Functional spectrum of contractions ✤

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Abstract

In this paper, we introduce a new kind of spectrum for the \( C_0 \)-class contractions. Since elements in this spectrum are functions, rather than numbers, we shall call it functional spectrum. Functional spectrum is a “large” closed subset of the Hardy space over the unit disk, and in many cases there is a canonical embedding of classical spectrum into functional spectrum. The study is carried out in the setting of the Hardy space over the bidisk \( H^2(D^2) \), on which every \( C_0 \)-class contraction has a representation. A key tool is reduction operator. The reduction operator also gives rise to an equivalent statement of the Invariant Subspace Problem.

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1. Introduction

For a bounded linear operator \( A \) on a complex separable Hilbert space \( \mathcal{H} \), its spectrum \( \sigma(A) \) is probably the most important base of analysis on \( A \). Being a compact subset of the complex plane \( \mathbb{C} \), \( \sigma(A) \) is easy to calculate in many cases; on the other hand, however, it is not a good representation of \( A \) which indeed has a nature of infinite dimensionality. For instance, it is easy to come up with two operators \( A_1 \) and \( A_2 \) such that \( \sigma(A_1) = \sigma(A_2) \) but \( A_1 \) and \( A_2 \) have completely nothing to do with each other. So are there other spectrum-like associates of \( A \) which will reflect \( A \) more faithfully? In this paper, we propose a new kind of spectrum \( \mathcal{E}(A) \) for a \( C_0 \)-class operator \( A \). Since elements in \( \mathcal{E}(A) \) are functions, we shall call \( \mathcal{E}(A) \) the functional spectrum of \( A \).

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A contraction $A$ on $H$ is said to be in $C_0$-class if $(A^*)^n$ converges to 0 in strong operator topology. $C_0$-class operators are indeed very general, for instance, every strict contraction is in $C_0$-class. To define functional spectrum, we first need to represent a $C_0$-class operator on the Hardy space over the bidisk $H^2(D^2)$. A classical result in Functional Operator Model Theory is that every $C_0$-class operator $A$ can be represented as a compression of the unilateral shift on a vector-valued Hardy space $H^2(E)$, where $E$ is an auxiliary separable Hilbert space, and $\sigma(A)$ can be calculated through its characteristic operator function $\theta_A(z)$. For details, we refer readers to [2–4] and [8], each of which has a comprehensive treatment on the subject. If $E$ is replaced by another copy of $H^2(D^2)$, then $H^2(E) = H^2(D^2)$. $H^2(D^2)$, being a two variable holomorphic function space, has much richer structure than $H^2(E)$ does. And it has always been a tantalizing question whether one can use the richer structure of $H^2(D^2)$ to do new studies of $C_0$-class operators. Some initial work is done in [11], where an important part of Functional Operator Model Theory is re-interpreted in $H^2(D^2)$. New successes along this line are reported in [7], where a representation of the Bergman shift on $H^2(D^2)$ creates new tools to study a reducing subspace problem. This paper is another exploration.

This paper is organized as follows. Section 1 is an introduction. Section 2 is a preparation, where we define our key tool—the reduction operator and display some of its properties. Section 3 deals with the genericity issue. Functional spectrum is defined in Section 4, and some work is also reported. In Section 5, we define spectral defect degree which measures how faithfully functional spectrum reflects the operator. An outstanding problem in Operator Theory is the invariant subspace problem, in Section 6 we will see how the problem can be re-formulated in this framework. Many examples will be given throughout the paper.

2. Preparation

In $H^2(D^2)$ with coordinates $z$ and $w$, multiplications by $z$ and $w$ (denoted by $T_z$ and $T_w$, respectively) are shift operators with infinite multiplicity. A closed subspace $M$ is said to be $z$ (or $w$)-invariant if $M$ is invariant under $T_z$ (or $T_w$, respectively), and $M$ is said to be invariant if it is invariant under both $T_z$ and $T_w$. This paper concerns mostly with $z$-invariant subspaces.

The classical one variable Hardy space $H^2(D)$ in the variable $z$ and that in the variable $w$ are different subspaces in $H^2(D^2)$, and we denote them by $H^2(z)$ and $H^2(w)$, respectively. The following definition brings up a key tool in this study.

**Definition.** For every $g \in H^2(z)$, the reduction operator $\pi_g : H^2(D^2) \rightarrow H^2(w)$ is defined as

$$\pi_g(h)(w) = \int_T h(z, w)\overline{g}(z)\,dm(z), \quad h \in H^2(D^2),$$

where $T$ is the unit circle and $dm(z)$ is the normalized Lebesgue measure on $T$.

Clearly, reduction operator reduces a two variable function to a one variable function. It is easy to check that $\pi_g$ is well defined and bounded. In fact, one verifies that

$$\pi_g^* f = g(z)f(w), \quad f \in H^2(w),$$

and hence $\|\pi_g\| = \|g\|$. We now look at two examples.
Example 2.1. For an \( h \in H^2(D^2) \), if we write \( h = \sum_{j \geq 0} h_j(z)w^j \), where \( h_j \in H^2(z) \), then
\[
\pi_g h = \sum_{j \geq 0} \langle h_j, g \rangle w^j.
\]
It is clear \( \pi_g h = 0 \) if and only if \( g \) is orthogonal to \( h_j \) for every \( j \geq 0 \).

Example 2.2. If \( g(z) = \frac{1}{1 - \lambda z} \), where \( \lambda \in D \), then by Cauchy integral formula
\[
\pi_g h = \int T h(z, w) \frac{1}{1 - \lambda z} dm(z) = h(\lambda, w).
\]
For simplicity, \( \pi_{\frac{1}{1 - \lambda z}} \) is denoted by \( L(\lambda) \).

Given a \( z \)-invariant \( M \), the operator \( S_z \) on \( N := H^2(D^2) \ominus M \) is defined as
\[
S_z f = P_N zf, \quad f \in N,
\]
where \( P_N \) is the orthogonal projection from \( H^2(D^2) \) onto \( N \). For convenience, we let \( S \) denote the collection of the pairs \((S_z, M)\), where \( M \) is \( z \)-invariant and \( S_z \) is as defined above. By Functional Operator Model Theory, every \( C_{0,0} \)-class operator is unitarily equivalent to \( S_z \) for some \((S_z, M) \in S \), though this representation may not be unique.

