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Journal of Algebra 304 (2006) 1114–1129

JOURNAL OF Algebra

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# On a class of Koszul algebras associated to directed graphs

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Received 10 August 2005

Available online 19 December 2005

Communicated by Efim Zelmanov

**Abstract**

In [I. Gelfand, V. Retakh, S. Serconek, R.L. Wilson, On a class of algebras associated to directed graphs, Selecta Math. (N.S.) 11 (2005), math.QA/0506507] I. Gelfand and the authors of this paper introduced a new class of algebras associated to directed graphs. In this paper we show that these algebras are Koszul for a large class of layered graphs.
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*Keywords:* Koszul algebras; Directed graphs

## 0. Introduction

In [GRSW] I. Gelfand and the authors of this paper associated to any layered graph $\Gamma$ an algebra $A(\Gamma)$ and constructed a basis in $A(\Gamma)$ when the graph is a layered graph with a unique minimal vertex.

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[1] Partially supported by NSA.

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doi:10.1016/j.jalgebra.2005.11.005

The algebra $A(\Gamma)$ is a natural generalization of universal algebra $Q_n$ of pseudo-roots of noncommutative polynomials introduced in [GRW]. In fact, $A(\Gamma)$ is isomorphic to $Q_n$ when $\Gamma$ is the hypercube of dimension $n$, i.e. the graph of all subsets of a set with $n$ elements.

The algebras $Q_n$ have a rich and interesting structure related to factorizations of polynomials over noncommutative rings. On one hand, $Q_n$ is a "big algebra" (in particular, it contains free subalgebras on several generators and so has an exponential growth). On the other hand, it is rather "tame": it is a quadratic algebra, one can construct a linear basis in $Q_n$ [GRW], compute its Hilbert series [GGRSW], prove that $Q_n$ is Koszul [SW,Pi], and construct interesting quotients of $Q_n$ [GGR].

Since the algebra $A(\Gamma)$ is a natural generalization of $Q_n$ one would expect that for a "natural" class of graphs the algebra $A(\Gamma)$ is Koszul. In this paper we prove this assertion when $\Gamma$ is a *uniform layered graph*; see Definition 3.3. The Hasse graph of ranked modular lattices with a unique minimal element is an example of such a graph.

Compared to the proof given in [SW] for the algebra $Q_n$, our proof is much simpler, and more geometric.

## 1. The algebra $A(\Gamma)$ as a quotient of $T(V^+)$

We begin by recalling (from [GRSW]) the definition of the algebra $A(\Gamma)$. Let $\Gamma = (V, E)$ be a *directed graph*. That is, $V$ is a set (of vertices), $E$ is a set (of edges), and $\mathbf{t}: E \to V$ and $\mathbf{h}: E \to V$ are functions. ($\mathbf{t}(e)$ is the *tail* of $e$ and $\mathbf{h}(e)$ is the *head* of $e$.)

We say that $\Gamma$ is *layered* if $V = \bigcup_{i=0}^{n} V_i$, $E = \bigcup_{i=1}^{n} E_i$, $\mathbf{t}: E_i \to V_i$, $\mathbf{h}: E_i \to V_{i-1}$. Let $V^+ = \bigcup_{i=1}^{n} V_i$.

We will assume throughout the remainder of the paper that $\Gamma = (V, E)$ is a layered graph with $V = \bigcup_{i=0}^{n} V_i$, that $V_0 = \{*\}$, and that, for every $v \in V^+$, $\{e \in E \mid \mathbf{t}(e) = v\} \neq \emptyset$.

If $v, w \in V$, a *path* from $v$ to $w$ is a sequence of edges $\pi = \{e_1, e_2, \dots, e_m\}$ with $\mathbf{t}(e_1) = v$, $\mathbf{h}(e_m) = w$ and $\mathbf{t}(e_{i+1}) = \mathbf{h}(e_i)$ for $1 \leqslant i < m$. We write $v = \mathbf{t}(\pi)$, $w = \mathbf{h}(\pi)$. We also write $v > w$ if there is a path from $v$ to $w$. Define

$$P_\pi(t) = (1 - te_1)(1 - te_2) \cdots (1 - te_m) \in T(E)[t]/\bigl(t^{n+1}\bigr)$$

and write

$$P_\pi(t) = \sum_{j=0}^{n} e(\pi, j) t^j.$$

Recall (from [GRSW]) that $R$ denotes the ideal of $T(E)$, the tensor algebra on $E$ over the field $F$, generated by

$$\bigl\{e(\pi_1, k) - e(\pi_2, k) \mid \mathbf{t}(\pi_1) = \mathbf{t}(\pi_2),\ \mathbf{h}(\pi_1) = \mathbf{h}(\pi_2),\ 1 \leqslant k \leqslant l(\pi_1)\bigr\}.$$

Also, by Lemma 2.5 of [GRSW], $R$ is actually generated by the smaller set

$$\bigl\{e(\pi_1, k) - e(\pi_2, k) \mid \mathbf{t}(\pi_1) = \mathbf{t}(\pi_2),\ \mathbf{h}(\pi_1) = \mathbf{h}(\pi_2) = *,\ 1 \leqslant k \leqslant l(\pi_1)\bigr\}.$$

**Definition 1.1.** $A(\Gamma) = T(E)/R$.

Note that $e(\pi, k) \in T(E)_k$. Thus $R = \sum_{j=1}^{\infty} R_j$ is a graded ideal in $T(E)$. We write $\tilde{e}(\pi, k)$ for the image of $e(\pi, k)$ in $A(\Gamma)$.

In fact, $A(\Gamma)$ may also be expressed as a quotient of $T(V^+)$. To verify this we need a general result about quotients of the tensor algebra by graded ideals. Let $W$ be a vector space over $F$ and let $I = \sum_{j=1}^{\infty} I_j$ be a graded ideal in the tensor algebra $T(W)$. Let $\psi$ denote the canonical map from $W$ to the quotient space $W/I_1$ and let $\langle I_1 \rangle$ denote the ideal in $T(W)$ generated by $I_1$. Then $\psi$ induces a surjective homomorphism of graded algebras

$$\phi : T(W) \to T(W/I_1) \cong T(W)/\langle I_1 \rangle.$$

Consequently, by the Third Isomorphism Theorem, we have:

**Proposition 1.2.**

$$T(W)/I \cong T(W/I_1)/\phi(I),$$

*where* $\phi(I) = \sum_{j=2}^{\infty} \phi(I_j)$ *is a graded ideal of* $T(W/I_1)$.

