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On fixed point sets of distinguished collections for groups of parabolic characteristic

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ABSTRACT

We determine the nature of the fixed point sets of groups of order p, acting on complexes of distinguished p-subgroups (those p-subgroups containing p-central elements in their centers). The case when G has parabolic characteristic p is analyzed in detail.

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1. Introduction

The subgroup complexes associated to suitably chosen collections of p-subgroups are relevant to the understanding of the p-local structure of the underlying group G, and also provide valuable tools for investigating the modular representation theory and the mod-p cohomology of the group G.

This paper continues the systematic study, started in [11], of certain collections of p-subgroups, which we call distinguished. These are subcollections of the standard collections of p-subgroups and which consist of those p-subgroups which contain p-central elements in their centers.

A group G has parabolic characteristic p if all the p-local subgroups which contain a Sylow p-subgroup of G have characteristic p (see Definitions 2.4 and 2.7). If G has parabolic characteristic p, then the collection of distinguished p-radical subgroups equals the collection of p-centric p-radical subgroups (Proposition 3.7(b)). This latter collection has been studied by several authors, including Dwyer [5], Yoshiara [20] and Sawabe [15], and it has been suggested as a "best" geometry for a finite group, generalizing the Tits building for a finite group of Lie type in characteristic p.

The structure of fixed point sets leads to information about the reduced Lefschetz module, a virtual module given by an alternating sum of chain complexes. For example, Webb [19] uses an assumption about certain fixed point sets having Euler characteristic one to conclude that the reduced Lefschetz module is projective, and he obtains an alternating sum formula for the Tate cohomology of the group.

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Thévenaz [17] generalizes this to a situation in which the reduced Lefschetz module is projective relative to a collection of subgroups.

Our main result in this paper concerns the homotopy type of the fixed point sets of subgroups of order p acting on the complex Δ of distinguished p-radical subgroups of G, for a group G of parabolic characteristic p. Let $P = \langle x \rangle$ be such a p-subgroup. If x is p-central then the fixed point set is contractible. If x is not of central type, then the homotopy type of the corresponding fixed point set depends on the group structure of $C_G(x)$. If $C_G(x)$ has characteristic p, then Δ^P is again contractible. If we assume that $C_G(x)$ does not have characteristic p, and that $\overline{C} = C_G(x)/O_p(C_G(x))$ has parabolic characteristic p, then the fixed point set Δ^P is equivariantly homotopy equivalent to the complex of distinguished p-radical subgroups of $\overline{C}$.

In Section 2, notation is introduced and a few basic results are reviewed. In Section 3, two varieties of collections of p-subgroups are defined and various homotopy properties are described; the fixed point sets of p-central elements acting on the corresponding complexes are shown to be contractible. In Section 4, under certain hypotheses, the fixed point sets of subgroups of order p of noncentral type are shown to be equivariantly homotopy equivalent to the complex for a quotient of the centralizer. In Section 5 we consider a few examples where G is a sporadic finite simple group; applications to modular representation theory are also given.

2. Notation, terminology and standard results

Throughout this paper G is a finite group and p a prime dividing its order.

A p-subgroup R of G is called *p-radical* if $R = O_p(N_G(R))$, where $O_p(H)$ is the largest normal p-subgroup of H. Every p-subgroup Q of G is contained in a p-radical subgroup of G uniquely determined by Q and G. This is called the *radical closure* of Q in G and it is the last term R_Q of the chain $P_{i+1} = O_p(N_G(P_i))$ starting with $Q = P_0$. It is easy to see that $N_G(Q) \leqslant N_G(R_Q)$. A p-subgroup R is called *p-centric* if $Z(R)$ is a Sylow p-subgroup of $C_G(R)$, in which case $C_G(R) = Z(R) \times H$, with H a subgroup of order relatively prime to p. A *p-local* subgroup is the normalizer of a nontrivial p-subgroup.

A *collection* $\mathcal{C}$ of p-subgroups of G is a set of p-subgroups which is closed under conjugation; a collection is a G-poset under the inclusion relation with G acting by conjugation. The order complex or the nerve $|\mathcal{C}|$ is the simplicial complex which has as simplices proper inclusion chains in $\mathcal{C}$; the correspondence $\mathcal{C} \to |\mathcal{C}|$ allows assignment of topological concepts to posets [13, Section 1]. A collection $\mathcal{C}$ is contractible if $|\mathcal{C}|$ is contractible. A poset map is a G-homotopy equivalence if and only if the induced map on H-fixed points is a homotopy equivalence for all $H \leqslant G$; see [18, 1.3].

Notation 2.1. For $P \leqslant G$ a subgroup let $\mathcal{C}^P = \{Q \in \mathcal{C} \mid P \leqslant N_G(Q)\}$ denote the subcollection of $\mathcal{C}$ fixed under the action of P. Next $\mathcal{C}_{>P} = \{Q \in \mathcal{C} \mid P < Q\}$. Similarly define $\mathcal{C}_{\geqslant P}$ and also $\mathcal{C}_{<P}$ and $\mathcal{C}_{\leqslant P}$. We will also use the notation $\mathcal{C}_{>P}^{\leqslant H}$ for the set $\mathcal{C}_{>P}^{\leqslant H} = \{Q \in \mathcal{C} \mid P < Q \leqslant H\}$. The notation $F \geqslant Id_{\mathcal{C}}$ used below means that $F(P) \geqslant P$ for all $P \in \mathcal{C}$.