In this paper, a \( z \)-invariant subspace \( M \) is said to be generic if \( M \cap H^2(w) = \{0\} \), in other words, if \( M \) does not contain non-trivial functions in the variable \( w \) only. The orthogonal difference \( M \ominus zM \) plays an important role here. For simplicity, we denote \( M \ominus zM \) by \( \partial M \). In most places, we will be concerned with restrictions of reduction operators to \( \partial M \) and to \( N \), and we will use the same notation \( \pi_g \) to denote these restrictions when there is no danger of confusion. One simple fact worth mentioning is that since \( M \cap H^2(w) = \partial M \cap H^2(w) \), \( M \) is generic if and only if \( \|L(0)f\| < \|f\| \) for every non-zero \( f \in \partial M \), i.e., \( L(\lambda)|_{\partial M} \) is purely contractive. \((S_z, M)\) will be said to be generic if \( M \) is generic.

The restriction \( T_z^*|_{\partial M} \) is also important, and for simplicity it is denoted by \( D_z \). Clearly,
\[
D_z f(z, w) = \bar{z}(f(z, w) - f(0, w)),
\]
and it is not hard to check that \( D_z \) maps \( \partial M \) into \( N \) (cf. [11]).

For a contraction \( A \) acting on \( \mathcal{H} \), its defect operators are \( D_A = (1 - A^*A)^{1/2} \) and \( D_A^* = (1 - AA^*)^{1/2} \), and the associated characteristic operator function is
\[
\theta_A(\lambda) = \left[ -A + \lambda D_A^* (1 - \lambda A^*)^{-1} D_A \right]|_{D_A}, \quad \lambda \in D,
\]
where \( D_A = \overline{R(A)} \). The next two lemmas show that the operators \( L(0)|_N \) and \( D_z \) explicitly express the defect operators of \( S_z \), and \( L(\lambda)|_{\partial M} \) in generic cases coincides with the characteristic operator function \( \theta_{S_z}(\lambda) \) for \( S_z \) (cf. [11]).

Lemma 2.3. For \((S_z, M) \in S, on N:\)
\[ \begin{align*}
& \text{(a) } S_z^* S_z + D_z D_z^* = I; \\
& \text{(b) } S_z S_z^* + (L(0)|_N)^* (L(0)|_N) = I; \\
& \text{and on } \partial M, \\
& \text{(c) } D_z^* D_z + (L(0)|_{\partial M})^* (L(0)|_{\partial M}) = I.
\end{align*} 

Lemma 2.4. For \((S_z, M) \in \mathcal{S}\), there are constant unitaries \(U\) and \(V\), and possibly \(W\), such that

\[ L(\lambda)|_{\partial M} = U \theta S_z(\lambda) V \oplus W, \quad \lambda \in \mathcal{D}. \]

And the operator \(W\) appears only if \(M\) is non-generic, in which case, the rank of \(W\) is equal to \(\dim(M \cap H^2(w))\).

3. Genericity

As remarked earlier for a \(C_{0}\)-class operator \(A\), its representation \((S_z, M)\) may not be unique. As a matter of fact, there are examples of \(C_{0}\)-class operators \(A\) which have two representations \((S_1, M_1)\) and \((S_2, M_2)\) such that \(M_1\) is generic but \(M_2\) is not. This section aims to clarify some ambiguities.

Proposition 3.1. Let \(A\) be a \(C_{0}\)-class operator on \(\mathcal{H}\). Then \(A\) has a generic representation \((S_z, M) \in \mathcal{S}\) if and only if \(I - AA^*\) is not of finite rank.

Proof. We first check that if \(M\) is generic, then \(L(0)(N)\) is dense in \(H^2(w)\). To see this, we let \(f \in H^2(w)\) such that

\[ \langle L(0)h, f \rangle = 0, \]

for every \(h \in N\). Then it follows that

\[ \int_{T^2} h(z, w) \overline{f(w)} \, dm(z) \, dm(w) = \int_{T} h(0, w) \overline{f(w)} \, dm(w) = 0, \]

which implies that \(f \in M\). Since \(M\) is generic, \(f = 0\). Now since \(L(0)(N)\) is dense in \(H^2(w)\), by Lemma 2.3(b), \(I - S_z S_z^*\) has infinite rank, and hence \(I - AA^*\) has infinite rank.

For the other direction, we assume \(I - AA^*\) is of infinite rank and \(A \cong S_z\) for some \((S_z, M) \in \mathcal{S}\). There is nothing to show if \(M\) is generic. In the case \(M\) is not generic, we construct another representation \((S_z', M') \in \mathcal{S}\) that is generic. To this end, we write \(\partial M = E_1 \oplus E_2\), where \(E_2 = \partial M \cap H^2(w)\) and \(E_1\) is the orthogonal complement of \(E_2\) in \(\partial M\). One checks easily that \(z^i E_1\) is orthogonal to \(z^j E_2\) for all integers \(i, j \geq 0\), and hence

\[ M = \bigoplus_{j \geq 0} z^j \partial M = \bigoplus_{j \geq 0} z^j (E_1 \oplus E_2) = \left( \bigoplus_{j \geq 0} z^j E_1 \right) \oplus H^2(z) \otimes E_2. \]
Therefore,

\[ N = \left( H^2(z) \otimes (H^2(w) \oplus E_2) \right) \ominus \left( \bigoplus_{j \geq 0} z^j E_1 \right). \]  

(3.1)

Since \((L(0)|_N)^* (L(0)|_N) = I - S_z S^*_z\) has infinite rank, \(L(0)|_N\) has infinite rank. So \(\dim (H^2(w) \oplus E_2) = \infty\). In this case, we pick any unitary \(J : H^2(w) \oplus E_2 \to H^2(w)\), and extend it to \(J : H^2(z) \otimes (H^2(w) \oplus E_2) \to H^2(D^2)\) by

\[ J z^j f = z^j J f, \quad f \in H^2(w) \ominus E_2, \quad j \geq 0. \]

Let \(M' = J(\bigoplus_{j \geq 0} z^j E_1), \ N' = H^2(D^2) \ominus M'\) and the operator \(S_z\) on \(N'\) be denoted by \(S'_z\). It is not hard to see that \(M'\) is generic. First of all, \(M' \ominus z M' = J(E_1)\). For any element \(h = \sum_{j \geq 0} h_j (w) z^j \in E_1\), there is a \(j_0 \geq 1\) such that \(h_{j_0}\) is not the constant 0. Therefore, \(J h_{j_0}\) is not the constant 0, and hence \(J h = \sum_{j \geq 0} z^j J h_j (w)\) is not a function in \(w\) only.