We now apply this to the presentation of the algebra $A(\Gamma)$. Recall that for each vertex $v \in V^+$ there is a distinguished edge $e_v$ with $\mathbf{t}(e_v) = v$. Recall further that for $v \in V^+$ we define $v^{(0)} = v$ and $v^{(i+1)} = \mathbf{h}(e_{v^{(i)}})$ for $0 \leqslant i < |v| - 1$ and that we set $e(v, 1) = e_{v^{(0)}} + e_{v^{(1)}} + \cdots + e_{v^{(|v|-1)}}$. Thus

$$e_v = e(v, 1) - e\big(v^{(1)}, 1\big) = e\big(\mathbf{t}(e_v), 1\big) - e\big(\mathbf{h}(e_v), 1\big).$$

Let $E' = \{e_v \mid v \in V^+\}$. Define $\tau : FE \to FE'$ by

$$\tau(f) = e\big(\mathbf{t}(f), 1\big) - e\big(\mathbf{h}(t), 1\big).$$

Then $\tau$ is a projection of $FE$ onto $FE'$ with kernel $R_1$.

Now define $\eta : FE' \to FV^+$ by

$$\eta : e_v \mapsto v.$$

Then $\eta$ is an isomorphism of vector spaces and $\eta\tau$ induces an isomorphism

$$\nu : FE/R_1 \to FV^+.$$

As above, $\nu$ induces a surjective homomorphism of graded algebras

$$\theta : T(E) \to T\big(V^+\big).$$

Thus Proposition 1.2 gives:

**Corollary 1.3.** $A(\Gamma) \cong T(V^+)/\theta(R)$.

It is important to write generators for the ideal $\theta(R)$ explicitly. Since $R$ is generated by $R_1$ together with the elements of the form $e(\pi_1, k) - e(\pi_2, k)$ it will be sufficient to write $\theta(e(\pi, k))$ explicitly. Let $\pi = \{e_1, e_2, \dots, e_m\}$ be a path, let $\mathbf{t}(e_i) = v_{i-1}$ for $1 \leqslant i \leqslant m$ and let $\mathbf{h}(e_m) = v_m$. Then

$$e(\pi, k) = (-1)^k \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant m} e_{i_1} \cdots e_{i_k}.$$

Now $\nu(e_i) = e_{v_{i-1}} - e_{v_i}$ and so $\eta\nu(e_i) = v_{i-1} - v_i$. Since $\theta$ is induced by $\eta\nu$ we have:

**Lemma 1.4.**

$$\theta\bigl(e(\pi, k)\bigr) = (-1)^k \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant m} (v_{i_1-1} - v_{i_1}) \cdots (v_{i_k-1} - v_{i_k}).$$

## 2. A presentation of gr $A(\Gamma)$

Let $W = \sum_{k=0}^{\infty} W_k$ be a graded vector space. We begin by recalling some basic properties of $T(W)$.

$T(W)$ is bi-graded, that is, in addition to the usual grading (by degree in the tensor algebra), there is another grading induced by the grading of $W$. Thus

$$T(W) = \sum_{i=0}^{\infty} T(W)_{[i]}$$

where

$$T(W)_{[i]} = \operatorname{span}\{w_1 \cdots w_r \mid r \geqslant 0,\ w_j \in W_{[l_j]},\ l_1 + \cdots + l_r = i\}.$$

This grading induces a filtration on $T(W)$. Namely

$$\begin{aligned} T(W)_i &= T(W)_{[i]} + T(W)_{[i-1]} + \cdots + T(W)_{[0]} \\ &= \operatorname{span}\{w_1 \cdots w_r \mid r \geqslant 0,\ w_j \in W_{[l_j]},\ l_1 + \cdots + l_r \leqslant i\}. \end{aligned}$$

Since $T(W)_i / T(W)_{i-1} \cong T(W)_{[i]}$ we may identify $T(W)$ with its associated graded algebra. Define a map

$$\operatorname{gr} : T(W) \to T(W) = \operatorname{gr} T(W)$$

by

$$\operatorname{gr} : \lambda \mapsto \lambda$$

for $\lambda \in F.1$ and

$$\mathrm{gr}: u = \sum_{i=0}^{k} u_k \mapsto u_k,$$

where $k > 0$, $u_i \in T(W)_{[i]}$ and $u_k \neq 0$.

**Lemma 2.1.** *Let $W$ be a graded vector space and $I$ be an ideal in $T(W)$. Then*

$$\mathrm{gr}\big(T(W)/I\big) \cong T(W)/(\mathrm{gr}\, I).$$

**Proof.** We have

$$(\mathrm{gr}\, I)_{[k]} = T(W)_{[k]} \cap \big(T(W)_{k-1} + I\big).$$

Therefore

$$\begin{aligned}
\mathrm{gr}\big(T(W)/I\big)_{[k]} &= \big(T(W)/I\big)_k / \big(T(W)/I\big)_{k-1} \\
&= \big(\big(T(W)_k + I\big)/I\big) / \big(\big(T(W)_{k-1} + I\big)/I\big) \\
&\cong \big(T(W)_k + I\big) / \big(T(W)_{k-1} + I\big) \\
&= \big(T(W)_{[k]} + T(W)_{k-1} + I\big) / \big(T(W)_{k-1} + I\big) \\
&\cong T(W)_{[k]} / \big(T(W)_{[k]} \cap \big(T(W)_{k-1} + I\big)\big) \\
&= T(W)_{[k]} / (\mathrm{gr}\, I)_{[k]}. \qquad \square
\end{aligned}$$

The decomposition of $V$ into layers induces a grading of the vector space $FV^+$. Thus the tensor algebra $T(V^+)$ is graded and filtered as above. The following lemma shows that this filtration on $T(V^+)$ agrees with that induced by the filtration on $T(E)$.

**Lemma 2.2.** *For all $i \geqslant 0$, $T(V^+)_i = \theta(T(E)_i)$.*

**Proof.** This holds for $i = 0$ since $T(V^+)_0 = T(E)_0 = F$. Furthermore, $T(E)_1$ is spanned by 1 and $\{f \mid f \in E_1\}$. For $f \in E_1$ we have $\tau(f) = e(\mathbf{t}(f), 1) - e(\mathbf{h}(f), 1)$, but $\mathbf{h}(f) = *$ and $e(*, 1) = 0$ so $\tau(f) = e(\mathbf{t}(f), 1)$. Hence $\eta\tau(f) = \mathbf{t}(f)$. Thus $T(V^+)_1 = \theta(T(E)_1)$.

