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Note

Some Notes on Feinberg's k-Independence Problem

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A short proof is given of a recent theorem of M. Feinberg on representable matroids. The result is shown not to hold for nonrepresentable matroids of rank 3.

Following Feinberg [1], we call a matroid **M** on a set S *k-independent in degrees* if there is an ordering $x_1,..., x_m$ of the elements of S such that, for each i ($i = 1,..., m$), there exist k not necessarily distinct hyperplanes of **M** whose union contains $x_1,..., x_{i-1}$ but not x_i. This implies that **M** has no loops or parallel elements (the effect of the hypothesis when $i = 1$ being solely to ensure that x_1 is not a loop).

M is *k-independent* if the above statement is true for *every* ordering of the elements of S; that is, if, for each element x of S, there exist k not necessarily distinct hyperplanes of **M** whose union is exactly $S\backslash\{x\}$. So "1-independent" and "1-independent in degrees" both mean exactly the same as "independent"; and, in general, "k-independent" implies "k-independent in degrees," but not *vice versa*.

The following theorem was proved by Feinberg in [1].

THEOREM 1. *Let* **M** *be a representable matroid of rank r on a set S. If* **M** *is k-independent in degrees (and hence, a fortiori, if* **M** *is k-independent), then* $|S| \leqslant \binom{r+k-1}{k}$.

Proof. Choose a representation for **M** as a set of $|S|$ distinct vectors in a vector space V of dimension r over some field K, so that independence in **M** corresponds to linear independence in V. Let $V^{(k)}$ be a vector space of

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dimension $\binom{r+k-1}{k}$ over K, fix bases for V and $V^{(k)}$, and define a map $f: V \to V^{(k)}$ as follows: if $\mathbf{x} = (x_1,..., x_r) \in V$, then the coordinates of $f(\mathbf{x})$ are all the distinct products of k not necessarily distinct x_i's arranged in lexicographic order of suffixes. (For example, if $r = k = 3$, then $\binom{r+k-1}{k} = 10$ and

$$f(\mathbf{x}) = (x_1^3, x_1^2x_2, x_1^2x_3, x_1x_2^2, x_1x_2x_3, x_1x_3^2, x_2^3, x_2^2x_3, x_2x_3^2, x_3^3).)$$

To prove the theorem, it suffices to prove that, if a subset X of V is k-independent in degrees, then $\{f(\mathbf{x}): \mathbf{x} \in X\}$ is linearly independent in $V^{(k)}$.

To do this, let $\mathbf{y} \in Y \subseteq X$ and suppose that $Y \backslash \{\mathbf{y}\}$ is covered by k not necessarily distinct hyperplanes none of which contains $\mathbf{y}$. Let the equations of these hyperplanes be $h_1 = 0,..., h_k = 0$, where each h_i is a linear function of coordinates, $h_i = a_{i1}x_1 + \cdots + a_{ir}x_r$. Then $h_1 \cdots h_k = 0$ is a homogeneous equation of degree k that is satisfied by all the points of $Y \backslash \{\mathbf{y}\}$ but not by $\mathbf{y}$. It can thus be reinterpreted as a linear equation that is satisfied by all the points of $\{f(\mathbf{x}): \mathbf{x} \in Y \backslash \{\mathbf{y}\}\}$ but not by $f(\mathbf{y})$.

Now, if X is k-independent in degrees, then there is an ordering $\mathbf{x}_1,..., \mathbf{x}_m$ of its elements such that, for each i ($i = 1,..., m$), there are k hyperplanes of V whose union contains $\mathbf{x}_1,..., \mathbf{x}_{i-1}$ but not $\mathbf{x}_i$. It follows from the previous paragraph that, for each i ($i = 1,..., m$), there is a hyperplane of $V^{(k)}$ that contains $f(\mathbf{x}_1),..., f(\mathbf{x}_{i-1})$ but not $f(\mathbf{x}_i)$. Thus $\{f(\mathbf{x}): \mathbf{x} \in X\}$ is linearly independent in $V^{(k)}$. This completes the proof of Theorem 1. ∎

This result clearly holds for non-representable matroids of ranks 1 and 2, but not (as we shall see in Example 2) for those of rank 3. For arbitrary matroids we have the following rather trivial upper bound.