Now we show that \(A\) and \(S'_z\) are unitarily equivalent by showing \(S_z\) and \(S'_z\) are unitarily equivalent. For every \(h = \sum_{j \geq 0} h_j (w) z^j \in N\),

\[ S'_z J h = S'^*_z \left( \sum_{j \geq 0} z^j J h_j (w) \right) = \sum_{j \geq 1} z^{j-1} J h_j (w) = J \left( \sum_{j \geq 1} z^{j-1} h_j (w) \right) = J S^*_z h, \]

i.e. \(S_z\) and \(S'_z\) are unitarily equivalent. So \((S'_z, M')\) is a generic representation of \(A\). \qed

**Proposition 3.2.** Let \((S_1, M_1)\) and \((S_2, M_2)\) be two generic elements in \(S\). Then \(S_1\) is unitarily equivalent to \(S_2\) if and only if there are unitaries \(U\) and \(V\) such that the diagram

\[
\begin{array}{ccc}
\partial M_1 & \xrightarrow{U} & \partial M_2 \\
\downarrow \pi_g & & \downarrow \pi_g \\
H^2(w) & \xrightarrow{V} & H^2(w)
\end{array}
\]

commutes for every \(g \in H^2(z)\).

**Proof.** By [8], \(S_1\) is unitarily equivalent to \(S_2\) if and only if \(\theta_{S_1}\) and \(\theta_{S_2}\) coincide, i.e., there are constant unitaries \(U_1\) and \(V_1\) such that \(V_1 \theta_{S_1}(\lambda) = \theta_{S_2}(\lambda) U_1\). By Lemma 2.4, it is the case if and only if there are constant unitaries \(U\) and \(V\) such that the diagram

\[
\begin{array}{ccc}
\partial M_1 & \xrightarrow{U} & \partial M_2 \\
\downarrow L(\lambda) & \xrightarrow{L(\lambda)} & \downarrow L(\lambda) \\
H^2(w) & \xrightarrow{V} & H^2(w)
\end{array}
\]

commutes for each \(\lambda \in D\). The proposition then follows directly from the inequality \(\| \pi_g \|_{\mathcal{A}D} \leq \|g\|\) and the fact that \(\text{span}\{ \frac{1}{1-z} \lambda \in D \}\) is dense in \(H^2(z)\). \qed
4. Functional spectrum

It is a classical fact in Functional Operator Model Theory that for a $\lambda \in \mathbb{D}$, $S_z - \lambda I$ is invertible if and only if $\theta_{S_z}(\lambda)$, and hence $L(\lambda) : \partial M \to H^2(w)$, is invertible. In other words,

$$\sigma(S_z) \cap \mathbb{D} = \{ \lambda \in \mathbb{D} \mid \pi_{\frac{1}{1-\lambda z}} : \partial M \to H^2(w) \text{ is not invertible} \}.$$ 

This observation motivates the definition of functional spectrum.

**Definition.** For $(S_z, M) \in \mathcal{S}$, the functional spectrum of $S_z$ is the set

$$E(S_z) := \{ g \in H^2(z) \mid \pi_g : \partial M \to H^2(w) \text{ is not invertible} \}.$$ 

Examples indicate that the related sets

$$E_c(S_z) := \{ g \in H^2(z) \mid \pi_g : \partial M \to H^2(w) \text{ is compact} \}$$

and, for $0 < p < \infty$,

$$E_p(S_z) := \{ g \in H^2(z) \mid \pi_g : \partial M \to H^2(w) \text{ is in Shatten-}p \text{ class} \}$$

are also of great interest. This paper concerns mostly with $E$ and $E_c$, though other class are also calculated in a few examples. It is clear that $E_c$ is a subset of $E$. And there is an embedding from $\sigma(S_z) \cap \mathbb{D}$ into $E(S_z)$ defined by

$$\lambda \rightarrow \frac{1}{1 - \lambda z}.$$ 

It will be called the canonical embedding in this paper.

**Proposition 4.1.** $E(S_z)$ is a closed subset of $H^2(z)$, and $E_c(S_z)$ is a closed space in $E(S_z)$.

**Proof.** Since $\|\pi_{g-f}|_{\partial M}\| \leq \|g-f\|$, if $\pi_g$ is invertible, then $\pi_f$ is invertible for every $f$ in a small neighborhood of $g$, i.e., the set complement $E^c(S_z)$ is open.

It is clear that for $f, g \in E_c(S_z)$ any linear combinations of $f$ and $g$ are in $E_c(S_z)$. The closedness of $E_c(S_z)$ also follows directly from the fact that $\|\pi_g|_{\partial M}\| \leq \|g\|$ for every $g \in H^2(z)$. 

The following corollary is an immediate consequence of Proposition 3.2.

**Corollary 4.2.** Let $(S_1, M_1)$ and $(S_2, M_2)$ be two generic elements in $\mathcal{S}$. If $S_1$ and $S_2$ are unitarily equivalent then $E(S_1) = E(S_2)$, $E_c(S_1) = E_c(S_2)$, and for each $0 < p < \infty$, $E_p(S_1) = E_p(S_2)$.

Now let us look at a few examples.

**Example 4.3.** If $M$ is non-generic, then we let $E_2 = \partial M \cap H^2(w)$ and $E_1 = \partial M \ominus E_2$. For $g \in H^2(z)$, and $f_j \in E_j$, $j = 1, 2$,

$$\pi_g(f_1 + f_2) = \pi_g(f_1) + g(0)f_2.$$
One observes that since $\langle f_1, z^j f_2 \rangle = 0$ for all $j \geq 0$,

$$\langle \pi_g(f_1), f_2 \rangle = \int_{T^2} f_1(z, w) g(z) f_2(w) \, dm(z) \, dm(w) = 0.$$ 

So $\pi_g$ is invertible if and only if $g(0) \neq 0$ and $\pi_g : E_1 \to H^2(w) \ominus E_2$ is invertible, i.e.

$$\mathcal{E}(S_z) = \{ g \mid \pi_g : E_1 \to H^2(w) \ominus E_2 \text{ not invertible} \} \cup \{ g \mid g(0) = 0 \}.$$ 

So, in particular, $zH^2(z)$ is a subset of $\mathcal{E}$.

The unilateral shift and the Bergman shift are well-known examples of $C_0$-class operators. The next two examples calculate their functional spectra.

**Example 4.4.** Consider $M = wH^2(D^2)$. In this case $N = H^2(z)$ and $S_z$ is multiplication by $z$, i.e., $S_z$ is the unilateral shift. One verifies that $\partial M = wH^2(w)$. So for any $g \in H^2(z)$ and $f \in H^2(w)$,

$$\pi_g(wf) = g(0)wf.$$ 

This shows that $\pi_g|_{\partial M} = 0$ if $g(0) = 0$, and the range $R(\pi_g|_{\partial M}) = wH^2(w)$ if $g(0) \neq 0$. In any case $\pi_g$ is not surjective, so $g \in \mathcal{E}(S_z)$. This shows that $\mathcal{E}(S_z) = H^2(z)$. The fact that $\mathcal{E}_c(S_z) = zH^2(z)$ is also easy to see. It is also clear from the calculation that $\pi_g|_{\partial M}$ is Fredholm when $g(0) \neq 0$, and ind($\pi_g|_{\partial M}$) = $-1$.