Now assume $T(V^+)_{i-1} = \theta(T(E)_{i-1})$. Then $\theta(T(E)_i)$ is spanned by

$$\begin{aligned}
&\theta\big(\big\{e_1 \cdots e_r \mid r \geqslant 0,\ |e_1| + \cdots + |e_r| \leqslant i\big\}\big) \\
&\quad = \big\{\big(\mathbf{t}(e_1) - \mathbf{h}(e_1)\big) \cdots \big(\mathbf{t}(e_r) - \mathbf{h}(e_r)\big) \mid r \geqslant 0,\ |e_1| + \cdots + |e_r| \leqslant i\big\}.
\end{aligned}$$

Let $u = (\mathbf{t}(e_1) - \mathbf{h}(e_1)) \cdots (\mathbf{t}(e_r) - \mathbf{h}(e_r))$. Then if $|e_1| + \cdots + |e_r| \leqslant i$ we have

$$u \equiv \mathbf{t}(e_1) \cdots \mathbf{t}(e_r) \mod T\big(V^+\big)_{i-1}.$$

The lemma then follows by induction. $\square$

**Corollary 2.3.** *$A(\Gamma) \cong T(V^+)/\theta(R)$ as filtered algebras.*

If $u \in A(\Gamma)_i$, $u \notin A(\Gamma)_{i-1}$ we write $|u| = i$.

As before, let $\pi = \{e_1, e_2, \ldots, e_m\}$ be a path and let $\mathbf{t}(e_i) = v_{i-1}$ for $1 \leqslant i \leqslant m$ and $\mathbf{h}(e_m) = v_m$. For $1 \leqslant k \leqslant m+1$ set

$$v(\pi, k) = v_0 \cdots v_{k-1}.$$

**Lemma 2.4.** *Let $\pi_1, \pi_2$ be paths with $\mathbf{t}(\pi_1) = \mathbf{t}(\pi_2)$ and let $1 \leqslant k \leqslant l(\pi_1)$. Then*

$$v(\pi_1, k) - v(\pi_2, k) \in \operatorname{gr} \theta(R).$$

**Proof.** We may extend $\pi_1, \pi_2$ to paths $\mu_1, \mu_2$ such that $\mathbf{h}(\mu_1) = \mathbf{h}(\mu_2) = *$. Then $e(\mu_1, k) - e(\mu_2, k) \in R$. The result now follows from Lemma 1.4. □

Let $R_{\mathrm{gr}}$ denote the ideal generated by

$$\bigl\{v(\pi_1, k) - v(\pi_2, k) \mid \mathbf{t}(\pi_1) = \mathbf{t}(\pi_2),\ 2 \leqslant k \leqslant l(\pi_1)\bigr\}.$$

**Proposition 2.5.** $\operatorname{gr} A(\Gamma) \cong T(V^+)/R_{\mathrm{gr}}$.

**Proof.** We begin by recalling the description of a basis for $\operatorname{gr} A(\Gamma)$.

We say that a pair $(v, k)$, $v \in V$, $0 \leqslant k \leqslant |v|$ can be *composed* with the pair $(u, l)$, $u \in V$, $0 \leqslant l \leqslant |u|$, if $v > u$ and $|u| = |v| - k$. If $(v, k)$ can be composed with $(u, l)$ we write $(v, k) \models (u, l)$. Let $\mathbf{B_1}(\Gamma)$ be the set of all sequences

$$\mathbf{b} = \bigl((b_1, m_1), (b_2, m_2), \ldots, (b_k, m_k)\bigr),$$

where $k \geqslant 0$, $b_1, b_2, \ldots, b_k \in V$, $0 \leqslant m_i \leqslant |b_i|$ for $1 \leqslant i \leqslant k$. Let

$$\mathbf{B}(\Gamma) = \bigl\{\mathbf{b} = \bigl((b_1, m_1), (b_2, m_2), \ldots, (b_k, m_k)\bigr) \in \mathbf{B}_1(\Gamma) \\ \mid (b_i, m_i) \not\models (b_{i+1}, m_{i+1}),\ 1 \leqslant i < k\bigr\}.$$

For

$$\mathbf{b} = \bigl((b_1, m_1), (b_2, m_2), \ldots, (b_k, m_k)\bigr) \in \mathbf{B}_1(\Gamma)$$

set

$$\tilde{e}(\mathbf{b}) = \tilde{e}(b_1, m_1) \cdots \tilde{e}(b_k, m_k).$$

Clearly $\{\tilde{e}(\mathbf{b}) \mid \mathbf{b} \in \mathbf{B_1}(\Gamma)\}$ spans $A(\Gamma)$. Writing $\bar{e}(\mathbf{b}) = \tilde{e}(\mathbf{b}) + A(\Gamma)_{i-1} \in \operatorname{gr} A(\Gamma)$ where $|\tilde{e}(\mathbf{b})| = i$, Corollary 4.4 of [GRSW] shows that $\{\bar{e}(\mathbf{b}) \mid \mathbf{b} \in \mathbf{B}(\Gamma)\}$ is a basis for $\operatorname{gr} A(\Gamma)$.

Let

$$\check{} : T(V^+) \to T(V^+)/R_{\mathrm{gr}}$$

denote the canonical mapping. Write $\check{e}(b,m)$ for the image of $e(b,m)$ and $\check{e}(\mathbf{b})$ for the image of $e(\mathbf{b})$.

By Lemma 2.1 and Corollary 1.3, we have $\operatorname{gr} A(\Gamma) \cong T(V^+)/(\operatorname{gr}\theta(R))$. Since $R_{\mathrm{gr}} \subseteq \operatorname{gr}\theta(R)$ (by Lemma 2.4) the canonical map

$$T(V^+) \to T(V^+)/(\operatorname{gr}\theta(R))$$

induces a homomorphism

$$\alpha : T(V^+)/R_{\mathrm{gr}} \to \operatorname{gr} A(\Gamma).$$

Clearly

$$\alpha : \check{e}(\mathbf{b}) \mapsto \bar{e}(\mathbf{b}).$$

Also, since $\{\bar{e}(\mathbf{b}) \mid \mathbf{b} \in \mathbf{B}(\Gamma)\}$ is a basis for $\operatorname{gr} A(\Gamma)$, there is a linear map

$$\beta : \operatorname{gr} A(\Gamma) \to T(V^+)/R_{\mathrm{gr}}$$

defined by

$$\beta : \bar{e}(\mathbf{b}) \mapsto \check{e}(\mathbf{b}).$$

As $\alpha$ and $\beta$ are inverse mappings, the proof is complete. □

## 3. The quadratic algebra $A(\Gamma)$

We will now see that, for certain graphs $\Gamma$, $R$ is generated by $R_1 + R_2$.

**Definition 3.1.** Let $\Gamma$ be a layered graph and $v \in V_j$, $j \geqslant 2$. For $1 \leqslant i \leqslant j$ define $\mathcal{S}_i(v) = \{w \in V_{j-i} \mid v > w\}$.

**Definition 3.2.** For $v \in V_j$, $j \geqslant 2$, let $\sim_v$ denote the equivalence relation on $\mathcal{S}_1(v)$ generated by $u \sim_v w$ if $\mathcal{S}_1(u) \cap \mathcal{S}_1(w) \neq \emptyset$.