Theorem 2.2. *Let G be a finite group and $\mathcal{C} \subseteq \mathcal{D}$ two collections of subgroups.*

(1) [18, Proposition 1.7] *Assume that $\mathcal{D}_{>P}$ is $N_G(P)$-contractible for all $P \in \mathcal{D} \setminus \mathcal{C}$. Then the inclusion $\mathcal{C} \hookrightarrow \mathcal{D}$ is a G-homotopy equivalence.*
(2) [18, Theorem 1] *Assume either that $\mathcal{C}_{\geqslant P}$ is $N_G(P)$-contractible for all $P \in \mathcal{D}$, or that $\mathcal{C}_{\leqslant P}$ is $N_G(P)$-contractible for all $P \in \mathcal{D}$. Then the inclusion $\mathcal{C} \hookrightarrow \mathcal{D}$ is a G-homotopy equivalence.*
(3) [10, 2.2(3)] *Suppose that F is a G-equivariant poset endomorphism of $\mathcal{C}$ satisfying either $F \geqslant Id_{\mathcal{C}}$ or $F \leqslant Id_{\mathcal{C}}$. Then, for any collection $\mathcal{C}'$ containing the image of F, the inclusions $F(\mathcal{C}) \subseteq \mathcal{C}' \subseteq \mathcal{C}$ are G-homotopy equivalences.*
(4) [10, proof of Lemma 2.7(1)] *Let $\mathcal{C}$ be a collection of p-subgroups that is closed under passage to p-overgroups. Let H be an arbitrary p-subgroup in G. Then the inclusion $\mathcal{C}_{\geqslant H} \hookrightarrow \mathcal{C}^H$ is a homotopy equivalence.*

In what follows $\mathcal{A}_p(G)$ will denote the Quillen collection of nontrivial elementary abelian p-subgroups, $\mathcal{S}_p(G)$ the Brown collection of nontrivial p-subgroups and $\mathcal{B}_p(G)$ the Bouc collection of nontrivial p-radical subgroups. The inclusions $\mathcal{A}_p(G) \subseteq \mathcal{S}_p(G)$ and $\mathcal{B}_p(G) \subseteq \mathcal{S}_p(G)$ are G-homotopy equivalences [18, Theorem 2].

Let $\mathcal{C}e_p(G)$ denote the subcollection of $\mathcal{S}_p(G)$ consisting of nontrivial p-centric subgroups and let $\mathcal{B}_p^{\mathrm{cen}}(G) = \mathcal{C}e_p(G) \cap \mathcal{B}_p(G)$ be the collection of nontrivial p-radical and p-centric subgroups. These two collections are not in general homotopy equivalent with $\mathcal{S}_p(G)$; however the inclusion map $\mathcal{B}_p^{\mathrm{cen}}(G) \subseteq \mathcal{C}e_p(G)$ is a G-homotopy equivalence; see [10, Theorem 1.1].

Because certain upward-closed collections will be important in the paper, as suggested by part (4) of Theorem 2.2, we explicitly state the following property of the collection of p-centric subgroups.

2.3. The collection $\mathcal{C}e_p(G)$ is closed under passage to p-overgroups; this means that if $P \in \mathcal{C}e_p(G)$ and Q is a p-subgroup of G which contains P then $Q \in \mathcal{C}e_p(G)$.

Definition 2.4. The group G has *characteristic* p if $C_G(O_p(G)) \leqslant O_p(G)$. If all p-local subgroups of G have characteristic p then G has *local characteristic* p.

We remark here that our notion of “local characteristic p” is what group theorists usually call “characteristic p type”.

Proposition 2.5. *Assume G has characteristic p. If P is a p-subgroup of G and H a subgroup of G with $PC_G(P) \leqslant H \leqslant N_G(P)$, then H has characteristic p* [16, Lemma 1]. *In particular, G has local characteristic p* [8, 12.6].

Proposition 2.6. *Let G be a finite group.*

(1) [8, 5.12] *Let $P \leqslant G$ be a p-subgroup. Then $C_G(P)$ has characteristic p if and only if $N_G(P)$ has characteristic p.*
(2) [16] *Let Q be a p-subgroup of a finite group G, with $C_G(Q)$ of characteristic p. Let P be a p-subgroup of G containing Q. Then $C_G(P)$ has characteristic p.*

Definition 2.7. A *parabolic subgroup* of G is defined to be a subgroup which contains a Sylow p-subgroup of G. The group G has *parabolic characteristic* p if all p-local, parabolic subgroups of G have characteristic p.

We would like to mention that our usage of the term “parabolic” differs from the classical notion of parabolics, in the special case of a Lie type group over a field of characteristic p, for in the latter, the parabolics are actually the overgroups not of just a full unipotent group U, but of a full Borel subgroup, i.e. of the normalizer $N_G(U)$. Our notion of “parabolic characteristic p” appears elsewhere in group theory: notably for $p = 2$ it appears as “even characteristic” in Aschbacher and Smith [1, p. 3]; this provides further evidence for the naturality of this concept.

Remark 2.8. Examples of groups of local characteristic p, but not of “global” characteristic p, are the groups of Lie type defined over fields of characteristic p, some of the sporadic groups (such as M_{22}, M_{24}, Co_2 for $p = 2$, M_{11}, McL for $p = 3$, McL, Ly for $p = 5$), and any groups with self-centralizing Sylow p-subgroup of order p, such as $\mathrm{Alt}(p)$.

Remark 2.9. Any group of local characteristic p has parabolic characteristic p. Some examples of sporadic groups of parabolic characteristic p, but not of local characteristic p, are: M_{12}, J_2, Co_1 for $p = 2$, M_{12}, J_3, Co_1 for $p = 3$, J_2, Co_1, Co_2 for $p = 5$. Further examples of sporadic groups of parabolic characteristic p appear in Section 5.

3. Distinguished collections of p-subgroups

An element x of order p in G is *p-central* if x is in the center of a Sylow p-subgroup of G. Let $\Gamma_p(G)$ denote the family of p-central elements of G.