**Example 4.5.** Consider $M = [z - w]$, where $[z - w]$ is the closure of the principal ideal $(z - w) \subset \mathbb{C}[z, w]$. It is well known that $S_z$ in this case is unitarily equivalent to the Bergman shift (cf. [5]). It is indicated in [10] that

$$\varphi_n = \frac{1}{\sqrt{n+2}} \left( \sqrt{n+1} w^{n+1} - \frac{z}{\sqrt{n+1}} \left( z^n + w^{n-1}w + \cdots + wz^{n-1} + w^n \right) \right), \quad n \geq 0,$$ 

is an orthonormal basis for $\partial M$. For any $g \in H^2(z)$,

$$\pi_g(\varphi_n) = \frac{\sqrt{n+1}}{\sqrt{n+2}} g(0) w^{n+1} - \frac{1}{\sqrt{(n+1)(n+2)}} \sum_{k=0}^{n} \langle z^{n-k+1}, g \rangle w^k$$ 

and it follows that

$$\|\pi_g(\varphi_n)\|^2 = \frac{n+1}{n+2} |g(0)|^2 + \frac{1}{(n+1)(n+2)} \sum_{k=0}^{n} |\langle z^{n-k+1}, g \rangle|^2$$

$$= \left( \frac{n+1}{n+2} - \frac{1}{(n+1)(n+2)} \right) |g(0)|^2 + \frac{1}{(n+1)(n+2)} \sum_{k=0}^{n+1} |\langle z^k, g \rangle|^2$$

$$= \frac{n}{n+1} |g(0)|^2 + \frac{1}{(n+1)(n+2)} \sum_{k=0}^{n+1} |\langle z^k, g \rangle|^2.$$
So if $g$ is a non-trivial function with $g(0) = 0$, then

$$\|\pi_g(\varphi_n)\|_2^2 = \sum_{n=0}^{\infty} \frac{|\langle z^n, g \rangle|^2}{(n+1)(n+2)} \leq \frac{\|g\|^2}{(n+1)(n+2)},$$

which implies that $\pi_g$ is a non-trivial Hilbert–Schmidt operator with

$$0 < \|\pi_g\|_2^2 = \sum_{n \geq 0} \|\pi_g(\varphi_n)\|_2^2 \leq \|g\|^2.$$

For $g(0) \neq 0$, we write $g(z) = g(0) + (g(z) - g(0))$. Then

$$\pi_g = g(0)\pi_1 + \pi_{g-g(0)}.$$

Since $\pi_{g-g(0)}$ is Hilbert–Schmidt, and $\pi_1 = L(0)$ is Fredholm with $\text{ind}(L(0)) = -1$ by virtue of Lemma 2.4, $\pi_g$ is Fredholm with $\text{ind}(\pi_g) = -1$. In particular, $\pi_g$ is not invertible. So $E = H^2(z)$, and $E_2 = E_c = zH^2(z)$.

It is well known that classical spectrum can hardly tell the differences between the unilateral shift and the Bergman shift. Examples 4.4 and 4.5 show that their functional spectra also coincide. But there is a notable difference: for a non-trivial $g$ with $g(0) = 0$, in Example 4.4 $\pi_g|_{\partial M} = 0$, but in Example 4.5, $\pi_g|_{\partial M}$ is a non-trivial Hilbert–Schmidt operator. So in the two examples, although the functional spectra are the same, they have different structure. We will take up this issue in Section 5.

The next example shows that functional spectrum contains much more information about the operator.

**Example 4.6.** Let $q_j(z)$ be a sequence of inner functions with $q_{j+1}|q_j$, $j \geq 0$. Set

$$M = \bigoplus_{j=0}^{\infty} q_j(z)H^2(z)w^j.$$

It is easy to check that $M$ is $z$-invariant (in fact it is also $w$-invariant), and some studies are made in [6]. In particular, it is shown that in this case $S_z = \bigoplus_{j=0}^{\infty} S(q_j)$, where $S(q_j)$ is the Jordan block defined on $H^2(z) \ominus q_jH^2(z)$ by

$$S(q_j)f = P_jzf, \quad f \in H^2(z) \ominus q_jH^2(z),$$

where $P_j$ is the orthogonal projection from $H^2(z)$ onto $H^2(z) \ominus q_jH^2(z)$. By classical operator theory $\sigma(S_z) = \sigma(S(q_0))$, and it is the closure of the union of zeros of $q_0$ in $D$ with the support of $q_0$’s singular measure. This shows that although $S_z$ is dependent on the sequence $\{q_j\}$, its spectrum $\sigma(S_z)$ only reflects the first function $q_0$.

Now let us look at functional spectrum. It is not hard to check (cf. [6]) that in this case

$$\partial M = \bigoplus_{j=0}^{\infty} Cq_j(z)w^j,$$
and \( \{q_j(z)w^j\}_{j \geq 0} \) is an orthonormal basis for \( \partial M \). For every \( g \in H^2(z) \) and every function \( h = \sum_{j \geq 0} c_j q_j(z)w^j \in \partial M \),

\[
\pi_g(h) = \sum_{j \geq 0} c_j \langle q_j, g \rangle w^j,
\]

and hence

\[
\|\pi_g(h)\|^2 = \sum_{j \geq 0} |c_j|^2 |\langle q_j, g \rangle|^2.
\]

So \( \pi_g \) is invertible if and only if there is a positive constant \( \eta \) such that

\[
|\langle q_j, g \rangle| \geq \eta > 0, \quad \forall j \geq 0.
\]

Therefore,

\[
\mathcal{E}(S_z) = \left\{ g \in H^2(z) \mid \inf_{j \geq 0} \{ |\langle q_j, g \rangle| \} = 0 \right\}.
\]

Clearly, \( \mathcal{E}(S_z) \) has something to do with each \( q_j \). This fact indicates that \( \mathcal{E}(S_z) \) captures much more information about \( S_z \) than \( \sigma(S_z) \) does. Now we compute \( \mathcal{E}_c(S_z) \). To this end, we let \( q_\infty(z) \) be the greatest common divisor of all \( q_j(z), \, j \geq 0 \). In fact, \( q_\infty \) is the inner function such that

\[
q_\infty H^2(z) = \bigcup_{j \geq 0} q_j H^2(z).
\]

If we assume that for each \( j \) the first non-zero coefficient of \( q_j \)'s Fourier series is positive, then we can check that \( q_j \) converges to \( q_\infty \) in \( H^2(z) \). Without loss of generality, we assume \( q_\infty = 1 \). Then,

\[
P_j 1 = 1 - q_j(0)q_j(z),
\]

and hence \( \|P_j 1\| = 1 - q_j(0)^2 \). By (4.2), \( \lim_{j \to \infty} \|P_j 1\| = 0 \), so \( \lim_{j \to \infty} q_j(0) = 1 \), and it follows that \( \lim_{j \to \infty} q_j = 1 \) in \( H^2(z) \).