**Definition 3.3.** The layered graph $V$ is said to be *uniform* if, for every $v \in V_j$, $j \geqslant 2$, all elements of $\mathcal{S}_1(v)$ are equivalent under $\sim_v$.

**Lemma 3.4.** *Let $\Gamma$ be a uniform layered graph. Then $R$ is generated by $R_1 + R_2$, in fact, $R$ is generated by $R_1 \cup \{e(\tau_1, 2) - e(\tau_2, 2) \mid \mathbf{t}(\tau_1) = \mathbf{t}(\tau_2),\ \mathbf{h}(\tau_1) = \mathbf{h}(\tau_2),\ |\tau_1| = 2\}$.*

**Proof.** Let $S$ denote the ideal of $T(E)$ generated by $R_1 \cup \{e(\tau_1, 2) - e(\tau_2, 2) \mid \mathbf{t}(\tau_1) = \mathbf{t}(\tau_2),\ \mathbf{h}(\tau_1) = \mathbf{h}(\tau_2),\ |\tau_1| = 2\}$.

We must show that if $\pi_1, \pi_2$ are paths in $\Gamma$ with $\mathbf{t}(\pi_1) = \mathbf{t}(\pi_2)$, and $\mathbf{h}(\pi_1) = \mathbf{h}(\pi_2) = *$, then $P_{\pi_1}(t) - P_{\pi_2}(t) \in S[t]$, or, equivalently, $P_{\pi_1}(t) \in (1 + S[t])P_{\pi_2}(t)$. This is clear if $l(\pi_1) \leqslant 2$. We will proceed by induction on $l(\pi_1)$. Thus we will assume that $k \geqslant 3$, that $l(\pi_1) = k$, and that whenever $\mu_1, \mu_2$ are paths in $\Gamma$ with $\mathbf{t}(\mu_1) = \mathbf{t}(\mu_2)$, and $\mathbf{h}(\mu_1) = \mathbf{h}(\mu_2) = *$, and $l(\mu_1) < k$, then $P_{\mu_1}(t) - P_{\mu_2}(t) \in S[t]$.

Write $\pi_1 = (e_1, e_2, \ldots, e_k), \pi_2 = (f_1, f_2, \ldots, f_k)$. We first consider the special case in which $\mathbf{h}(e_1) > \mathbf{h}(f_2)$ (and so there is an edge, say $g$, with $\mathbf{t}(g) = \mathbf{h}(e_1)$, $\mathbf{h}(g) = \mathbf{h}(f_2)$). Consequently, $P_{(e_1,g)}(t) \in (1 + S[t])P_{(f_1,f_2)}(t)$. Write $\pi_1 = (e_1, e_2)\nu_1$ and $\pi_2 = (f_1, f_2)\nu_2$. Then

$$P_{\pi_1}(t) = (1 - te_1)(1 - te_2)P_{\nu_1}(t)$$

and

$$P_{\pi_2}(t) = (1 - tf_1)(1 - tf_2)P_{\nu_2}(t)$$

so

$$\begin{aligned} P_{\pi_2}(t) = {} & (1 - tf_1)(1 - tf_2)\big((1 - tg)^{-1}(1 - te_1)^{-1}(1 - te_1)(1 - tg)\big)P_{\nu_2}(t) \\ & \times P_{\nu_1}(t)^{-1}\big((1 - te_2)^{-1}(1 - te_1)^{-1}P_{\pi_1}(t)\big). \end{aligned}$$

Let $\mu_1 = e_2\nu_1$ and $\mu_2 = g\nu_2$. Then, by the induction assumption,

$$(1 - tg)P_{\nu_2}(t)P_{\nu_1}(t)^{-1}(1 - te_2)^{-1} = P_{\mu_2}(t)P_{\mu_1}(t)^{-1} \in 1 + S[t].$$

Consequently,

$$\begin{aligned} & (1 - te_1)(1 - tg)P_{\nu_2}(t)P_{\nu_1}(t)^{-1}(1 - te_2)^{-1}(1 - te_1)^{-1} \\ & \quad \in (1 - te_1)\big(1 + S[t]\big)(1 - te_1)^{-1} \\ & \quad = 1 + S[t] \end{aligned}$$

and so we have

$$P_{\pi_2}(t) \in \big(1 + S[t]\big)P_{\pi_1}(t).$$

In the general case, let $\mathbf{h}(e_1) = u$ and $\mathbf{h}(f_1) = w$. Then $u, w \in \mathcal{S}_1(v)$ so, since $\Gamma$ is uniform, there exist $a_1, \ldots, a_{l+1} \in \mathcal{S}_1(v)$ with $a_1 = u, a_{l+1} = w$ and $b_1, \ldots, b_l \in V$ with $b_i \in \mathcal{S}_1(a_i) \cap \mathcal{S}_1(a_{i+1})$ for $1 \leqslant i \leqslant l$. For $1 \leqslant i \leqslant l$, let $\tau_i$ be a path from $b_i$ to $*$. For $2 \leqslant i \leqslant l$ let $g_i \in E$ satisfy $\mathbf{t}(g_i) = \mathbf{t}(\pi_1), \mathbf{h}(g_i) = a_i$. For $1 \leqslant i \leqslant l$ let $r_i \in E$ satisfy $\mathbf{t}(r_i) = a_i$ and $\mathbf{h}(r_i) = b_i$ and let $s_i \in E$ satisfy $\mathbf{t}(s_i) = a_{i+1}, \mathbf{h}(s_i) = b_i$. Then the previously considered case shows that

$$P_{\pi_1}(t) \in \big(1 + S[t]\big) P_{g_2 s_1 \tau_1}(t);$$
$$P_{g_i s_{i-1} \tau_{i-1}}(t) \in \big(1 + S[t]\big) P_{g_{i+1} s_i \tau_i}(t)$$

for $2 \leqslant i \leqslant l-1$;

$$P_{g_l s_{l-1} \tau_{l-1}}(t) \in \big(1 + S[t]\big) P_{f_1 s_l \tau_l}(t);$$

and

$$P_{f_1 s_l \tau_l}(t) \in \big(1 + S[t]\big) P_{\pi_2}(t),$$

proving the lemma. □

Now assume that $\Gamma$ is a uniform layered graph. Then $R$ is generated by $R_1 + R_2$, in fact, $R$ is generated by $R_1 \cup \mathcal{R}_2$ where $\mathcal{R}_2 = \{e(\tau_1, 2) - e(\tau_2, 2) \mid \mathbf{t}(\tau_1) = \mathbf{t}(\tau_2), \mathbf{h}(\tau_1) = \mathbf{h}(\tau_2), |\tau_1| = 2\}$. Set $R_V = \langle \theta(\mathcal{R}_2) \rangle$.