3.1. For a p-subgroup P of G define:

$$\widehat{P} = \langle x \mid x \in \Omega_1 Z(P) \cap \Gamma_p(G) \rangle.$$

Further, for $\mathcal{C}_p(G)$ a collection of p-subgroups of G denote:

$$\widehat{\mathcal{C}}_p(G) = \{ P \in \mathcal{C}_p(G) \mid \widehat{P} \neq 1 \}$$

the collection of subgroups in $\mathcal{C}_p(G)$ which contain p-central elements in their centers. We call $\widehat{\mathcal{C}}_p(G)$ the *distinguished $\mathcal{C}_p(G)$ collection*. We shall refer to the subgroups in $\widehat{\mathcal{C}}_p(G)$ as *distinguished subgroups*. Also, denote

$$\widetilde{\mathcal{C}}_p(G) = \{ P \in \mathcal{C}_p(G) \mid P \cap \Gamma_p(G) \neq \emptyset \}$$

the collection of subgroups in $\mathcal{C}_p(G)$ which contain a p-central element. Obviously $\widehat{\mathcal{C}}_p(G) \subseteq \widetilde{\mathcal{C}}_p(G) \subseteq \mathcal{C}_p(G)$ and notice that:

3.2. The collection $\widetilde{\mathcal{S}}_p(G)$ is closed under passage to p-overgroups.

Unlike $\widetilde{\mathcal{S}}_p(G)$, the distinguished collection $\widehat{\mathcal{S}}_p(G)$ is not necessarily closed under p-overgroups. But facts such as Lemma 3.3 are almost as useful in the proofs.

Lemma 3.3. *Let $P \in \mathcal{S}_p(G)$ and $Q \in \widehat{\mathcal{S}}_p(G)_{>P}$. Then $N_Q(P) \in \widehat{\mathcal{S}}_p(G)$.*

Proof. Since $P < Q$ it follows that $Z(Q) \leqslant N_Q(P)$ so that $Z(Q) \leqslant Z(N_Q(P))$ and $N_Q(P)$ is a distinguished p-subgroup of G. $\square$

Proposition 3.4. *All p-centric subgroups of G are distinguished, that is $\mathcal{C}e_p(G) \subseteq \widehat{\mathcal{S}}_p(G)$. Consequently the collection of distinguished p-radical subgroups contains the collection of p-centric and p-radical subgroups: $\mathcal{B}_p^{\text{cen}}(G) \subseteq \widehat{\mathcal{B}}_p(G)$.*

Proof. Let P be a centric p-subgroup of G and let S be any Sylow p-subgroup of G which contains P. Then $Z(S) \leqslant C_G(P)$ and so $Z(S) \leqslant Z(P)$. $\square$

Lemma 3.5. *Let $P \in \mathcal{S}_p(G)$ and assume that $N_G(P)$ has characteristic p. Then $C_G(O_p(N_G(P))) = Z(O_p(N_G(P)))$, and thus $O_p(N_G(P))$ is p-centric and distinguished.*

Proof. Since $P \leqslant O_p(N_G(P))$, we have $C_G(O_p(N_G(P))) \leqslant C_G(P) \leqslant N_G(P)$. Thus $C_G(O_p(N_G(P))) = C_{N_G(P)}(O_p(N_G(P))) \leqslant O_p(N_G(P))$, and so $C_G(O_p(N_G(P))) = Z(O_p(N_G(P)))$. Clearly $O_p(N_G(P))$ is p-centric, and Proposition 3.4 implies this group is distinguished. $\square$

Proposition 3.6. *Let G have local characteristic p. Then $\mathcal{B}_p^{\text{cen}}(G) = \mathcal{B}_p(G)$.*

Proof. Let $R \in \mathcal{B}_p(G)$, so that $R = O_p(N_G(R))$. Since $N_G(R)$ has characteristic p, it follows from 2.4 that $C_G(R) = Z(R)$ and so $R \in \mathcal{B}_p^{\text{cen}}(G)$. $\square$

Proposition 3.7. *Let G have parabolic characteristic p. Then:*

(a) *If $P \in \widetilde{\mathcal{S}}_p(G)$ then $N_G(P)$ has characteristic p.*
(b) $\mathcal{B}_p^{\text{cen}}(G) = \widehat{\mathcal{B}}_p(G) = \widetilde{\mathcal{B}}_p(G)$.

Proof. (a) Let $z \in P$ be a p-central element, so that $N_G(\langle z\rangle)$ contains a Sylow p-subgroup of G. Thus $N_G(\langle z\rangle)$ has characteristic p. By Proposition 2.6(1), $C_G(z)$ has characteristic p. By Proposition 2.6(2), $C_G(P)$ has characteristic p, and another application of Proposition 2.6(1) shows $N_G(P)$ has characteristic p.

(b) Note that $\mathcal{B}_p^{\text{cen}}(G) \subseteq \widehat{\mathcal{B}}_p(G) \subseteq \widetilde{\mathcal{B}}_p(G)$. Let $R \in \widetilde{\mathcal{B}}_p(G)$. Then $N_G(R)$ has characteristic p and so $C_G(R) = Z(R)$. Thus $R \in \mathcal{B}_p^{\text{cen}}(G)$. □

Remark 3.8. It follows from the above proposition that if G has parabolic characteristic p and $V \in \mathcal{B}_p(G) \setminus \widehat{\mathcal{B}}_p(G)$, then V does not contain any p-central elements.

In the next proposition (and in many later proofs) we will establish the homotopy equivalence between two collections by a sequence of poset maps, where usually the main part of the proof will be to establish that the indicated terms are actually in the stated collection. Each term in the sequence represents an equivariant poset map, and each inequality yields an equivariant homotopy equivalence via Theorem 2.2(3).