By (4.1), \( \pi_g \) on \( \partial M \) is compact if and only if

\[
\lim_{j \to \infty} \langle q_j, g \rangle = \langle q_\infty, g \rangle = 0.
\]

So

\[
\mathcal{E}_c(S_z) = \left\{ g \in H^2(z) : \langle q_\infty, g \rangle = 0 \right\}.
\]

This is quite interesting, since, on the contrary to \( \sigma(S_z) \) which reflects the leading function \( q_0 \), \( \mathcal{E}_c(S_z) \) reflects the end function \( q_\infty \).
Example 4.7. It is useful to single out a particular case of Example 4.6. When $q_j = q_0$ for all $j \geq 1$, one checks that $M = q_0H^2(D^2)$. So $N = (H^2(z) \ominus q_0H^2(z)) \otimes H^2(w)$ and correspondingly $S_z = S(q_0) \otimes I$. Moreover, $\partial M = q_0(z)H^2(w)$. So for $g \in H^2(z)$ and $h = q_0(z)f(w) \in \partial M$,

$$\pi_g(h) = \langle q_0, g \rangle f(w),$$

and it follows that

$$E = E_c = E_p = \{ g \in H^2(z) : \langle q_0, g \rangle = 0 \}, \quad \forall p > 0.$$

It is clear from the definition that 0 is an element in both $E$ and $E_c$, and we shall call 0 the trivial element. An important issue is whether $E$ and $E_c$ always have non-trivial elements. We will show that $E$ is not only non-trivial it is in fact fairly big. But we leave this issue to Section 5. In the later part of this section, we examine how $E$ is dependent on $S_z$ in some more general cases, and we will also display an example which indicates that $E_c$ can be trivial. The following lemma is useful to this end.

Lemma 4.8. On $H^2(D^2)$,

$$\pi_g = \overline{g(0)L(0)} + \pi_g'T_z^*,$$

where $g'(z) = \tilde{z}(g(z) - g(0))$.

Proof. Write $g(z) = g(0) + zg'(z)$. Then for every $h \in \partial M$,

$$\pi_g(h) = \pi_{g(0)+zg'}(h) = \int_{T} h(z, w)\overline{(g(0) + zg'(z))} dm(z)$$

$$= \overline{g(0)h(0, w)} + \int_{T} \tilde{z}h(z, w)\overline{g'} dm(z) = \overline{g(0)h(0, w)} + \int_{T} \tilde{z}(h(z, w) - h(0, w))\overline{g'} dm(z)$$

$$= \overline{g(0)L(0)h} + \pi_g'(T_z^*h). \quad \Box$$

Corollary 4.9. Let $M$ be $z$-invariant. Then on $\partial M$,

(a) $\pi_g = \overline{g(0)L(0)} + \pi_g'D_z$;
(b) $\pi_{z^{n+1}} = L(0)(S_z^*)^n D_z$, $n \geq 0$.

Proof. Since $D_z = T_z^*|_{\partial M}$, (a) is obvious. For (b), using Lemma 4.8 repetitively, we have

$$\pi_{z^{n+1}} = \pi_1(T_z^*)^n T_z^* = L(0)(T_z^*)^n T_z^*. \quad (4.4)$$

On $\partial M$, $D_z = T_z^*|_{\partial M}$, and it maps $\partial M$ into $N$. Since $N$ is invariant for $T_z^*$ and $T_z^*|_N = S_z^*$, we have

$$\pi_{z^{n+1}} = L(0)(S_z^*)^n D_z. \quad \Box$$
A contraction $A$ on $\mathcal{H}$ is said to be pure if $\|Ax\| < \|x\|$ for every non-zero $x \in \mathcal{H}$.

**Corollary 4.10.** If $S_z$ is pure and $\|S_z\| = 1$, then $z \in E(S_z)$.

**Proof.** We prove by contradiction. Lemma 4.9(b) implies

$$\pi_z = L(0)D_z$$

on $\partial M$. If $\pi_z|_{\partial M}$ is invertible, then $L(0) : N \rightarrow H^2(w)$ is onto. If $\ker(L(0)|_N) \neq \{0\}$, then there exists a non-trivial function $zf \in N$. This implies that $\|S_zf\| = \|zf\| = \|f\|$, which contradicts the fact that $S_z$ is pure. If $L(0) : N \rightarrow H^2(w)$ is injective, then $L(0)|_N$ is invertible, and it follows from Lemma 2.3(b) that $S_z$ is a strict contraction, in contradiction with the condition $\|S_z\| = 1$. $\square$

Corollary 4.10 is an interesting fact, since for a $C_0$-class pure contraction $A$, no specific things can be said about its classical spectrum $\sigma(A)$.

In all examples we have considered so far, $E$ is fairly big. This phenomenon happens in some other more general cases.

**Proposition 4.11.** If $\|L(0)|_{\partial M}\| = 1$, then $zH^2(z) \subset E(S_z)$.

**Proof.** For every non-trivial $g \in H^2(z)$ and $h \in \partial M$, Lemma 4.9(a) implies that

$$\pi_{zg}h = \pi_g(Dzh).$$

If $\pi_{zg}|_{\partial M}$ is invertible, then there exists a constant $\eta > 0$ such that

$$\eta \|h\| \leq \|\pi_{zg}h\| = \|\pi_g(Dzh)\| \leq \|\pi_g\| \|Dzh\|.$$ 

This implies that $D_z$ is injective with closed range. So $D_z^*D_z$ is invertible, and it follows from Lemma 2.3(c) that $L(0)|_{\partial M}$ is a strict contraction, in contradiction with the fact $\|L(0)|_{\partial M}\| = 1$. $\square$

**Proposition 4.12.** If $I - S_z^*S_z$ is compact and $\text{ind}(S_z) \neq 0$, then:

(a) $E(S_z) = H^2(z)$;
(b) $E_c(S_z) = zH^2(z)$.

**Proof.** We first prove (b). By Corollary 4.9(a), for every $g \in H^2(z)$,

$$\pi_{zg} = \pi_gD_z$$

on $\partial M$. Since $I - S_z^*S_z$ is compact, $D_z$ is compact by Lemma 2.3(a), and hence $\pi_{zg}$ is compact. This shows that $zH^2(z) \subset E_c(S_z)$.

