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Nonexistence of Eventually Positive Solutions of a Difference Inequality with Multiple and Variable Delays and Coefficients

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Abstract—In this paper, we consider the nonexistence of eventually positive solutions of the difference inequality

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i(n) x_{n-k_i(n)} \le 0.$$

Let m be a positive integer. Then for each positive integer i: $1 \le i \le m$, $\{k_i(n)\}_{n=0}^{\infty}$ and $\{p_i(n)\}_{n=0}^{\infty}$ are a sequence of positive integers and a sequence of nonnegative real numbers, respectively. A sufficient condition guaranteeing the nonexistence of eventually positive solutions is obtained with the help of a new method. As an application of the main result, a conjecture is proved.

Keywords—Difference equation, Inequality, Positive solution, Oscillation.

1. INTRODUCTION

Consider the difference inequality

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i(n) x_{n-k_i(n)} \le 0, \tag{1.1}$$

and the difference equation corresponding to (1.1)

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i(n) x_{n-k_i(n)} = 0, \tag{1.2}$$

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where m is a positive integer, for each $i : 1 \le i \le m$, $\{k_i(n)\}_{n=0}^{\infty}$ and $\{p_i(n)\}_{n=0}^{\infty}$ are sequences of positive integers and nonnegative real numbers, respectively. By a solution of (1.1) (respectively, (1.2)), we mean a sequence $\{x_n\}_{n=-q}^{+\infty}$, where the positive integer q is sufficiently large so that $\{x_n\}_{n=-q}^{+\infty}$ satisfies (1.1) (respectively, (1.2)) for $n \ge 0$. For the existence and general theory of solutions of inequality (1.1) and equation (1.2), we refer to [1,2].

A solution $\{x_n\}$ of equation (1.2) is called oscillatory if for any L (positive integer) there exist $n(L), \overline{n}(L) \ge L$ such that $x_n \cdot x_{\overline{n}} \le 0$. Otherwise, it is nonoscillatory. Equation (1.2) is said to be oscillatory if every solution of (1.2) is oscillatory. A solution $\{x_n\}$ of (1.1) is called an eventually positive solution (EPS, for short) if there is a positive integer N such that $n \ge N$ implies $x_n > 0$. Note that if $\{x_n\}$ is a solution of equation (1.2), then so is $\{-x_n\}$. From this, it is clear that the nonexistence of EPS of (1.1) implies that every solution of equation (1.2) is oscillatory.

Let $m = 1$ and set $k_1(n) = k_n$, $p_1(n) = p_n$. Then equation (1.2) becomes

$$x_{n+1} - x_n + p_n x_{n-k_n} = 0. \tag{1.3}$$

The oscillation of equation (1.3) has been studied in [3,4].

In [3], Philos proved the following results: if $p_n \ge 0$ and $\lim_{n\to\infty}(n - k_n) = \infty$, then

$$\liminf_{n\to\infty} \sum_{i=n-k_n}^{n-1} p_i > \limsup_{n\to\infty} \left(\frac{k_n}{k_n+1}\right)^{k_n+1} \tag{1.4}$$

implies equation (1.3) is oscillatory.

In [4], Yu proved that if

(i) $p_n \ge 0$;

(ii) $\{n - k_n\}_{n=0}^{\infty}$ is a monotone nondecreasing sequence and $\lim_{n\to\infty}(n - k_n) = \infty$;

(iii)

$$\liminf_{n\to\infty} \left(\frac{k_n+1}{k_n}\right)^{k_n+1} \sum_{i=n-k_n}^{n-1} p_i > 1, \tag{1.5}$$

then (1.3) is oscillatory.

Based on the above result, there arises a natural conjecture for (1.1) and (1.2).

CONJECTURE A. *If*

(i) $p_i(n) \ge 0$;

(ii) $\lim_{n\to\infty}(n - k_i(n)) = \infty$, *for each* $i : 1 \le i \le m$;

(iii)

$$\liminf_{n\to\infty} \sum_{i=1}^{m} p_i(n) \frac{(k_i(n)+1)^{k_i(n)+1}}{k_i(n)^{k_i(n)}} > 1, \tag{1.6}$$

then (1.1) has no EPS, *thus equation (1.2) is oscillatory.*

The purpose of this paper is to prove Conjecture A. Indeed, we will first establish a weaker sufficient condition for the nonexistence of EPS of (1.1) which is analogous to (1.6), that is, Theorem 1 in Section 2. Then, employing this weaker condition, we prove Conjecture A.

