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Direct Sums of Ordered Near-Rings

GÜNTER F. PILZ

University of Arizona, Tucson, Ariz. 85721

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In [5] J. L. Zemmer has shown that a direct sum of ordered rings can again be ordered if and only if all but at most one of the summands are zerorings. In this paper it will be shown that one can only obtain a weaker result for near-rings instead of rings. This result can be applied to get strong structure theorems for large classes of ordered near-rings. "Ordered" shall always mean "fully ordered".

1. DIRECT SUMS

An *ordered near-ring* is a non-trivial left near-ring $(N, +, \cdot)$ such that $(N, +)$ is an ordered group under the order relation $\leqslant$ and $n_1 \geqslant 0$, $n_2 \geqslant 0$ implies $n_1 n_2 \geqslant 0$ for all $n_1, n_2 \in N$. If $n_1 > 0$, $n_2 > 0$ implies $n_1 n_2 > 0$, we call N *strictly ordered.* Examples of strictly ordered near-rings are the polynomial near-rings of all $\sum_{i=1}^{n} a_i x^i$ $(a_n \neq 0)$ with addition and substitution of polynomials as composition and coefficients from an ordered ring. $\sum_{i=1}^{n} a_i x^i$ is then defined to be greater than 0 if a_n is greater than 0. For the background of ordered near-rings see [4].

THEOREM 1. *Let N be the finite direct sum of the near-rings $N_1, \ldots, N_s$. If N is ordered, then in all but at most one of the near-rings N_i all positive elements (in the induced order of N under the projection map) annihilate N_i from the left.*

Proof. Assume that there exist two of these near-rings, say N_i and N_j $(i \neq j,\ 1 \leqslant i, j \leqslant s)$ containing positive elements n_i and n_j which are not left nullifiers. Then one can choose $n_i' \in N_i$ and $n_j' \in N_j$, $n_i' > 0$, $n_j' > 0$ such that $n_i n_i' \neq 0$ and $n_j n_j' \neq 0$. This implies $n_i n_i' > 0$ and $n_j n_j' > 0$. We will show that n_i' and n_j' turn out to be incomparable, which contradicts the assumption that N is ordered. In fact, if $n_i' < n_j'$ holds then one gets $0 < n_i n_i' \leqslant n_i n_j' = 0$, a contradiction; similarly, if $n_i' > n_j'$.

Remark. More cannot be obtained: take for N_1 an arbitrary strictly ordered near-ring and for N_2 an ordered group which is made into a near-ring by defining n_2n_2' to be 0 if n_2 is greater than or equal to 0 and to be n_2' otherwise ($n_2, n_2' \in N_2$). Ordering $N_i = N_1 \oplus N_2$ lexicographically yields a counter-example to the exact analogue of Zemmer's result.

COROLLARY 1. *Let $N = N_1 \oplus \cdots \oplus N_s$ be an ordered near-ring with right multiplicative identity e. Then $s = 1$.*

Proof. e can uniquely be decomposed into $e = e_1 + \cdots + e_s$ ($e_i \in N_i$). A simple calculation shows that e_i is a right identity in N_i for all $i \in \{1,..., s\}$. If s is greater than 1, then by theorem 1 all positive elements $p_i \in N_i$ annihilate N_i from the left. In particular, $p_i = p_ie_i = 0$, which implies $N_i = 0$ for all but at most one i. The following corollary follows immediately.

COROLLARY 2. *Let $N = N_1 \oplus \cdots \oplus N_s$ be strictly ordered. Then again $s = 1$.*

2. STRUCTURE-THEORETICAL APPLICATIONS OF THEOREM 1

Assuming the additional postulate $0n = 0$ for all $n \in N$, Blackett defined in [2] a near-ring N to be *semi-simple* if in N the right modules (subgroups M of $(N, +)$ with $MN \subseteq M$) fulfill the descending chain condition and if there exist no non-zero nilpotent right modules. A near-ring is called *simple*, if N has no proper two-sided ideals, satisfies the descending chain condition on right modules and if N has no non-zero right modules M fulfilling $MN = 0$. Deskins [3] and Betsch [1] defined *radicals* rad(N) for near-rings N which in the case, when the right modules fulfill the d.c.c., are equal to 0 if and only if N is semi-simple. Since a semisimple near-ring with right identity and satisfying the d.c.c. for right modules is a direct sum of ideals of which is simple [2] and Corollary 1 we get

THEOREM 2. *Each ordered semi-simple near-ring with right identity is simple.*

An element n_0 of an arbitrary near-ring N is said to be *constant*, if $nn_0 = n_0$ for all $n \in N$. N is called *constant*, if all $n \in N$ are constant.

COROLLARY 3. *Let N be an ordered near-ring with identity, fulfilling the d.c.c. for right modules and containing no non-zero nilpotent right modules. Then N is simple.*

Proof. By the corollary of Theorem 8 in [4] an ordered near-ring with identity contains no non-zero constant elements. Therefore $0n = 0$ for all $n \in N$, since otherwise $0n$ would be a non-zero constant element. By assumption, the conditions for semi-simplicity are satisfied. Theorem 2 implies that N is simple.

THEOREM 3. *Let N be a strictly ordered near-ring with d.c.c. for right modules. Then N is either constant or simple.*

Proof. If there exists a non-zero constant element of N, then there also exists a strictly positive one, say k_0. If $n \in N$, then $n > 0n$ implies $k_0 n > 0n$ and $n < 0n$ implies $k_0 n < 0n$. But by [5] $k_0 n = 0n$, since k_0 is a positive constant element. Therefore $n = 0n$, $n_1 n = n_1(0n) = (n_1 0)\, n = 0n = n$ for all $n_1 \in N$ and N is constant.

If N is not constant, then $0n$ is equal to 0 for all $n \in N$. Let $M \neq 0$ be a right module. If $m \in M$ is greater than 0, then $mm > 0$ which proves that $MM \neq 0$. The right modules fulfill the d.c.c., therefore N is semi-simple and can be written as a direct sum of s simple near-rings. But $s = 1$ by Corollary 2. Therefore N is simple.

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