each point lies on lines of ψn different colors, a point may lie on at most $\lambda=n-\psi n<\frac{1}{7}(2n-8)$ lines of the same color. The theorem will follow by Lemma 4 if we can find $K>\frac{1}{7}(n+3)>\frac{1}{2}(\lambda+2)$ disjoint good paths. We can use Lemma 3 to find disjoint good paths, at each step letting m be the number of points remaining. Let K be the number of disjoint good paths we can find in this fashion. Then $n-3K\leq 2(1-\psi)n+1\Rightarrow 3K\geq 2\psi n-n-1>\frac{1}{7}(3n+9)\Rightarrow K>\frac{1}{7}(n+3)$ as desired.

It is known (see [2]) that Theorem 1 fails for $n=2\lambda+1$ and Theorem 2 fails for $\psi n=\frac{1}{2}(n-1)$. No better negative results are known to the author.

One can also look for subgraphs other than cycles with adjacent lines having different colors. In this regard we make the following conjecture.

Conjecture. Let λ, α be fixed. Let n be sufficiently large. Let G be a graph on n points containing at most αn paths of length 2. Let K_n be colored so no line lies on more than λ edges of the same color. Then K_n contains a subgraph isomorphic to G with adjacent lines having different colors.

References

[1] B. Bollobas and P. Erdös, Alternating Hamiltonian cycles, Israel J. Math. 23 (1976) 126–131.
[2] D.E. Daykin, Graphs with cycles having adjacent lines different colors, J. Combinatorial Theory (B) 20 (1976) 149–152.