2. MAIN RESULT AND PROOF

THEOREM 1. *If*

(i)

$$p_i(n) \ge 0, \qquad n = 0, 1, 2, \dots; \tag{2.1}$$

(ii)

$$\lim_{n\to\infty} (n - k_i(n)) = \infty, \qquad i : 1 \le i \le m; \tag{2.2}$$

(iii)

$$\liminf_{n\to\infty}\left\{\inf_{0<\lambda<1}\sum_{i=1}^{m}p_i(n)\left[(1-\lambda)\lambda^{k_i(n)}\right]^{-1}\right\}>1, \tag{2.3}$$

then (1.1) has no EPS *and equation (1.2) is oscillatory.*

PROOF. Set $k_n = \max_{1\le i\le m} k_i(n)$ for $n = 0, 1, 2, \dots$. We know from (2.2) that

$$\lim_{n\to\infty}(n-k_n)=\infty. \tag{2.4}$$

We note that (2.3) implies that there exist $C_0 > 1$ and $\overline{n}$ such that for $n \ge \overline{n}$ and $\lambda \in (0,1)$

$$\sum_{i=1}^{m}p_i(n)\left[(1-\lambda)\lambda^{k_i(n)}\right]^{-1}\ge C_0. \tag{2.5}$$

Assume, for the sake of contradiction, that (1.1) has an EPS, say $\{x_n\}$. Then there exists $\overline{n_0} \ge \overline{n}$ so that $x_n > 0$ for $n \ge \overline{n}_0$.

So for $n \ge \overline{n}_0$, we can rewrite (1.1) as

$$\frac{x_{n+1}}{x_n}-1+\sum_{i=1}^{m}p_i(n)\frac{x_{n-k_i(n)}}{x_n}\le 0. \tag{2.6}$$

Furthermore, we may assume by (2.4) that there exists $\overline{n}_1 > \overline{n}_0$ so that $n - k_n \ge \overline{n}_0$ for $n \ge \overline{n}_1$, that is, for each $i : 1 \le i \le m$ and $n \ge \overline{n}_1$, we have $x_{n-k_i(n)} > 0$. Combining this result with (1.1), one obtains $x_{n+1} - x_n \le 0$, i.e., $(x_{n+1})/x_n \le 1$ for $n \ge \overline{n}_1$. In a similar way, from (2.4) we can find $\overline{n}_2 > \overline{n}_1$ so that $n - k_n \ge \overline{n}_1$ for $n \ge \overline{n}_2$. Thus, for all $n \ge \overline{n}_2$ and each $i : 1 \le i \le m$,

$$\frac{x_{n-k_i(n)}}{x_n}=\prod_{j=1}^{k_i(n)}\frac{x_{n-j}}{x_{n-j+1}}\ge 1.$$

This result and (2.6) lead to

$$\frac{x_{n+1}}{x_n}-1+\sum_{i=1}^{m}p_i(n)\le 0. \tag{2.7}$$

We may assume from (2.3) that for $n \ge \overline{n}_2, \sum_{i=1}^{m} p_i(n) > 0$. Combining this and (2.7), we get $(x_{n+1})/x_n < 1$ for $n \ge \overline{n}_2$. In a similar fashion, we find $\overline{n}_3 > \overline{n}_2$ so that $n - k_n \ge \overline{n}_2$ for $n \ge \overline{n}_3$. So we have

$$\frac{x_{n-j+1}}{x_{n-j}}<1,\quad \text{for } n\ge\overline{n}_j,\qquad j:0\le j\le k_n. \tag{2.8}$$

Dividing (2.6) by $(1 - (x_{n+1})/x_n)$ yields

$$\sum_{i=1}^{m}p_i(n)\left[\left(1-\frac{x_{n+1}}{x_n}\right)\frac{x_n}{x_{n-k_i(n)}}\right]^{-1}\le 1. \tag{2.9}$$

For each $n \ge \overline{n}_3$, we define $a(n) : 1 \le a(n)$ such that

$$\frac{x_{n-a(n)+1}}{x_{n-a(n)}}=\max_{1\le j\le k_n}\frac{x_{n-j+1}}{x_{n-j}}. \tag{2.10}$$