**Proposition 3.5.** *Let $\Gamma$ be a uniform layered graph. Then $A(\Gamma) \cong T(V^+)/R_V$ is a quadratic algebra and $R_V$ is generated by*

$$\left\{ v(u-w) - u^2 + w^2 + (u-w)x \;\middle|\; v \in \bigcup_{i=2}^{n} V_i,\ u, w \in \mathcal{S}_1(v),\ x \in \mathcal{S}_1(u) \cap \mathcal{S}_1(w) \right\}.$$

**Proof.** By Lemma 3.4, $R_V$ is generated by

$$\theta\big\{e(\tau_1, 2) - e(\tau_2, 2) \mid \mathbf{t}(\tau_1) = \mathbf{t}(\tau_2),\ \mathbf{h}(\tau_1) = \mathbf{h}(\tau_2),\ |\tau_1| = 2\big\}.$$

Let $\tau_1 = (e, f)$, $\tau_2 = (e', f')$, $\mathbf{t}(e) = \mathbf{t}(e') = v$, $\mathbf{h}(e) = u$, $\mathbf{h}(e') = w$, $\mathbf{h}(f) = \mathbf{h}(f') = x$. Then

$$\begin{aligned} \theta\big(e(\tau_1, 2) - e(\tau_2, 2)\big) &= (v-u)(u-x) - (v-w)(w-x) \\ &= v(u-w) - u^2 + w^2 + (u-w)x. \qquad \square \end{aligned}$$

Combining this proposition with the results of the previous section we obtain the following presentation for $\operatorname{gr} A(\Gamma)$.

**Proposition 3.6.** *Let $\Gamma$ be a uniform layered graph. Then $\operatorname{gr} A(\Gamma) \cong T(V^+)/R_{\mathrm{gr}}$ is a quadratic algebra and $R_{\mathrm{gr}}$ is generated by*

$$\left\{ v(u-w) \;\middle|\; v \in \bigcup_{i=2}^{n} V_i,\ u, w \in \mathcal{S}_1(v),\ \mathcal{S}_1(u) \cap \mathcal{S}_1(w) \neq \emptyset \right\}.$$

## 4. gr $A(\Gamma)$ is a Koszul algebra

If $W$ is a graded subspace of $V^2$ we write

$$W^{(k)} = \bigcap_{i=0}^{k-2} V^i W V^{k-i-2}$$

so that

$$(\operatorname{gr} W)^{(k)} = \bigcap_{i=0}^{k-2} V^i (\operatorname{gr} W) V^{k-i-2}.$$

Then, by Proposition 3.6,

$$(\operatorname{gr} R_{\mathrm{gr}})^{(k)} \subset \operatorname{span}\{v(\pi,k) \mid \pi \text{ is a path}, \ l(\pi) \geqslant k\}.$$

To simplify notation, we will write $\mathbf{V}$ for $V^+$ and $\mathbf{R}$ for $R_{\mathrm{gr}}$. Note that if $\pi$ is a path with $l(\pi) \geqslant k$ and $v(\pi,k) = v_0 v_1 \cdots v_{k-1}$ then $|v_{k-1}| = |v_k| + 1 \geqslant 1$. Thus $v(\pi,k) \in \mathbf{V}^k$.

**Definition 4.1.**

$$\mathrm{Path}_k = \operatorname{span}\{v(\pi,k) \mid \pi \text{ is a path}, \ l(\pi) \geqslant k\}$$

and

$$\mathrm{Path}_k(v) = v\mathbf{V}^{k-1} \cap \mathrm{Path}_k.$$

Let $f : \mathbf{V} \to F$ be defined by $f(v) = 1$ for all $v \in \mathbf{V}$ and $I^l$ denote $I \otimes \cdots \otimes I$, taken $l$ times. Let $g_l : \mathbf{V}^l \to \mathbf{V}^{l-1}$ be defined by $g_l = f \otimes I^{l-1}$.

For any vertex $v \in \mathbf{V}$ and any $l \geqslant 0$ define

$$S_l(v) = \operatorname{span}\{u \mid v > u, \ |u| = |v| - l\}$$

and

$$P_l(v) = \operatorname{span}\{u - w \mid v > u, \ v > w, \ |u| = |w| = |v| - l\}.$$

Note that $P_l(v) = S_l(v) = (0)$ if $l > |v|$, that $S_0(v) = \operatorname{span}\{v\}$, and that $P_0(v) = (0)$.

Note also that $P_l(v) = \ker f|_{S_l(v)}$ and therefore

$$P_l(v)\mathbf{V}^m = \ker g_{m+1}|_{S_l(v)\mathbf{V}^m}$$

for all $l, m \geqslant 0$. Combining this with Proposition 3.6, we have

$$\operatorname{gr} \mathbf{R}_2 = \operatorname{span}\{v(u-w) \mid u, w \in \mathcal{S}_1(v), \ v \in \mathbf{V}\} = \sum_{v \in \mathbf{V}} v P_1(v).$$

**Lemma 4.2.** *For $k \geqslant 2$,*

$$\mathbf{R}^{(k)} = \mathrm{Path}_k \cap \bigcap_{i=0}^{k-2} \ker\big(I^{i+1} \otimes f \otimes I^{k-i-2}\big).$$

**Proof.** Since

$$\mathbf{R}^{(k)} = \bigcap_{i=0}^{k-2} \mathbf{V}^i \mathbf{R} \mathbf{V}^{k-i-2}$$

we have

$$\mathbf{R}^{(k)\perp} = \sum_{i=0}^{k-2} \mathbf{V}^{*i} \mathbf{R}^{\perp} \mathbf{V}^{*k-i-2}.$$

We also have that

$$\begin{aligned} \mathbf{R}^{\perp} &= \mathrm{span}\big\{\big\{v^* u^* \mid |u| \neq |v| - 1 \text{ or } v \not> u\big\} \cup \big\{v^* f \mid v \in \mathbf{V}\big\}\big\} \\ &= \mathrm{span}\big\{v^* u^* \mid |u| \neq |v| - 1 \text{ or } v \not> u\big\} + \mathbf{V}^* f. \end{aligned}$$

Let

$$M = \mathrm{span}\big\{v^* u^* \mid |u| \neq |v| - 1 \text{ or } v \not> u\big\}.$$

Then

$$\begin{aligned} \mathbf{R}^{(k)\perp} &= \sum_{i=0}^{k-2} \big\{\mathbf{V}^{*i} M \mathbf{V}^{*k-i-2} + \mathbf{V}^{*i+1} f \mathbf{V}^{*k-i-2}\big\} \\ &= \sum_{i=0}^{k-2} \mathbf{V}^{*i} M \mathbf{V}^{*k-i-2} + \sum_{i=0}^{k-2} \mathbf{V}^{*i+1} f \mathbf{V}^{*k-i-2} \\ &= \left(\left(\sum_{i=0}^{k-2} \mathbf{V}^{*i} M \mathbf{V}^{*k-i-2}\right)^{\perp} \cap \left(\sum_{i=0}^{k-2} \mathbf{V}^{*i+1} f \mathbf{V}^{*k-i-2}\right)^{\perp}\right)^{\perp}. \end{aligned}$$