THEOREM 2. *Let* **M** *be a matroid of rank r on a set S. If* **M** *is k-independent in degrees (and hence, a fortiori, if* **M** *is k-independent), then* $|S| \leqslant (k^r - 1)/(k - 1)$.

Proof. The proof is performed by induction on r. The result is clearly true if $r = 1$ or 2; so suppose $r \geqslant 3$. Let $x_1,..., x_m$ be the elements of S ordered in such a way that, for each i ($i = 1,..., m$), there are k hyperplanes of **M** whose union contains $x_1,..., x_{i-1}$ but not x_i. Let $H_1,..., H_k$ be these k hyperplanes in the case $i = m$. It is easy to see that, for each such hyperplane H_j, $\mathbf{M}|H_j$ is a matroid of rank $r - 1$ that is k-independent in degrees. So, by the induction hypothesis, $|H_j| \leqslant (k^{r-1} - 1)/(k - 1)$ for each j, whence

$$|S| \leqslant k \cdot \frac{k^{r-1} - 1}{k - 1} + 1 = \frac{k^r - 1}{k - 1},$$

as required. This completes the proof of Theorem 2. ∎

We now give two constructions, the first of which is used in the second and also shows that Theorem 1 is best possible. (This example appears in [1].)

EXAMPLE 1. Let V be a vector space of dimension r over any sufficiently large field. Choose $r+k-1$ hyperplanes in "general position" in V (that is, such that only the zero vector belongs to r or more of them). Each subset of $r-1$ of these hyperplanes then determines (intersects in) a one-dimensional subspace. If X is any set of $\binom{r+k-1}{r-1} = \binom{r+k-1}{k}$ non-zero vectors, one chosen from each of these one-dimensional subspaces, then it is clear that, for each $\mathbf{x}$ in X, $X \backslash \{\mathbf{x}\}$ can be covered by k hyperplanes that do not contain $\mathbf{x}$. Thus X is k-independent (hence, k-independent in degrees).

For use in Example 2, consider the configuration of Example 1 in the case $r=3$, $k=t$ (say). The analogous affine or projective configuration consists of the $\binom{t+2}{2}$ points of intersection of $t+2$ lines in "general position" in a plane (meaning that no three lines are concurrent). If n is a positive integer, let $S(n)$ denote any matroid obtained from such a configuration by deleting any $\binom{t+2}{2} - n$ points, where t is as small as possible subject to $\binom{t+2}{2} \geqslant n$.

EXAMPLE 2. Consider an affine plane Π with n^2 points and n^2+n lines. Let the n lines in one particular parallel class be $L_1, ..., L_n$, and replace each of these by a configuration of type $S(n)$. Then the resulting configuration Π' is $(n+t)$-independent. For, let P and Q be two points of L_1 (say). The n other lines of Π through Q cover all the points of Π' not in L_1 (and do not cover P); and, from the definition of the $S(n)$ that replaced L_1, there are t lines in this $S(n)$ whose union is $L_1 \backslash \{P\}$. Thus there are $n+t$ lines in Π' whose union is $\Pi' \backslash \{P\}$. This holds for each point P in Π', and so Π' is $(n+t)$-independent, as stated.

To show that the result of Theorem 1 does not apply to non-representable matroids, we can take $n=19$, $t=5$ in Example 2, and we get a 24-independent matroid of rank 3 on $19^2 = 361$ points, whereas $\binom{24+2}{24} = \binom{26}{2} = 325$. By combining the results of Example 2 and Theorem 2, one can show that the largest number of elements in a k-independent matroid of rank 3 is $k^2 + o(k^2)$ as $k \to \infty$. It would be nice to reduce the error term in this expression.

It would also be nice to generalize this construction to larger values of r. It is still possible that Theorem 1 holds without the word "representable" if $r \geqslant 4$.

REFERENCES

1. M. FEINBERG, On a generalization of linear independence in finite-dimensional vector spaces, *J. Combin. Theory Ser. B* **30** (1981), 61–69.