By (2.8), we obtain

$$\frac{x_{n-a(n)+1}}{x_{n-a(n)}}<1,\qquad \text{for } n\ge\overline{n}_3. \tag{2.11}$$

At the same time, for $n \geq \overline{n}_3$ and each $i : \leq i \leq m$, we have

$$\frac{x_n}{x_{n-k_i(n)}} = \prod_{l=1}^{k_i(n)} \frac{x_{n-l+1}}{x_{n-l}} \leq \left(\frac{x_{n-a(n)+1}}{x_{n-a(n)}}\right)^{k_i(n)}. \tag{2.12}$$

From (2.9), (2.11), and (2.12), it is easily obtained that

$$\sum_{i=1}^{m} p_i(n) \left[\left(1 - \frac{x_{n-a(n)+1}}{x_{n-a(n)}}\right)\left(\frac{x_{n-a(n)+1}}{x_{n-a(n)}}\right)^{k_k(n)}\right]^{-1} \left[\frac{1-(x_{n-a(n)+1}/x_{n-a(n)})}{1-(x_{n+1})/x_n}\right] \leq 1. \tag{2.13}$$

Now, combining (2.11) with (2.13) and (2.5), one obtains

$$C_0 \left[\frac{1-(x_{n-a(n)+1}/x_{n-a(n)})}{1-(x_{n+1})/x_n}\right] \leq 1.$$

From this, for $n \geq \overline{n}_3$, we have

$$\frac{x_{n+1}}{x_n} < \frac{x_{n-a(n)+1}}{x_{n-a(n)}}. \tag{2.14}$$

To complete the proof of Theorem 1, we need the following lemma.

LEMMA 1. *If (2.1)–(2.3) hold, then*

$$\limsup_{n\to\infty} \frac{x_{n-a(n)+1}}{x_{n-a(n)}} = \limsup_{n\to\infty} \frac{x_{n+1}}{x_n} = \alpha < 1. \tag{2.15}$$

PROOF OF LEMMA 1. Let $\alpha = \limsup_{n\to\infty}(x_{n+1}/x_n)$. Then we know from (2.4) and (2.14) that $\limsup_{n\to\infty}(x_{n-a(n)+1}/x_{n-a(n)}) = \alpha$. It is sufficient to prove $\alpha < 1$. To do so, we let

$$u_n = \max_{\overline{n}_2 \leq m \leq n-1} \frac{x_{m+1}}{x_m}. \tag{2.16}$$

It is easy to see that $u_{n+1} \geq u_n$ and $u_n < 1$ for $n \geq \overline{n}_3$. On the other hand, (2.16) gives

$$\begin{aligned} u_{n+1} &= \max_{\overline{n}_2 \leq m \leq n} \frac{x_{m+1}}{x_m} \\ &= \max\left\{\frac{x_{n+1}}{x_n}, \max_{\overline{n}_2 \leq m \leq n-1} \frac{x_{m+1}}{x_m}\right\} \\ &= \max\left\{\frac{x_{n+1}}{x_n}, u_n\right\}. \end{aligned}$$

But for $n \geq \overline{n}_3$, we have $\overline{n}_2 \leq n - k_n \leq n - a(n) \leq n - 1$. We then derive, from (2.10) and (2.14), that

$$\frac{x_{n+1}}{x_n} < \max_{\overline{n}_2 \leq m \leq n-1} \frac{x_{m+1}}{x_m} = u_n.$$

Thus, $u_{n+1} = u_n$ for $n \geq \overline{n}_3$, that is, for all $n \geq \overline{n}_3, u_n = u_{\overline{n}_3}$. Moreover, we actually obtain that, for all $n \geq \overline{n}_3$,

$$\frac{x_{n+1}}{x_n} \leq u_{n+1} = u_{\overline{n}_3} < 1.$$

So we have $\alpha = \limsup_{n\to\infty}(x_{n+1}/x_n) \leq u_{\overline{n}_3} < 1$. This complete the proof of Lemma 1.