So

$$\mathbf{R}^{(k)} = \left(\sum_{i=0}^{k-2} \mathbf{V}^{*i} M \mathbf{V}^{*k-i-2}\right)^{\perp} \cap \left(\sum_{i=0}^{k-2} \mathbf{V}^{*i+1} f \mathbf{V}^{*k-i-2}\right)^{\perp}.$$

Now

$$\left(\sum_{i=0}^{k-2} \mathbf{V}^{*i} M \mathbf{V}^{*k-i-2}\right)^{\perp} = \bigcap_{i=0}^{k-2} \mathbf{V}^{i} M^{\perp} \mathbf{V}^{k-i-2} = \mathrm{Path}_k$$

and

$$\begin{aligned}\left(\sum_{i=0}^{k-2} \mathbf{V}^{*i} f \mathbf{V}^{*k-i-2}\right)^{\perp} &= \bigcap_{i=0}^{k-2} \mathbf{V}^{i+1} \langle f\rangle^{\perp} \mathbf{V}^{k-i-2} = \bigcap_{i=0}^{k-2} \mathbf{V}^{i+1} (\ker f) \mathbf{V}^{k-i-2} \\ &= \bigcap_{i=0}^{k-2} \ker\bigl(I^{i+1} \otimes f \otimes I^{k-i-2}\bigr),\end{aligned}$$

giving the result. □

We will need the following result, whose proof is straightforward.

**Lemma 4.3.** *Let $W_1$ and $W_2$ be $F$-vector spaces, $h: W_1 \to W_2$ a linear transformation, $A \subseteq W_1$, $C \subseteq W_2$, subspaces, and $B = h^{-1}(C)$. Then*

$$h(A) \cap h(B) = h(A \cap B).$$

**Lemma 4.4.** *If $l \geqslant 0$, $j \geqslant 1$, and $v \in \bigcup_{j=2}^{n} V_j$, then*

$$P_j(v)\mathbf{V}^{l+1} \cap \mathbf{R}^{(l+2)} = g_{l+3}\bigl(S_{j-1}(v)\mathbf{V}^{l+2} \cap \mathbf{R}^{(l+3)}\bigr).$$

**Proof.** Note that

$$(f \otimes I)\bigl(S_{j-1}(v)\mathbf{V} \cap \mathbf{R}\bigr) \subseteq P_j(v)$$

and

$$\bigl(f \otimes I^{l+2}\bigr)\bigl(\mathbf{V}\mathbf{R}^{(l+2)}\bigr) \subseteq \mathbf{R}^{(l+2)}.$$

Consequently,

$$g_{l+3}\bigl(S_{j-1}(v)\mathbf{V}^{l+2} \cap \mathbf{R}^{(l+3)}\bigr) \subseteq P_j(v)\mathbf{V}^{l+1} \cap \mathbf{R}^{(l+2)}.$$

To prove the reversed inclusion we note that by Lemma 4.2,

$$P_j(v)\mathbf{V}^{l+1} \cap \mathbf{R}^{(l+2)} = P_j(v)\mathbf{V}^{l+1} \cap \mathrm{Path}_{l+2} \cap \left(\bigcap_{i=0}^{l} \ker\bigl(I^{i+1} \otimes f \otimes I^{l-i}\bigr)\right)$$

and

$$g_{l+3}\big(S_{j-1}(v)\mathbf{V}^{l+2}\cap \mathbf{R}^{(l+3)}\big)$$
$$= g_{l+3}\left(S_{j-1}(v)\mathbf{V}^{l+2}\cap \mathrm{Path}_{l+3}\cap\left(\bigcap_{i=0}^{l+1}\ker\big(I^{i+1}\otimes f\otimes I^{l+1-i}\big)\right)\right).$$

Let $g = g_{l+4}\cdots g_{l+j+2}$. Then for any subspace $W\subseteq \mathbf{V}^{l+2}$ we have

$$g^{-1}(W) = \mathbf{V}^{j-1}W$$

and

$$(g_{l+3}g)^{-1}(W) = \mathbf{V}^{j}W.$$

Then

$$P_j(v)\mathbf{V}^{l+1}\cap \mathrm{Path}_{l+2}\cap\left(\bigcap_{i=0}^{l}\ker\big(I^{i+1}\otimes f\otimes I^{l-i}\big)\right)$$
$$= (g_{l+3}g)\big(\mathrm{Path}_{j+1}(v)\mathbf{V}^{l+1}\big)\cap (g_{l+3}g)\big(\mathbf{V}^{j}\big(\ker\big(f\otimes I^{l+1}\big)\big)\big)$$
$$\cap\, (g_{l+3}g)\big(\mathbf{V}^{j}\,\mathrm{Path}_{l+2}\big)\cap\left(\bigcap_{i=0}^{l}\mathbf{V}^{j}\big(\ker\big(I^{i+1}\otimes f\otimes I^{l-i}\big)\big)\right)$$
$$= (g_{l+3}g)\left(\mathrm{Path}_{j+1}(v)\mathbf{V}^{l+1}\cap \mathbf{V}^{j}\,\mathrm{Path}_{l+2}\cap\left(\bigcap_{i=0}^{l+1}\mathbf{V}^{j}\big(\ker\big(I^{i}\otimes f\otimes I^{l+1-i}\big)\big)\right)\right)$$
$$= (g_{l+3}g)\left(\mathrm{Path}_{j+l+2}(v)\cap\left(\bigcap_{i=0}^{l+1}\mathbf{V}^{j}\big(\ker\big(I^{i}\otimes f\otimes I^{l+1-i}\big)\big)\right)\right).$$