Proposition 3.9. *If G has parabolic characteristic p, then the collections $\widehat{\mathcal{B}}_p(G)$, $\widehat{\mathcal{A}}_p(G)$ and $\widehat{\mathcal{S}}_p(G)$ are G-homotopy equivalent.*

Proof. We first show that the inclusion map $\widehat{\mathcal{A}}_p(G) \hookrightarrow \widehat{\mathcal{S}}_p(G)$ is a G-homotopy equivalence. We attain this result by showing that $\widehat{\mathcal{A}}_p(G)_{\leqslant P}$ is $N_G(P)$-contractible for any $P \in \widehat{\mathcal{S}}_p(G)$ and then applying Theorem 2.2(2). Consider the string of poset maps given by $Q \leqslant Q\widehat{P} \geqslant \widehat{P}$. Here $Q \in \widehat{\mathcal{A}}_p(G)_{\leqslant P}$, and $\widehat{P}$ is defined in 3.1 to be a subgroup of $Z(P)$. Observe that $Q\widehat{P} \in \widehat{\mathcal{A}}_p(G)_{\leqslant P}$ as $\widehat{P} \leqslant Z(P)$, so $\widehat{P}$ centralizes Q. The $N_G(P)$-contractibility follows from the fact that the two inequalities correspond to poset maps which are $N_G(P)$-equivariant, for example $F(Q) = Q\widehat{P}$.

To show that $\widehat{B}_p(G)$ is G-homotopy equivalent to $\widehat{S}_p(G)$, we will use Theorem 2.2(1). Thus we have to prove that for each $P \in \widehat{\mathcal{S}}_p(G) \setminus \widehat{\mathcal{B}}_p(G)$, the subcollection $\widehat{\mathcal{S}}_p(G)_{>P}$ is $N_G(P)$-contractible. Denote $O_{NP} = O_p(N_G(P))$ and recall since $P \notin \mathcal{B}_p(G)$ that $P < O_{NP}$. Let $Q \in \widehat{\mathcal{S}}_p(G)_{>P}$, and consider the string of poset maps $\widehat{\mathcal{S}}_p(G)_{>P} \to \widehat{\mathcal{S}}_p(G)_{>P}$ given by:

$$Q \geqslant N_Q(P) \leqslant N_Q(P)O_{NP} \geqslant O_{NP}.$$

We have the poset maps $F_1(Q) = N_Q(P)$, $F_2(R) = R \cdot O_{NP}$ for R in the image of F_1, and $F_3(R) = O_{NP}$ for R in the image of F_2. Since $P < Q$, it follows that $P < N_Q(P) \leqslant N_G(P)$ and by Lemma 3.3, the subgroup $N_Q(P)$ is distinguished. As P is a distinguished p-subgroup, $N_G(P)$ has characteristic p, by Proposition 3.7(a). By Lemma 3.5, O_{NP} is p-centric and distinguished. As noted in 2.3, the collection of p-centric groups is closed under p-supergroups, and so $N_Q(P)O_{NP}$ is also p-centric and is distinguished by Proposition 3.4. □

Proposition 3.10. *Assume that G has parabolic characteristic p; then the inclusion $\widehat{\mathcal{S}}_p(G) \hookrightarrow \widetilde{\mathcal{S}}_p(G)$ is a G-homotopy equivalence.*

Proof. We will show that the subcollection $\widehat{\mathcal{S}}_p(G)_{\geqslant P}$ is $N_G(P)$-contractible for every $P \in \widetilde{\mathcal{S}}_p(G)$ and then apply Theorem 2.2(2). Denote by R_P the radical closure of P defined in Section 2. Let $Q \in \widehat{\mathcal{S}}_p(G)_{\geqslant P}$ and consider the string of $N_G(P)$-equivariant poset maps $\widehat{\mathcal{S}}_p(G)_{\geqslant P} \to \widehat{\mathcal{S}}_p(G)_{\geqslant P}$ given by:

$$Q \geqslant N_Q(P) \leqslant N_Q(P)R_P \geqslant R_P.$$

We have $P \leqslant N_Q(P) \leqslant Q$ and by Lemma 3.3, $N_Q(P)$ is a distinguished p-subgroup. Note that $N_Q(P)R_P$ is a subgroup in $N_G(R_P)$ since $N_Q(P) \leqslant N_G(P) \leqslant N_G(R_P)$. We have $P \in \widetilde{\mathcal{S}}_p(G)$, a collection closed under p-overgroups (3.2), so that $P \leqslant R_P$ implies that $R_P \in \widetilde{\mathcal{S}}_p(G)$ also. By Proposition 3.7(a), $N_G(R_P)$ has characteristic p and R_P is p-centric. Thus (see 2.3) the p-overgroup $N_Q(P)R_P$ is also p-centric, and both R_P and $N_Q(P)R_P$ are distinguished, by Proposition 3.4. □

Proposition 3.11. *The fixed point set $\widetilde{\mathcal{S}}_p^Z(G)$ is $N_G(Z)$-contractible whenever $Z = \langle z\rangle$ with z a p-central element in G.*

Proof. First note that $Z \in \widetilde{\mathcal{S}}_p^Z(G)$. If $P \in \widetilde{\mathcal{S}}_p^Z(G)$ then $PZ \in \widetilde{\mathcal{S}}_p^Z(G)$ since Z normalizes PZ. There is a contracting homotopy $P \leqslant PZ \geqslant Z$ via $N_G(Z)$-equivariant maps. □

Remark 3.12. According to Propositions 3.9 and 3.10, $\widehat{\mathcal{S}}_p(G), \widehat{\mathcal{A}}_p(G), \widehat{\mathcal{B}}_p(G)$ and $\widetilde{\mathcal{S}}_p(G)$ are G-homotopy equivalent, for G of parabolic characteristic p, and so under this condition $\widehat{\mathcal{S}}_p^Z(G)$, $\widehat{\mathcal{A}}_p^Z(G)$ and $\widehat{\mathcal{B}}_p^Z(G)$ are $N_G(Z)$-contractible as well.