Let us return to proof of Theorem 1. Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ so that

$$\lim_{n_j\to\infty} \frac{x_{n_j+1}}{x_{n_j}} = \alpha. \tag{2.17}$$

Then we obtain from (2.15) and (2.17) that

$$\begin{aligned}\liminf_{n_j\to\infty}\frac{\left(1-x_{n_j-a_{(n_j)}+1}\right)/x_{n_j-a_{(n_j)}}}{1-x_{n_j+1}/x_{n_j}} &= \liminf_{n_j\to\infty}\frac{\left(1-x_{n_j-a_{(n_j)}+1}\right)/x_{n_j-a_{(n_j)}}}{\lim\limits_{n_j\to\infty}\left(1-x_{n_j+1}/x_{n_j}\right)}\\ &= \frac{1-\limsup\limits_{n_j\to\infty} x_{n_j-a_{(n_j)}+1}/x_{n_j-a_{(n_j)}}}{1-\alpha}\\ &\geq \frac{1-\alpha}{1-\alpha}\\ &= 1.\end{aligned}$$

In summary, we have established the following:

$$\liminf_{n_j\to\infty}\frac{1-\left(x_{n_j-a(n_j)+1}/x_{n_j-a(n_j)}\right)}{1-\left(x_{n_j+1}/x_{n_j}\right)}\geq 1. \tag{2.18}$$

Putting $\{x_{n_j}\}$ into (2.13) and employing (2.18), we have

$$\begin{aligned}1 &\geq \liminf_{n_j\to\infty}\left\{\sum_{i=1}^{m}p_i(n_j)\left[\left(1-\frac{x_{n_j-a(n_j)+1}}{x_{n_j-a(n_j)}}\right)\left(\frac{x_{n_j-a(n_j)+1}}{x_{n_j-a(n_j)}}\right)^{k_i(n)}\right]^{-1}\right.\\ &\qquad\left.\left[\frac{1-\left(x_{n_j-a(n_j)+1}/x_{n_j-a(n_j)}\right)}{1-\left(x_{n_j+1}/x_{n_j}\right)}\right]\right\}\\ &\geq \liminf_{n_j\to\infty}\left\{\sum_{i=1}^{m}p_i(n_j)\left[\left(1-\frac{x_{n_j-a(n_j)+1}}{x_{n_j-a(n_j)}}\right)\left(\frac{x_{n_j-a(n_j)+1}}{x_{n_j-a(n_j)}}\right)^{k_j(n)}\right]^{-1}\right\}\\ &\qquad\liminf_{n_j\to\infty}\left\{\frac{1-\left(x_{n_j-a(n_j)+1}/x_{n_j-a(n_j)}\right)}{1-\left(x_{n_j+1}\right)/x_{n_j}}\right\}\\ &\geq \liminf_{n_j\to\infty}\left\{\inf_{0<\lambda<1}\sum_{i=1}^{m}p_i(n_j)\left[(1-\lambda)\lambda^{k_i(n_j)}\right]^{-1}\right\}.\end{aligned}$$

Putting these inequalities together, we get

$$\liminf_{n_j\to\infty}\left\{\inf_{0<\lambda<1}\sum_{i=1}^{m}p_i(n)\left[(1-\lambda)\lambda^{k_i(n)}\right]\right\}\leq 1.$$

Then, using (2.3), we obtain a contradiction. The proof of Theorem 1 is completed.

We are now in the position to prove Conjecture A by virtue of Theorem 1.

THEOREM 2. *If*

(i) $p_i(n)\geq 0,\ i:1\leq i\leq m,\ n=0,1,2,\dots;$

(ii) $\lim_{n\to\infty}(n-k_i(n))=\infty,$ *for each* $i:1\leq i\leq m;$

(iii)

$$\liminf_{n\to\infty}\sum_{i=1}^{m}p_i(n)\frac{(k_i(n)+1)^{k_i(n)+1}}{k_i(n)^{k_i(n)}}>1, \tag{2.19}$$

then (1.1) has no EPS, that is, Conjecture A is true.

PROOF. The proof is merely a verification for

$$\min_{0<\lambda<1}\sum_{i=1}^{m}p_i(n)\left[(1-\lambda)\lambda^{k_i(n)}\right]^{-1}\geq\sum_{i=1}^{m}p_i(n)\frac{(k_i(n)+1)^{k_i(n)+1}}{k_i(n)^{k_i(n)}},$$

which is easily obtained by noting that

$$\min_{0<\lambda<1}\left[(1-\lambda)\lambda^{k_i(n)}\right]^{-1}=\frac{(k_i(n)+1)^{k_i(n)+1}}{k_i(n)^{k_i(n)}}.$$

This completes the proof of Theorem 2.