If $g(0) \neq 0$, we write $g(z) = g(0) + zg'(z)$, and hence

$$\pi_g = \overline{g(0)}L(0) + \pi_{zg'}.$$
Since $S_z$ is semi-Fredholm with $\text{ind}(S_z) \neq 0$, by virtue of Lemma 2.4 $L(0)|_{\partial M}$ is semi-Fredholm with $\text{ind}(L(0)|_{\partial M}) = \text{ind}(S_z) \neq 0$ (cf. [11]). Therefore $\pi_g$ is semi-Fredholm with $\text{ind}(\pi_g) = \text{ind}(L(0)|_{\partial M}) \neq 0$. This implies that $\pi_g$ is not compact nor invertible, thus (a) and (b) are both established. \qed

One observes that Examples 4.4 and 4.5 are special cases of Proposition 4.12. We need to point out that it is easy to come up with examples for which $zH^2(z)$ is not a subset of $\mathcal{E}$. The fact that $\mathcal{E}$ is big in so many cases is not accidental. We will prove in Section 5 that this is true in general.

Next we give an example in which $\mathcal{E}_c$ is trivial.

**Example 4.13.** Let $\{F_j\}_{j \geq 0}$ be a sequence of orthogonal closed subspaces of $H^2(w)$ such that $\dim F_j = \infty$ for each $j$ and $\bigoplus_{j=0}^{\infty} F_j = H^2(w)$. Set

$$E_k = \bigoplus_{j=0}^{k} F_j, \quad k \geq 0,$$

and let

$$M = \bigoplus_{k=0}^{\infty} z^k E_k.$$

It is not hard to check that $M$ is $z$-invariant. Moreover,

$$\partial M = E_0 \oplus z(E_1 \oplus E_0) \oplus z^2(E_2 \oplus E_1) \cdots = \bigoplus_{k=0}^{\infty} z^k F_k.$$

So for any $g \in H^2(z)$,

$$\pi_g|_{z^k F_k} = \langle z^k, g \rangle I_k,$$

where $I_k$ is the identity on $F_k$. So $\pi_g|_{\partial M}$ is compact if and only if $\langle z^k, g \rangle = 0$ for each $k$, i.e., $g = 0$.

Although, by the definition of functional spectrum, there is no reason to expect that $\mathcal{E}$ shall be a space, it is the case in all the examples above, except possibly Example 4.6 in which $\mathcal{E}$ is not a space if a particular sequences $\{q_0, q_1, \ldots\}$ are selected. This phenomenon makes one ponder the following question.

**Question.** For what type of $(S_z, M)$ is $\mathcal{E}(S_z)$ a space?

At this moment, there is no good guess.
5. Spectral defect degree

One motivation behind the idea of functional spectrum is the expectation that it will reflect the operator more faithfully. As indicated by Examples 4.4 and 4.5, the structure of functional spectrum, rather than the set itself, does the job. So a certain type of equivalence relation for functional spectrum is needed. One good candidate is suggested by Proposition 3.2.

**Definition.** Given \((S_i, M_i) \in \mathcal{S}, i = 1, 2\), the functional spectra \(E(S_1)\) and \(E(S_2)\) are said to be equivalent if \(E(S_1) = E(S_2)\) and there are unitaries \(U\) and \(V\) such that the diagram

\[
\begin{array}{ccc}
\partial M_1 & \xrightarrow{U} & \partial M_2 \\
\downarrow \pi_g & & \downarrow \pi_g \\
H^2(w) & \xrightarrow{V} & H^2(w)
\end{array}
\]

commutes for every \(g \in \mathcal{E}\).

So if \(\mathcal{E} = H^2(z)\), then by virtue of Proposition 3.2 the operator \(S_z\), up to a unitary equivalence of operators, is completely determined by the structure of its functional spectrum. For a \((S_z, M) \in \mathcal{S}\), the numbers

\[
n(S_z) := \dim(H^2(z) \ominus E(S_z)) \quad \text{and} \quad n_c(S_z) := \dim(H^2(z) \ominus E_c(S_z))
\]

are called **spectral defect degrees** (or simply defect degrees). In view of Proposition 3.2, \(n(S_z)\) and \(n_c(S_z)\) measures how faithfully \(\mathcal{E}\), and respectively \(E_c\), reflects \(S_z\). If \(n(S_z) = 0\), then \(\text{span} \mathcal{E}\) is dense in \(H^2(z)\), and the diagram in the above definition extends to every \(g \in H^2(z)\), and hence as remarked earlier, \(S_z\) is determined by its functional spectrum. If \(n(S_z) = 1\) then, intuitively speaking, \(\mathcal{E}\) becomes slightly less faithful. However, more concrete relations are yet to be discovered.

One important question left unanswered in Section 4 is whether \(\mathcal{E}\) is always non-trivial. We address this issue now. An interesting lemma is needed for this purpose. Let \(E\) be a Hilbert space and \(A_1, A_2\) be bounded linear operators from \(\mathcal{H}\) to \(E\). Define

\[
\Omega(A_1, A_2) = \{ (\lambda_1, \lambda_2) \in \mathbb{C}^2 : \lambda_1 A_1 + \lambda_2 A_2 \text{ is not invertible} \}.
\]

It is clear that \((0, 0)\) is a trivial element in \(\Omega(A_1, A_2)\).

**Lemma 5.1.** \(\Omega(A_1, A_2)\) is always non-trivial.

**Proof.** First of all, it is not hard to check that \(\Omega(A_1, A_2)\) is closed. Now let

\[
F(\lambda_1, \lambda_2) = (\lambda_1 A_1 + \lambda_2 A_2)^{-1}, \quad (\lambda_1, \lambda_2) \in \Omega^c(A_1, A_2).
\]

Then
\[ F(\lambda_1, \lambda_2) - F(\lambda'_1, \lambda_2) \]
\[ = (\lambda_1 A_1 + \lambda_2 A_2)^{-1} (I - (\lambda_1 A_1 + \lambda_2 A_2)(\lambda'_1 A_1 + \lambda_2 A_2)^{-1}) \]
\[ = (\lambda_1 A_1 + \lambda_2 A_2)^{-1} (I - (\lambda'_1 A_1 + \lambda_2 A_2 + (\lambda_1 - \lambda'_1) A_1)(\lambda'_1 A_1 + \lambda_2 A_2)^{-1}) \]
\[ = - (\lambda_1 - \lambda'_1)(\lambda_1 A_1 + \lambda_2 A_2)^{-1} A_1(\lambda'_1 A_1 + \lambda_2 A_2)^{-1}. \]