Remark 3.13. If G is a group of Lie type in characteristic p, the building Δ is the simplicial complex associated to the poset of parabolic subgroups of G (that is the overgroups of the Borel subgroups of G). The building of G is equivariantly homotopy equivalent to the Quillen complex [13] and therefore to the Brown and the Bouc complexes as well; see [18] for example. Furthermore, it follows from Quillen's proof that the fixed point sets Δ^P are contractible for any p-subgroup P of G. A group of Lie type in characteristic p has local characteristic p (see 2.8) so by Proposition 3.6, $\mathcal{B}_p^{\mathrm{cen}}(G) = \mathcal{B}_p(G)$ and according to Proposition 3.7(b), $\mathcal{B}_p^{\mathrm{cen}}(G) = \widehat{\mathcal{B}}_p(G) = \widetilde{\mathcal{B}}_p(G)$. Finally, using Proposition 3.9, we conclude that the building Δ is also G-homotopy equivalent to $\widehat{\mathcal{S}}_p(G)$.

4. Fixed point sets for noncentral elements

We shall investigate the fixed point set of an element of order p of noncentral type; these are elements of order p in G which are not conjugate to any element in the center of a Sylow p-subgroup of G. Under certain hypotheses, we will prove that the fixed point set is equivariantly homotopy equivalent to the complex for a quotient of the centralizer. This will require a combination of nine homotopy equivalences.

Notation 4.1. Throughout this section, T will be a subgroup of order p of noncentral type in G. We will use the shorthand notations $C = C_G(T)$ and $O_C = O_p(C)$. The quotient group C/O_C will be denoted by $\overline{C}$; the quotient map is $q : C \to \overline{C}$. For $H \leqslant C$, let $\overline{H} = q(H)$. For $Q \leqslant C$, denote $O_Q = O_p(N_C(Q))$; for $\overline{Q} \leqslant \overline{C}$, denote $O_{\overline{Q}} = O_p(N_{\overline{C}}(\overline{Q}))$. Let $S_T \in Syl_p(C)$, and extend it to $S \in Syl_p(G)$. Since $T \leqslant S_T \leqslant S$, we have $Z(S) \leqslant C$; thus $Z(S) \leqslant S_T$ and in fact $Z(S) \leqslant Z(S_T)$. Note that $\overline{S_T} = q(S_T) \in Syl_p(\overline{C})$.

Remark 4.2. The proof of our main result, Theorem 4.14 will require the following hypotheses:

(1) G is a finite group of parabolic characteristic p;
(2) $C = C_G(T)$ does not have characteristic p;
(3) The quotient group $\overline{C} = C_G(T)/O_p(C_G(T))$ has parabolic characteristic p.

But most of our preliminary results will only specify a subset of these three hypotheses.

We first recall a result which is due to Grodal [9, pp. 420–421], see also Sawabe [15, Theorem 1]. For completeness we provide a proof.

Proposition 4.3. *(See proof of [9, Theorem 1.1].) Let $\mathcal{C}$ be a collection of nontrivial p-subgroups of G, which is closed under passage to p-overgroups. Let $Q \in \mathcal{S}_p(G)$. If $\mathcal{C}'$ is a collection satisfying: $\mathcal{C} \cap \mathcal{B}_p(G) \subseteq \mathcal{C}' \subseteq \mathcal{C}$ then $\mathcal{C}_{>Q}$ is $N_G(Q)$-homotopy equivalent to $\mathcal{C}'_{>Q}$.*

Proof. To simplify the notation, we shall denote by $\mathcal{X} = \mathcal{C}_{>Q}$ and by $\mathcal{Y} = \mathcal{C}'_{>Q}$. We will prove that $\mathcal{X}_{>P}$ is $N_G(P)$-contractible for all $P \in \mathcal{X} \setminus \mathcal{Y}$. Then, an application of Theorem 2.2(1) will give the result. Note that $\mathcal{X}_{>P} = \{R \in \mathcal{C} \mid P < R\}$. Denote $O_{NP} = O_p(N_G(P))$; since P is not p-radical, $P < O_{NP}$. Let $R \in \mathcal{X}_{>P}$ and consider the string of poset maps:

$$R \geqslant N_R(P) \leqslant N_R(P)O_{NP} \geqslant O_{NP}.$$

By elementary group theory, for $P < R$, we have $P < N_R(P) \leqslant R$. Since $\mathcal{C}$ is closed under passage to p-overgroups, the subgroups $N_R(P), N_R(P)O_{NP}$ and O_{NP} are also in $\mathcal{X}_{>P}$. In this way, we obtain a contracting homotopy given by the above string of $N_G(P)$-equivariant poset maps $\mathcal{X}_{>P} \to \mathcal{X}_{>P}$. Hence $\mathcal{X}_{>P}$ is $N_G(P)$-contractible. $\square$

Proposition 4.4. *Let G be a finite group of parabolic characteristic p. The inclusion $\widehat{\mathcal{S}}_p(G)_{>T}^{\leqslant C} \hookrightarrow \widehat{\mathcal{S}}_p^T(G)$ is a $N_G(T)$-homotopy equivalence.*

Proof. We verify the following chain of $N_G(T)$-homotopy equivalences:

$$\widehat{\mathcal{S}}_p^T(G) \simeq \widetilde{\mathcal{S}}_p^T(G) \simeq \widetilde{\mathcal{S}}_p(G)_{\geqslant T} = \widetilde{\mathcal{S}}_p(G)_{>T} \simeq \widehat{\mathcal{S}}_p(G)_{>T} \simeq \widehat{\mathcal{S}}_p(G)_{>T}^{\leqslant N_G(T)} = \widehat{\mathcal{S}}_p(G)_{>T}^{\leqslant C}.$$