3. A COMPARISON BETWEEN THEOREM 1 AND CONJECTURE A

From the proof of Theorem 2, we see that Condition (iii) in Theorem 1 is no stronger than Condition (iii) in Conjecture A. In this section, we give an example to show that Condition (iii) in Theorem 1 is indeed weaker than Condition (iii) in Conjecture A.

Consider the nonexistence of EPS of the difference inequality

$$x_{n+1} - x_n + p_n x_{n-j_n} + q_n x_{n-k_n} \leq 0. \tag{3.1}$$

We have the following theorem.

THEOREM 3. *If* $p_n \geq 0$, $q_n \geq 0$, $n = 0, 1, 2, \ldots$, *and*

(i) *there exists a positive integer* k *so that*

$$1 \leq j_n \leq k, \qquad \text{for } n = 0, 1, 2, \ldots; \tag{3.2}$$

(ii)

$$\lim_{n\to\infty} k_n = \infty \quad \text{and} \quad \lim_{n\to\infty} (n - k_n) = \infty; \tag{3.3}$$

(iii)

$$\liminf_{n\to\infty} p_n \cdot \liminf_{n\to\infty} q_n \cdot \frac{(k_n+1)^{k_n+1}}{k_n^{k_n}} \neq 0, \tag{3.4}$$

then (3.1) has no EPS.

PROOF. We first show that $\limsup_{n\to\infty} p_n > 1$ implies that (3.1) has no EPS. Indeed, if this were false, let $\beta = \limsup_{n\to\infty} p_n > 1$ and $\{x_n\}$ be an EPS of (3.1). Then

$$x_{n+1} - x_n + p_n x_{n-j_n} \leq 0. \tag{3.5}$$

Let $\{p_{n_l}\}$ be a subsequence of $\{p_n\}$ so that $\lim_{n_l\to\infty} p_{n_l} = \beta$. Choosing $\beta_0 : 1 < \beta_0 < \beta$, then there exists N such that $n \geq N$ implies $p_{n_l} > \beta_0$. From this and (3.5), we have

$$x_{n_l+1} - x_{n_l} + \beta_0 x_{n_l - j_{n_l}} \leq 0. \tag{3.6}$$

But on the other hand, we know from (2.8) that for sufficient large n_l, $x_{n_l} < x_{n_l - j_{n_l}}$, i.e. $-x_{n_l} + \beta_0 x_{n_l - j_{n_l}} > 0$. So $x_{n_l+1} < 0$ for sufficient large n_l, which is a contradiction. Thus, we assume without loss of generality,

$$u = \limsup_{n\to\infty} p_n \leq 1, \tag{3.7}$$

$$v = \liminf_{n\to\infty} p_n > 0, \tag{3.8}$$

$$w = \liminf_{n\to\infty} q_n \frac{(k_n+1)^{k_n+1}}{k_n^{k_n}} > 0. \tag{3.9}$$

Using the fact that $\lim_{n\to\infty}((n+1)/n)^n = e$, we obtain from (3.3) and (3.9) that $\liminf_{n\to\infty} q_n(k_n+1) = w/e$. Combining the previous estimates, we can find N_0 such that for $n \geq N_0$

$$v_0 \leq p_n \leq u_0, \tag{3.10}$$

$$q_n(k_n+1) \geq \frac{w_0}{e}, \qquad k_n > k, \tag{3.11}$$

where v_0 is such that $(1/2)v < v_0 < v$ and u_0 is such that $u < u_0 < 2$ and w_0 satisfies $(1/2)w < w_0 < w$. Let $f_n : (0,1) \to R$ be defined as follows:

$$f_n(\lambda) = p_n \left[(1-\lambda)\lambda^{j_n}\right]^{-1} + q_n \left[(1-\lambda)\lambda^{k_n}\right]^{-1}, \qquad 0 < \lambda < 1. \tag{3.12}$$

Noting that $\lim_{\lambda\to 0^+} f_n(\lambda) = \lim_{\lambda\to 1} f_n(\lambda) = \infty$, we may assume that exists $r_n : 0 < r_n < 1$ such that

$$f_n(r_n) = \inf_{0<\lambda<1} f_n(\lambda), \tag{3.13}$$

$$f_n'(r_n) = 0. \tag{3.14}$$

LEMMA 2. *The r_n defined by (3.13) and (3.14) satisfies*

$$\lim_{n\to\infty} r_n = 1. \tag{3.15}$$

PROOF OF LEMMA 2. By (3.12), it is easy to verify that $f_n'(r_n) = 0$ is equivalent to

$$(1+j_n)\left(r_n - \frac{j_n}{j_n+1}\right)p_n = [q_n(1+k_n)]\left(\frac{k_n}{k_n+1} - r_n\right)r_n^{-(k_n-j_n)},$$

that is,

$$\left(\frac{k_n}{k_n+1} - r_n\right) = (1+j_n)\left(r_n - \frac{j_n}{j_n+1}\right)p_n[q_n(1+k_n)]^{-1}r_n^{(k_n-j_n)}. \tag{3.16}$$