This shows that \( F \) is analytic in \( \lambda_1 \). Likewise, \( F \) is analytic in \( \lambda_2 \). The lemma then follows directly from Hartog’s extension theorem. □

We believe \( \Omega(A_1, A_2) \) is interesting in its own right. As a matter of fact, Lemma 5.1 is equivalent to the non-emptiness of classical spectrum \( \sigma \). For one direction, one checks that for any bounded operator \( A \), non-trivial elements \( (\lambda_1, \lambda_2) \) in \( \Omega(A, I) \) must have \( \lambda_1 \neq 0 \), and
\[ \sigma(A) = \{ \lambda_2/\lambda_1 \mid (\lambda_1, \lambda_2) \in \Omega(A, I), \lambda_1 \neq 0 \}. \]

For the other direction, if \( A_1 \) is not invertible, then \((1, 0) \in \Omega(A_1, A_2)\); if \( A_1 \) is invertible, then for any \( \lambda \in \sigma(A_1^{-1} A_2) \),
\[ \lambda A_1 + A_2 = A_1(\lambda I + A_1^{-1} A_2) \]
is not invertible, hence \((\lambda, 1) \in \Omega(A_1, A_2)\). Lemma 5.1 and its proof provide a new and two variable point of view on the non-emptiness of \( \sigma(A) \).

Now we prove the non-triviality of \( \mathcal{E} \).

**Theorem 5.2.** For any \((S_z, M) \in \mathcal{S}, n(S_z) \) is either 0 or 1.

**Proof.** It is clear from Examples 4.5 and 4.7 that 0 and 1 are possible values for \( n(S_z) \). So it suffices to show that \( n(S_z) < 2 \). Let \( g_1 \) and \( g_2 \) be any two non-zero functions in \( H^2(z) \ominus \mathcal{E} \). By Lemma 5.1, \( \Omega(\pi g_1, \pi g_2) \) contains a non-trivial element \((\lambda_1, \lambda_2)\). So
\[ \pi_{\lambda_1 g_1 + \lambda_2 g_2} = \lambda_1 \pi g_1 + \lambda_2 \pi g_2 \]
is not invertible, hence \( \lambda_1 g_1 + \lambda_2 g_2 \in \mathcal{E} \) which is possible only if \( \lambda_1 g_1 + \lambda_2 g_2 = 0 \), and it concludes that
\[ n(S_z) = \dim(H^2(z) \ominus \mathcal{E}) < 2. \] □

So \( \mathcal{E}(S_z) \) is not only non-trivial but is in fact very big, which is in an interesting contrast to the one-dimensional nature of classical spectrum. Theorem 5.2 divides \( \mathcal{S} \) into two classes. Elements \((S_z, M) \in \mathcal{S} \) for which \( n(S_z) = 0 \) are said to be of type 0, those with \( n(S_z) = 1 \) are said to be of type 1. Type 0 elements are easy to construct, for instance, Examples 4.4 and 4.5. In fact, by virtue of canonical embedding, every \((S_z, M) \) such that \( \sigma(S_z) \) contains an accumulation point in \( \mathcal{D} \) is of type 0. Example 4.7 is a case of type 1 class. For a type 1 \((S_z, M) \in \mathcal{S}, \)
\[ H^2(z) \ominus \mathcal{E} = \mathbb{C} \phi, \]
for some non-trivial $\phi$. This fact associates with every type 1 element $(S_z, M)$ a function $\phi \in H^2(z)$ unique up to a non-zero scalar. In Example 4.7, this $\phi$ is a scalar multiple of $q_0$ which is the \textit{minimal function} of $S_z$. Of course, there is no reason to believe that every type 1 element $(S_z, M)$ is like that in Example 4.7. Nevertheless, we suspect that in more general cases, especially in cases where $E$ is a space, this $\phi$ will also play an important role in the study of $(S_z, M)$. But more examples are needed before we can have a better picture.

$n_c(S_z)$, on the other hand, can take on any numbers including $+\infty$. For instance, in Example 4.13, $n_c(S_z) = \infty$. Example 4.13 can also be modified to give other cases.

Example 5.3. For any natural number $n$, we let $\{F_j\}_{j \geq 0}$ be a sequence of orthogonal closed subspaces of $H^2(w)$ such that $\dim F_j = \infty$ for each $j \leq n - 1$, $\dim F_j = 1$ for each $j \geq n$, and $\bigoplus_{j=0}^{\infty} F_j = H^2(w)$. $M$ is as constructed in Example 4.13. As in Example 4.13, for any $g \in H^2(z)$,

$$
\pi_{g \mid \partial M} = \langle z, g \rangle I_k,
$$

where $I_k$ is the identity on $F_k$. So $\pi_{g \mid \partial M}$ is compact if and only if $\langle z^k, g \rangle = 0$ for each $k \leq n - 1$. This implies that $E_c = z^n H^2(z)$, and hence $n_c(S_z) = n$.

The next theorem shows that $n_c(S_z)$ is small if $S_z$ itself is compact.

Theorem 5.4. Let $(S_z, M) \in S$ be generic. If $S_z$ is compact, then $n_c(S_z) = 1$.

Proof. By Corollary 4.9(b), $\pi_{z^k}$ on $\partial M$ is compact for each $k \geq 2$. Since $E_c$ is a closed space, $z^2 H^2(z) \subset E_c$. Moreover, since $M$ is generic, by Lemma 2.4, there are constant unitaries $U$ and $V$ such that

$$
U \theta_{S_z}(\lambda) V = L(\lambda) |_{\partial M}, \quad \lambda \in \mathcal{D}.
$$

In particular, $U \theta_{S_z}(0) V = L(0) |_{\partial M} = \pi_1 |_{\partial M}$. Now since $S_z$ is compact, expression (2.1) implies that $\theta_{S_z}(0)$, and hence $\pi_1 |_{\partial M}$, is compact, i.e. $1 \in E_c$.

We now check that $\pi_{z^k} |_{\partial M}$ is not compact. In fact, by Lemma 2.3 and the fact that $S_z$ is compact, $(L(0) |_N)^* (L(0) |_N)$ and $D_z D_z^*$ are both Fredholm. By Corollary 4.9(b),

$$
\pi_{z^k} |_{\partial M} = L(0) |_N D_z,
$$

so

$$
(L(0) |_N)^* \pi_{z^k} |_{\partial M} D_z^* = (L(0) |_N)^* (L(0) |_N) D_z D_z^*
$$

is Fredholm. Therefore $\pi_{z^k} |_{\partial M}$ is not compact. Since $E_c$ is a closed space, we conclude from these observations that

$$
E_c = \{ g \in H^2(z) \mid \langle z, g \rangle = 0 \}.
$$

So

$$
H^2(z) \ominus E_c = C_z,
$$

and hence $n_c(S_z) = 1$. \qed
6. Right reduction and the invariant subspace problem

Up to a scalar multiple, every bounded linear operator is a strict contraction, and hence is in $C_{00}$-class. So the pair $(S_z, M)$ and the idea of functional spectrum suggest a new framework for studying bounded linear operators on complex separable Hilbert spaces. A well-known outstanding problem here is the invariant subspace problem. So it will be interesting to see how the problem can be interpreted in this framework. To this end, we need to introduce the right reduction.