Similarly,

$$g_{l+3}\left(S_{j-1}(v)\mathbf{V}^{l+2}\cap \mathrm{Path}_{l+3}\cap\left(\bigcap_{i=0}^{l+1}\ker\big(I^{i+1}\otimes f\otimes I^{l+1-i}\big)\right)\right)$$
$$= g_{l+3}\left(g\big(\mathrm{Path}_{j}(v)\mathbf{V}^{l+2}\big)\cap g\big(\mathbf{V}^{j-1}\,\mathrm{Path}_{l+3}\big)\right.$$
$$\left.\cap\left(\bigcap_{i=0}^{l+1}g\big(\mathbf{V}^{j-1}\big(\ker\big(I^{i+1}\otimes f\otimes I^{l+1-i}\big)\big)\big)\right)\right)$$
$$= (g_{l+3}g)\left(\mathrm{Path}_{j}(v)\mathbf{V}^{l+2}\cap \mathbf{V}^{j-1}\,\mathrm{Path}_{l+3}\cap\left(\bigcap_{i=0}^{l+1}\mathbf{V}^{j-1}\big(\ker\big(I^{i+1}\otimes f\otimes I^{l+1-i}\big)\big)\right)\right)$$

$$= (g_{l+3}g)\left(\mathrm{Path}_{j+l+2}(v) \cap \left(\bigcap_{i=0}^{l+1} \mathbf{V}^j\bigl(\ker\bigl(I^i \otimes f \otimes I^{l+1-i}\bigr)\bigr)\right)\right),$$

proving the lemma. □

**Lemma 4.5.** *Suppose* $\{P_1(v)\mathbf{V}^k\} \cup \{\mathbf{V}^i\mathbf{R}\mathbf{V}^{k-i-1} \mid 0 \leqslant i \leqslant k-1\}$ *is distributive for any* $v \in \mathbf{V}$. *Then* $\{\mathbf{V}^i\mathbf{R}\mathbf{V}^{k-i} \mid 0 \leqslant i \leqslant k\}$ *is distributive.*

**Proof.** By [SW, Lemma 1.1] it is sufficient to prove that

$$\bigl\{v\mathbf{V}^{k+1} \cap \mathbf{V}^i\mathbf{R}\mathbf{V}^{k-i} \mid 0 \leqslant i \leqslant k\bigr\}$$

is distributive for all $v \in \mathbf{V}$. Now $g_{k+2}$ restricts to an isomorphism of $v\mathbf{V}^{k+1}$ onto $\mathbf{V}^{k+1}$. Since $g_{k+2}(v\mathbf{V}^{k+1} \cap \mathbf{R}\mathbf{V}^k) = P_1(v)\mathbf{V}^k$ and $g_{k+2}(v\mathbf{V}^{k+1} \cap \mathbf{V}^{i+1}\mathbf{R}\mathbf{V}^{k-i-1}) = \mathbf{V}^i\mathbf{R}\mathbf{V}^{k-i-1}$ for $0 \leqslant i \leqslant k-1$, the result follows. □

**Theorem 4.6.** *Let $\Gamma$ be a uniform layered graph with a unique minimal element. Then* $\{\mathbf{V}^i\mathbf{R}\mathbf{V}^{k-1} \mid 0 \leqslant i \leqslant k\}$ *generates a distributive lattice in* $T(\mathbf{V})$. *Consequently,* $\operatorname{gr} A(\Gamma)$ *is a Koszul algebra.*

In view of Lemma 4.5, this will follow from:

**Lemma 4.7.** $\{P_l(v)\mathbf{V}^k\} \cup \{\mathbf{V}^i\mathbf{R}\mathbf{V}^{k-i-1} \mid 0 \leqslant i \leqslant k-1\}$ *is distributive for all* $k \geqslant 1$ *and all* $l > 0$.

**Proof.** The proof is by induction on $k$, the result being trivial for $k = 1$. We assume $\{P_l(v)\mathbf{V}^m\} \cup \{\mathbf{V}^i\mathbf{R}\mathbf{V}^{m-i-1} \mid 0 \leqslant i \leqslant m-1\}$ is distributive for all $m < k$ and all $l > 0$.

First note that any proper subset of $\{P_l(v)\mathbf{V}^k\} \cup \{\mathbf{V}^i\mathbf{R}\mathbf{V}^{k-i-1} \mid 0 \leqslant i \leqslant k-1\}$ is distributive. Indeed, by Lemma 4.5, $\{\mathbf{V}^i\mathbf{R}\mathbf{V}^{k-i-1} \mid 0 \leqslant i \leqslant k-1\}$ is distributive. Hence it is sufficient to show that $\{P_l(v)\mathbf{V}^k\} \cup \{\mathbf{V}^i\mathbf{R}\mathbf{V}^{k-i-1} \mid 0 \leqslant i < j\} \cup \{\mathbf{V}^i\mathbf{R}\mathbf{V}^{k-i-1} \mid j < i \leqslant k-1\}$ is distributive for all $j, 0 \leqslant j \leqslant k-1\}$. Now let $\mathcal{K}_{j,1} = \{P_l(v)\mathbf{V}^j\} \cup \{\mathbf{V}^i\mathbf{R}\mathbf{V}^{j-i-1} \mid 0 \leqslant i \leqslant j-1\}$ and $\mathcal{K}_{j,2} = \{\mathbf{V}^i\mathbf{R}\mathbf{V}^{k-j-2-i} \mid 0 \leqslant i \leqslant k-j-2\}$. Then $\mathcal{K}_{j,1}$ and $\mathcal{K}_{j,2}$ are distributive by the induction assumption. Since $\{P_l(v)\mathbf{V}^k\} \cup \{\mathbf{V}^i\mathbf{R}\mathbf{V}^{k-i-1} \mid 0 \leqslant i < j\} \cup \{\mathbf{V}^i\mathbf{R}\mathbf{V}^{k-i-1} \mid j < i \leqslant k-1\} = \mathcal{K}_{j,1}\mathbf{V}^{k-j} \cup \mathbf{V}^{j+1}\mathcal{K}_{j,2}$, the assertion follows.

In view of [SW, Theorem 1.2], it is therefore sufficient to prove that

$$\begin{aligned}
&\bigl(P_l(v)\mathbf{V}^k \cap \mathbf{R}\mathbf{V}^{k-1} \cap \cdots \cap \mathbf{V}^i\mathbf{R}\mathbf{V}^{k-i-1}\bigr) \cap \bigl(\mathbf{V}^{i+1}\mathbf{R}\mathbf{V}^{k-i-2} + \cdots + \mathbf{V}^{k-1}\mathbf{R}\bigr) \\
&\quad = \bigl(P_l(v)\mathbf{V}^k \cap \mathbf{R}\mathbf{V}^{k-1} \cap \cdots \cap \mathbf{V}^{i+1}\mathbf{R}\mathbf{V}^{k-i-2}\bigr) \\
&\qquad + \bigl(P_l(v)\mathbf{V}^k \cap \mathbf{R}\mathbf{V}^{k-1} \cap \cdots \cap \mathbf{V}^i\mathbf{R}\mathbf{V}^{k-i-1} \cap \bigl(\mathbf{V}^{i+2}\mathbf{R}\mathbf{V}^{k-i-3} + \cdots + \mathbf{V}^{k-1}\mathbf{R}\bigr)\bigr).
\end{aligned}$$

Now write

$$\begin{aligned}
X_i &= S_l(v)\mathbf{V}^k \cap \mathbf{R}\mathbf{V}^{k-1} \cap \cdots \cap \mathbf{V}^i\mathbf{R}\mathbf{V}^{k-i-1}, \\
Y_i &= \mathbf{V}^{i+1}\mathbf{R}\mathbf{V}^{k-i-2},
\end{aligned}$$

and

$$Z_i = \mathbf{V}^{i+2}\mathbf{R}\mathbf{V}^{k-i-3} + \cdots + \mathbf{V}^{k-1}\mathbf{R}.$$

Then we need to show that

$$\ker g_{k+1}|_{X_i \cap (Y_i + Z_i)} = \ker g_{k+1}|_{X_i \cap Y_i} + \ker g_{k+1}|_{X_i \cap Z_i}.$$

The right-hand side is contained in the left-hand side, so it is enough to prove equality of dimensions.