So we get

$$\frac{j_n}{j_n+1} < r_n < \frac{k_n}{k_n+1}, \qquad \text{for } n \geq N_0.$$

From (3.10), we have

$$(1+j_n)\left(r_n - \frac{j_n}{j_n+1}\right)p_n < (1+j_n)\left(1 - \frac{j_n}{j_n+1}\right)p_n = p_n \leq u_0.$$

This together with (3.11),(3.16) gives

$$\frac{k_n}{k_n+1} - r_n < \frac{u_0 e}{w_0} r_n^{k_n-k}. \tag{3.17}$$

If there exists a subsequence of $\{r_n\}$, say $\{r_{n_l}\}$ so that

$$\lim_{n_l\to\infty} r_{n_l} = s < 1,$$

we choose $s_0 : s < s_0 < 1$. Then there exists $N_1 \geq N_0$ such that for $n_l \geq N_1$, $r_{n_l} < s_0$.

On the other hand, from $\lim_{n_l\to\infty}(k_{n_l}/(k_{n_l}+1) - r_{n_l}) = 1 - s$, we can find $N_2 \geq N_1$ such that for $n_l \geq N_2$,

$$\frac{k_{n_l}}{k_{n_l}+1} - r_{n_l} > 1 - s_0. \tag{3.18}$$

By (3.3), we assume $k_{n_l} > k$ for $n_l \geq N_2$. Combining (3.17) with (3.18), it is easy to deduce that

$$1 - s_0 < \frac{u_0 l}{w_0} s_0^{k_{n_l}-k}, \qquad \text{for } n_l \geq N_2. \tag{3.19}$$

Let $n_l \to \infty$. Then $k_{n_l} - k \to \infty$, and (3.19) gives: $1 - s_0 \leq 0$, that is, $s_0 \geq 1$. As this contradicts the fact that $s_0 < 1$, the proof of Lemma 2 is complete.

Let us return to the proof of Theorem 3. Now from (3.2), (3.8), and (3.12), it is easy to obtain

$$\begin{aligned}
\liminf_{n\to\infty} f_n(r_n) &= \liminf_{n\to\infty}\left\{p_n\left[(1-r_n)r_n^{j_n}\right]^{-1} + q_n\left[(1-r_n)r_n^{k_n}\right]^{-1}\right\} \\
&\geq \liminf_{n\to\infty} p_n \cdot \liminf_{n\to\infty}\left[(1-r_n)r_n\right]^{-1} \\
&= v \cdot \liminf_{n\to\infty}\ (1-r_n)^{-1} \\
&= v \cdot \lim_{n\to\infty}(1-r_n)^{-1} \\
&= \infty,
\end{aligned}$$

that is, $\liminf_{n\to\infty}\{\inf_{0<\lambda<1} f_n(\lambda)\} = \infty$.

Now, Theorem 1 implies the assertion of Theorem 3. The proof of Theorem 3 is completed.

Finally, if we specify $p_n = 1/8$, $j_n = 1$, $k_n = [\sqrt{n}]$, and $q_n = 1/(4([\sqrt{n}]+1)e)$, where $[\cdot]$ is the greatest integer function, then

$$\begin{aligned}
p_n\frac{(j_n+1)^{j_n+1}}{j_n^{j_n}} + q_n\frac{(k_n+1)^{k_n+1}}{k_n^{k_n}} &= \frac{1}{4} + \frac{1}{4e}\left(1 + \frac{1}{[\sqrt{n}]}\right)^{[\sqrt{n}]} \\
&\to \frac{1}{4} + \frac{1}{4e}\cdot e = \frac{1}{2}, \qquad \text{as } n \to \infty.
\end{aligned}$$

Thus, (1.6) is not satisfied, and hence, Theorem 2 does not apply to (3.1).

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