The subcollections $\widehat{\mathcal{S}}_p^T(G)$ and $\widetilde{\mathcal{S}}_p^T(G)$ are $N_G(T)$-homotopy equivalent because $\widetilde{\mathcal{S}}_p(G)$ and $\widehat{\mathcal{S}}_p(G)$ are G-homotopy equivalent, as proved in Proposition 3.10. The next step follows by an application of Theorem 2.2(4), whose hypothesis is fulfilled by 3.2; the inclusion $\widetilde{\mathcal{S}}_p(G)_{\geqslant T} \hookrightarrow \widetilde{\mathcal{S}}_p^T(G)$ is a homotopy equivalence. Next, since $T \notin \widetilde{\mathcal{S}}_p(G)$ it follows that $\widetilde{\mathcal{S}}_p(G)_{\geqslant T} = \widetilde{\mathcal{S}}_p(G)_{>T}$. The homotopy equivalence between $\widetilde{\mathcal{S}}_p(G)_{>T}$ and $\widehat{\mathcal{S}}_p(G)_{>T}$ follows from an application of Proposition 4.3 with $\mathcal{C} = \widetilde{\mathcal{S}}_p(G)$ and $\mathcal{C}' = \widehat{\mathcal{S}}_p(G)$; note that Proposition 3.7(b) provides the necessary hypothesis $\mathcal{C} \cap \mathcal{B}_p(G) \subseteq \mathcal{C}'$. To see that $\widehat{\mathcal{S}}_p(G)_{>T}^{\leqslant N_G(T)} \hookrightarrow \widehat{\mathcal{S}}_p(G)_{>T}$ is a homotopy equivalence, consider the poset map $\widehat{\mathcal{S}}_p(G)_{>T} \to \widehat{\mathcal{S}}_p(G)_{>T}$ given by $Q \mapsto N_Q(T)$ whose image lies in $\widehat{\mathcal{S}}_p(G)_{>T}^{\leqslant N_G(T)}$ by Lemma 3.3, and apply Theorem 2.2(3). The final equality follows from $T \leqslant P \leqslant N_G(T)$ if and only if $T \leqslant P \leqslant C$, since T of order p normal in P implies that $T \leqslant Z(P)$. $\square$

Proposition 4.5. *Let G be a finite group of parabolic characteristic p. The inclusion $\mathcal{X} = \widehat{\mathcal{S}}_p(G)_{>H}^{\leqslant C} \hookrightarrow \mathcal{Y} = \widetilde{\mathcal{S}}_p(G)_{>H}^{\leqslant C}$ is a $N_G(T)$-homotopy equivalence, where H satisfies $T \leqslant H \leqslant C$ and $N_G(T) \leqslant N_G(H)$.*

Proof. Let $P \in \mathcal{Y}$; we will show that $\mathcal{X}_{\geqslant P}$ is equivariantly contractible and apply Theorem 2.2(2). Let R_P be the radical closure of P and let $Q \in \mathcal{X}_{\geqslant P}$. Now consider the string of poset maps given by:

$$Q \geqslant N_Q(P) \leqslant N_Q(P)Z(R_P) \geqslant P \cdot Z(R_P).$$

Note that $T < P \leqslant R_P$, so that $Z(R_P) \leqslant C$. By Proposition 3.7(b), R_P is p-centric and distinguished. This implies that $Z(R_P)$ is distinguished, and also $P \cdot Z(R_P)$ is distinguished, since $Z(R_P) \leqslant Z(P \cdot Z(R_P))$. Next, Lemma 3.3 implies that $N_Q(P)$ is distinguished. Observe that $N_Q(P) \leqslant N_G(P) \leqslant N_G(R_P)$, implying $N_Q(P)Z(R_P)$ is a group. Choose $S_R \in Syl_p(N_G(R_P))$ satisfying $N_Q(P) \leqslant S_R$, and extend it to $S \in Syl_p(G)$. Note $R_P \leqslant S_R \leqslant S$. Then $Z(S) \leqslant C_G(N_Q(P))$ and $Z(S) \leqslant C_G(R_P) = Z(R_P)$, by Proposition 3.7(a). This implies that $Z(S) \leqslant Z(N_Q(P)Z(R_P))$, so that $N_Q(P)Z(R_P)$ is distinguished. Thus the above is a string of equivariant poset maps $\mathcal{X}_{\geqslant P} \to \mathcal{X}_{\geqslant P}$, which proves the equivariant contractibility of the subcollection $\mathcal{X}_{\geqslant P}$. $\square$

Lemma 4.6. *Let G be a finite group of parabolic characteristic p. Then $O_C \in \widetilde{\mathcal{S}}_p(G)$ if and only if C has characteristic p.*

Proof. If $O_C \in \widetilde{\mathcal{S}}_p(G)$ then $N_G(O_C)$ has characteristic p, by Proposition 3.7(a). As $T \leqslant O_C \trianglelefteq C \leqslant N_G(O_C)$, it follows that $C_{N_G(O_C)}(T) = C_G(T) = C$ and so $C = T \cdot C_{N_G(O_C)}(T)$. Thus C has characteristic p, by Proposition 2.5. Conversely, assume that C has characteristic p. Note $T \leqslant O_C$, so $C_G(O_C) \leqslant C$. Thus $C_G(O_C) = C_C(O_C) \leqslant O_C$, and so O_C is p-centric. Thus $O_C \in \widehat{\mathcal{S}}_p(G) \subseteq \widetilde{\mathcal{S}}_p(G)$, according to Proposition 3.4. $\square$

Note that if $O_C \in \widetilde{\mathcal{S}}_p(G)$, then $O_C \in \widehat{\mathcal{S}}_p(G)$. This may be compared with 3.8 concerning p-radical subgroups of a group G having parabolic characteristic p.