Hence it is enough to prove

$$\begin{aligned} &\dim X_i \cap (Y_i + Z_i) - \dim g_{k+1}\big(X_i \cap (Y_i + Z_i)\big) \\ &\quad = \dim X_i \cap Y_i - \dim g_{k+1}(X_i \cap Y_i) + \dim X_i \cap Z_i - \dim g_{k+1}(X_i \cap Z_i) \\ &\qquad - \dim X_i \cap Y_i \cap Z_i + \dim g_{k+1}(X_i \cap Y_i \cap Z_i). \end{aligned}$$

Now, by the induction assumption and Lemma 4.5, $\{X_i, Y_i, Z_i\}$ is distributive. Therefore, the desired equality is equivalent to

$$\begin{aligned} \dim g_{k+1}\big(X_i \cap (Y_i + Z_i)\big) &= \dim g_{k+1}(X_i \cap Y_i) + \dim g_{k+1}(X_i \cap Z_i) \\ &\quad - \dim g_{k+1}(X_i \cap Y_i \cap Z_i). \end{aligned}$$

But, since $\{X_i, Y_i, Z_i\}$ is distributive,

$$g_{k+1}\big(X_i \cap (Y_i + Z_i)\big) = g_{k+1}(X_i \cap Y_i + X_i \cap Z_i) = g_{k+1}(X_i \cap Y_i) + g_{k+1}(X_i \cap Z_i).$$

Hence we need only show that

$$\dim g_{k+1}(X_i \cap Y_i) \cap g_{k+1}(X_i \cap Z_i) = \dim g_{k+1}(X_i \cap Y_i \cap Z_i).$$

Since

$$g_{k+1}(X_i \cap Y_i \cap Z_i) \subseteq g_{k+1}(X_i \cap Y_i) \cap g_{k+1}(X_i \cap Z_i)$$

this is equivalent to

$$g_{k+1}(X_i \cap Y_i) \cap g_{k+1}(X_i \cap Z_i) = g_{k+1}(X_i \cap Y_i \cap Z_i)$$

Now by Lemma 4.4, the left-hand side of this expression is equal to

$$P_{l+1}(v)\mathbf{V}^{k-1} \cap \mathbf{R}\mathbf{V}^{k-2} \cap \cdots \cap \mathbf{V}^{i}\mathbf{R}\mathbf{V}^{k-i-2} \cap \big(\mathbf{V}^{i+1}\mathbf{R}\mathbf{V}^{k-i-3} + \cdots + \mathbf{V}^{k-2}\mathbf{R}\big).$$

In view of the distributivity of $\{S_l(v)\mathbf{V}^k\} \cup \{\mathbf{V}^t\mathbf{R}\mathbf{V}^{k-t-1} \mid 0 \leqslant t \leqslant k-1\}$, which follows from [SW, Lemma 1.1] and the induction assumption, the right-hand side of the expression may be written as

$$g_{k+1}\big(S_l(v)\mathbf{V}^k \cap \mathbf{R}\mathbf{V}^{k-1} \cap \cdots \cap \mathbf{V}^{i+2}\mathbf{R}\mathbf{V}^{k-i-3}\big)$$
$$+ g_{k+1}\big(S_l(v)\mathbf{V}^k \cap \mathbf{R}\mathbf{V}^{k-1} \cap \cdots \cap \mathbf{V}^{i+1}\mathbf{R}\mathbf{V}^{k-i-2} \cap \big(\mathbf{V}^{i+3}\mathbf{R}\mathbf{V}^{k-i-4} + \cdots + \mathbf{V}^{k-1}\mathbf{R}\big)\big).$$

By Lemma 4.4, this is equal to

$$P_{l+1}(v)\mathbf{V}^{k-1} \cap \mathbf{R}\mathbf{V}^{k-2} \cap \cdots \cap \mathbf{V}^{i+1}\mathbf{R}\mathbf{V}^{k-i-3}$$
$$+ P_{l+1}(v)\mathbf{V}^{k-1} \cap \mathbf{R}\mathbf{V}^{k-2} \cap \cdots \cap \mathbf{V}^{i}\mathbf{R}\mathbf{V}^{k-i-2} \cap \big(\mathbf{V}^{i+2}\mathbf{R}\mathbf{V}^{k-i-4} + \cdots + \mathbf{V}^{k-2}\mathbf{R}\big).$$

By the induction assumption, these expressions for the left- and right-hand sides are equal, so the proof is complete. □

## 5. Koszulity of $A(\Gamma)$

We will need the following lemma which is a special case of a more general result [PP, Proposition 3.7.1].

**Lemma 5.1.** *Let $A$ be a filtered quadratic algebra. If* gr $A$ *is quadratic and Koszul then $A$ is Koszul.*

**Theorem 5.2.** *Let $\Gamma$ be a uniform layered graph with a unique minimal element. Then $A(\Gamma)$ is a Koszul algebra.*

**Proof.** This follows from Theorem 4.6 and Lemma 5.1. □

## References


[GGR] I. Gelfand, S. Gelfand, V. Retakh, Noncommutative algebras associated to complexes and graphs, Selecta Math. (N.S.) 7 (2001) 525–531.

[GGRSW] I. Gelfand, S. Gelfand, V. Retakh, S. Serconek, R.L. Wilson, Hilbert series of quadratic algebras associated with decompositions of noncommutative polynomials, J. Algebra 254 (2002) 279–299.

[GRSW] I. Gelfand, V. Retakh, S. Serconek, R.L. Wilson, On a class of algebras associated to directed graphs, Selecta Math. (N.S.) 11 (2005), math.QA/0506507.

[GRW] I. Gelfand, V. Retakh, R.L. Wilson, Quadratic-linear algebras associated with decompositions of noncommutative polynomials and differential polynomials, Selecta Math. (N.S.) 7 (2001) 493–523.

[Pi] D. Piontkovski, Algebras associated to pseudo-roots of noncommutative polynomials are Koszul, Internat. J. Algebra Comput. 15 (2005) 643–648.

[PP] A. Polishchuk, L. Positselski, Quadratic Algebras, Amer. Math. Soc., Providence, RI, 2005.

[SW] S. Serconek, R.L. Wilson, Quadratic algebras associated with decompositions of noncommutative polynomials are Koszul algebras, J. Algebra 278 (2004) 473–493.