**Definition.** For every $g \in H^2(w)$, the right reduction operator $\pi^r_g : H^2(D^2) \to H^2(z)$ is defined as

$$\pi^r_g(h)(z) = \int_T h(z, w) \overline{g}(w) \, dm(w), \quad h \in H^2(D^2).$$

This definition is completely parallel to the definition of reduction operator $\pi_g$. So, in particular, $\|\pi^r_g\| = \|g\|$. It is also the restrictions $\pi^r_g|_N$ and $\pi^r_g|_{\partial M}$ that are important here.

**Proposition 6.1.** Let $(S_z, M)$ be a generic representation of a strict contraction $A$ on $\mathcal{H}$. Then $A$ has a non-trivial invariant subspace if and only if there exists a non-zero $g \in H^2(w)$ such that $\pi^r_g : N \to H^2(z)$ has non-trivial kernel.

**Proof.** First of all, Proposition 3.1 ensures the existence of a generic representation $(S_z, M)$, and it is equivalent to prove the statement for $S^*_z$.

We now check that $L(0)|_N$ is invertible in this case. By Lemma 2.3, $(L(0)|_N)^*(L(0)|_N)$ is invertible, and it follows that $L(0)|_N$ is injective with closed range. If there exists $f \in H^2(w)$, such that for all $h \in N$, $\int_T h(0, w) \overline{f}(w) \, dm(w) = 0$, then

$$\int_T h(z, w) \overline{f}(w) \, dm(z) \, m(w) = \int_T h(0, w) \overline{f}(w) \, dm(w) = 0$$

which means $f \in M$, and hence $f = 0$ because $M$ is generic. This shows that $L(0)|_N$ has a dense range, and one concludes that $L(0)|_N$ is invertible.

We now prove the sufficiency. If there exists a non-zero $g \in H^2(w)$ such that $\ker(\pi^r_g|_N)$ is non-trivial, then pick any non-trivial function $h \in \ker(\pi^r_g|_N)$, and write $h = \sum_{j=0}^{\infty} h_j(w)z^j$.

Since

$$0 = \pi^r_g(h) = \sum_{j=0}^{\infty} (h_j, g)z^j,$$
\[ \langle h_j, g \rangle = 0, \forall j \geq 0. \] Set
\[ E = \text{span}\{ (S_z^*)^j h : j \geq 0 \}. \]
One checks that \( L(0)(S_z^*)^j h = h_j(w), j \geq 0 \). It then follows that
\[ \{ L(0)(S_z^*)^j h, g \} = 0, \forall j \geq 0. \]

This indicates that \( L(0)(E) \) is a proper subspace in \( H^2(w) \). Now since \( L(0) : N \rightarrow H^2(w) \) is invertible, \( E \) is a proper subspace of \( N \). Clearly, \( E \) is invariant for \( S_z^* \).

The proof of necessity is similar. If \( E \) is a non-trivial invariant subspace for \( S_z^* \), then \( L(0)(E) \) is a closed proper subspace of \( H^2(w) \). Pick any non-trivial function \( h \in E \) and \( g \in H^2(w) \ominus L(0)(E) \). Writing \( h = \sum_{j=0}^{\infty} h_j(w)z^j \), one verifies that
\[ \langle h_j, g \rangle = \langle L(0)(S_z^*)^j h, g \rangle = 0, \forall j \geq 0. \]
So \( h \in \ker(\pi^r_g|_N) \).

So in this framework the invariant subspace problem is translated as whether some right reduction \( \pi^r_g \) on \( N \) has a non-trivial kernel. Proposition 6.1, in a certain sense, offers a two variable explanation on why the invariant subspace problem is difficult: the definition of \( (S_z, M) \) is based on the \( z \) variable only, there is no information on how \( M \), or its complement \( N \), is related to the variable \( w \), and hence it is difficult to determine whether or not for some non-zero \( g \in H^2(w) \) the right reduction \( \pi^r_g|_N \) has a non-trivial kernel. From this point of view, Proposition 6.1 suggests that to settle the invariant subspace problem, or some cases of it, it is necessary to have a good understanding of how functions in \( N \) depend on \( w \). Although difficult, it is not entirely intangible. At least, there is one step we can make towards an understanding of the right reduction \( \pi^r_g \) on \( N \). The following lemma is essentially a different statement of Lemma 1.1 in [9].

**Lemma 6.2.** For every \( z \)-invariant \( M \) and every \( g \in H^2(w) \), the right reduction \( \pi^r_g : \partial M \rightarrow H^2(z) \) is Hilbert–Schmidt.

**Corollary 6.3.** If \( S_z \) is a strict contraction, then \( \pi^r_g : N \rightarrow H^2(z) \) is Hilbert–Schmidt for every \( g \in H^2(w) \).

**Proof.** By Lemma 2.3(a), \( D_zD_z^* \) is invertible in this case, so it suffices to show that \( \pi^r_gD_z : \partial M \rightarrow H^2(z) \) is Hilbert–Schmidt for every \( g \in H^2(w) \). To this end, one checks that for every \( h \in \partial M \),
\[ \pi^r_gD_z h = \int_T \bar{z}(h(z, w) - h(0, w))\overline{g(w)} dm(w) \]
\[ = \bar{z}\pi^r_g(h) - \bar{z} \int_T h(0, w)\overline{g(w)} dm(w). \]
So by Lemma 6.2 and the fact that $\pi_g^r L(0)$ is of rank 1, $\pi_g^r D_z : \partial M \to H^2(z)$ is Hilbert–Schmidt. □

A deep result on the invariant subspace problem is that if a contraction $A$ has a “rich” spectrum then it has a non-trivial invariant subspace (cf. [1]). A functional spectrum $\mathcal{E}(S_z)$ is rich if the defect degree $n(S_z) = 0$. Furthermore, in the framework here, functional spectrum is defined through (left) reduction $\pi_g$ and the invariant subspace problem is linked with right reduction $\pi_g^r$. Whether this duality will give rise to new techniques for studying the invariant subspace problem is an appealing question.

References