Proposition 4.7. *Let G be a finite group of parabolic characteristic p, and assume that $O_C \in \widetilde{\mathcal{S}}_p(G)$. Then the fixed point set $\widehat{\mathcal{S}}_p(G)^T$ is contractible.*

Proof. Consider the poset map $\varphi : \widetilde{\mathcal{S}}_p(G)^{\leqslant C}_{>T} \to \widetilde{\mathcal{S}}_p(G)^{\leqslant C}_{>T}$ given by $\varphi(P) = P \cdot O_C$. By hypothesis, O_C contains p-central elements and the poset map φ has image equal to $\widetilde{\mathcal{S}}_p(G)^{\leqslant C}_{\geqslant O_C}$; this is contractible, a cone on O_C. Apply Theorem 2.2(3) to conclude $\widetilde{\mathcal{S}}_p(G)^{\leqslant C}_{>T}$ is contractible. Apply Proposition 4.5, with T in the role of H, to conclude that also $\widehat{\mathcal{S}}_p(G)^{\leqslant C}_{>T}$ is contractible. Finally, apply Proposition 4.4 to conclude that $\widehat{\mathcal{S}}^T_p(G)$ is contractible. □

Proposition 4.8. *Let G be a finite group of parabolic characteristic p, and assume that O_C contains no p-central elements. The inclusion $\widetilde{\mathcal{S}}_p(G)^{\leqslant C}_{>O_C} \hookrightarrow \widetilde{\mathcal{S}}_p(G)^{\leqslant C}_{>T}$ is a $N_G(T)$-homotopy equivalence.*

Proof. The poset map $\varphi : \widetilde{\mathcal{S}}_p(G)^{\leqslant C}_{>T} \to \widetilde{\mathcal{S}}_p(G)^{\leqslant C}_{>T}$ given by $\varphi(P) = P \cdot O_C$ now has image in $\widetilde{\mathcal{S}}_p(G)^{\leqslant C}_{>O_C}$, using 3.2 and that O_C is purely noncentral. The result follows by an application of Theorem 2.2(3). □

There is no obvious relationship among those elements which are p-central in G, or in C, or in $\overline{C}$. In order to overcome this difficulty, we define a subcollection $\mathfrak{S}$ of $\widehat{\mathcal{S}}_p(G)^{\leqslant C}_{>O_C}$ as follows:

Definition 4.9. Assume the notation from 4.1 and set:

$$\mathfrak{S} = \left\{ P \in \widehat{\mathcal{S}}_p(G)^{\leqslant C}_{>O_C} \;\middle|\; Z(P) \cap Z(S) \neq 1, \text{ for some } S_T \text{ and } S \text{ with } P \leqslant S_T \leqslant S \right\}.$$

Note that the condition for $\mathfrak{S}$ is stronger than "distinguished", so that the definition can equivalently be stated for any p-subgroup P with $O_C < P \leqslant C$.

4.10. The subcollection $\mathfrak{S}$ is also contained in $\widehat{\mathcal{S}}_p(C)$, and it contains all subgroups P of C which are p-centric in G and properly contain O_C.

The above is true since if $P \leqslant S_T \leqslant S$, then $Z(S) \leqslant C_G(P)$, implying $Z(S) \leqslant Z(P)$.

The following fact can be shown to be true by an underlying argument similar to the one used in Lemma 3.3:

4.11. Let P be in $\mathfrak{S}$ and let $Q \leqslant P$. Then $N_P(Q)$ is in $\mathfrak{S}$.

Proposition 4.12. *Let G be a finite group of parabolic characteristic p. The inclusion $\mathfrak{S} \hookrightarrow \widehat{\mathcal{S}}_p(G)^{\leqslant C}_{>O_C}$ is an $N_G(T)$-homotopy equivalence.*

Proof. We will apply Theorem 2.2(2) once again. We need to show that $\mathfrak{S}_{\geqslant Q}$ is equivariantly contractible whenever $Q \in \widehat{\mathcal{S}}_p(G)^{\leqslant C}_{>O_C}$. Set $O_Q = O_p(N_C(Q))$, assume that $P \in \mathfrak{S}_{\geqslant Q}$ and consider the contracting homotopy given by the following string of equivariant poset maps:

$$P \geqslant N_P(Q) \leqslant N_P(Q)O_Q \geqslant O_Q.$$

It remains to show that each of these subgroups lies in $\mathfrak{S}_{\geqslant Q}$. Since Q is a distinguished p-subgroup of G, $N_G(Q)$ has characteristic p, by Proposition 3.7(a). Also $QC_G(Q) \leqslant N_C(Q) \leqslant N_G(Q)$ since $T < Q \leqslant C$; by Proposition 2.5, it follows that $N_C(Q)$ has characteristic p. Next, according to Lemma 3.5, with C in the role of G, it follows that O_Q is p-centric in C. Now just observe that $C_G(O_Q) \leqslant C_G(T) = C$ so that O_Q is then also p-centric in G. For $P \leqslant C$, $N_P(Q)O_Q$ is a group which is also

p-centric, according to 2.3. Thus O_Q and $N_P(Q)O_Q$ lie in $\mathfrak{S}$, according to 4.10. Finally, by 4.11, it follows that $N_P(Q)$ lies in $\mathfrak{S}_{\geqslant Q}$. This concludes the proof of the proposition. □

Proposition 4.13. *Let G be a finite group of parabolic characteristic p, and assume that C does not have characteristic p. Also assume that $\overline{C}$ has parabolic characteristic p. Then the map $q_* : \mathfrak{S} \to \widehat{\mathcal{S}}_p(\overline{C})$ induced by the quotient map $q : C \to \overline{C}$ is a homotopy equivalence.*

Proof. To see that $q_*(\mathfrak{S}) \subseteq \widehat{\mathcal{S}}_p(\overline{C})$, let $P \in \mathfrak{S}$. Recall that $Z(S) \leqslant Z(S_T)$ and $\overline{S_T} \in Syl_p(\overline{C})$. Thus $\overline{Z(S)} \leqslant \overline{Z(S_T)} \leqslant Z(\overline{S_T})$. Since O_C is purely noncentral (Lemma 4.6), the map $q : C \to \overline{C}$ is injective on the elements of $Z(S)$ as they are p-central. Therefore $Z(S) \cap Z(P) \neq 1$ implies $Z(\overline{S_T}) \cap Z(\overline{P}) \neq 1$, and we have $q_*(P) = \overline{P} \in \widehat{\mathcal{S}}_p(\overline{C})$.

According to a result of Thévenaz and Webb [18, Theorem 1], the poset map $q_* : \mathfrak{S} \to \widehat{\mathcal{S}}_p(\overline{C})$ is an equivariant homotopy equivalence if $q_*^{-1}(\widehat{\mathcal{S}}_p(\overline{C})_{\geqslant \overline{Q}})$ is equivariantly contractible for any $\overline{Q} \in \widehat{\mathcal{S}}_p(\overline{C})$. Define $Q = q^{-1}(\overline{Q})$. Then $q_*^{-1}(\widehat{\mathcal{S}}_p(\overline{C})_{\geqslant \overline{Q}}) = \{P \in \mathfrak{S} \mid \overline{Q} \leqslant \overline{P}\} = \{P \in \mathfrak{S} \mid Q \leqslant P\}$, since $O_C \leqslant P$. This latter set is just $\mathfrak{S}_{\geqslant Q}$. Given $P \in \mathfrak{S}_{\geqslant Q}$, consider the string of equivariant poset maps $\mathfrak{S}_{\geqslant Q} \to \mathfrak{S}_{\geqslant Q}$ given by:

$$P \geqslant N_P(Q) \leqslant N_P(Q)O_Q \geqslant O_Q.$$

We need to show that all of these terms lie in $\mathfrak{S}_{\geqslant Q}$. Note that $N_P(Q) \in \mathfrak{S}$, by 4.11. Next, we have $N_C(Q) = q^{-1}(N_{\overline{C}}(\overline{Q}))$, using $O_C \leqslant Q$. Thus $O_Q := O_p(N_C(Q))$ is equal to $q^{-1}(O_{\overline{Q}})$ by the correspondence theorem for normal subgroups applied to $N_C(Q) \to N_{\overline{C}}(\overline{Q})$. Since $\overline{Q} \in \widehat{\mathcal{S}}_p(\overline{C})$, $N_{\overline{C}}(\overline{Q})$ has characteristic p, by Proposition 3.7(a). Then $C_{\overline{C}}(O_{\overline{Q}}) \leqslant O_{\overline{Q}}$. Since $T \leqslant Q \leqslant O_Q$, $C_G(O_Q) \leqslant C_G(T) = C$ and so $C_G(O_Q) = C_C(O_Q)$. Therefore $C_G(O_Q) = C_C(O_Q) \leqslant q^{-1}(C_{\overline{C}}(O_{\overline{Q}})) \leqslant q^{-1}(O_{\overline{Q}}) = O_Q$. The group O_Q is p-centric in G. According to 4.10, $\mathfrak{S}$ contains all the subgroups of C, properly containing O_C, which are p-centric in G. The p-overgroup $N_P(Q)O_Q$ is also p-centric, by 2.3 and similarly by 4.10 lies in $\mathfrak{S}$. □

We can now state the main result of this section.

Theorem 4.14. *Maintain the notation in* 4.1 *and the hypotheses in* 4.2. *There is an $N_G(T)$-equivariant homotopy equivalence $\widehat{\mathcal{S}}_p^T(G) \simeq \widehat{\mathcal{S}}_p(\overline{C})$.*

Proof. We have the chain of $N_G(T)$-homotopy equivalences:

$$\widehat{\mathcal{S}}_p^T(G) \simeq \widehat{\mathcal{S}}_p(G)_{>T}^{\leqslant C} \simeq \widetilde{\mathcal{S}}_p(G)_{>T}^{\leqslant C} \simeq \widetilde{\mathcal{S}}_p(G)_{>O_C}^{\leqslant C} \simeq \widehat{\mathcal{S}}_p(G)_{>O_C}^{\leqslant C} \simeq \mathfrak{S} \simeq \widehat{\mathcal{S}}_p(\overline{C}).$$

The first step is Proposition 4.4; then apply Proposition 4.5 with $H = T$. Next, use Proposition 4.8 (and Lemma 4.6), and then Proposition 4.5 again with $H = O_C$. Finally, a combination of Propositions 4.12 and 4.13 completes the proof of the theorem. □

5. Examples and Lefschetz modules

We will discuss three examples, and give an application to modular representation theory. Recall that if a group G acts on a simplicial complex Δ, we can construct the virtual Lefschetz module by taking the alternating sum of the vector spaces (over a field of characteristic p) spanned by the chains. To obtain the reduced Lefschetz module $\widetilde{L}_G(\Delta)$, subtract the trivial one-dimensional representation. Information about fixed point sets leads to details about the vertices of irreducible summands of this module.

A theorem due to Burry and Carlson [3, Theorem 5] and to Puig [12] was applied by Robinson [14, in Corollary 3.2] to Lefschetz modules to obtain the following result.

Lemma 5.1. *The number of indecomposable summands of $\widetilde{L}_G(\Delta)$ with vertex Q is equal to the number of indecomposable summands of $\widetilde{L}_{N_G(Q)}(\Delta^Q)$ with the same vertex Q.*

This result, also see [15, Lemma 1], reproves the Green correspondence, and the relationship to the Brauer correspondence permits a conclusion regarding the blocks in which the summands lie.

The Fischer group Fi_{22} and $p = 2$

We begin with the sporadic simple Fischer group Fi_{22}, which has parabolic characteristic 2 and has three conjugacy classes of involutions, denoted $2A, 2B$ and $2C$ in the Atlas [4]. The class $2B$ is 2-central. Their centralizers are $C_{Fi_{22}}(2A) = 2U_6(2)$, $C_{Fi_{22}}(2B) = (2 \times 2_+^{1+8} : U_4(2)) : 2$ and $C_{Fi_{22}}(2C) = 2^{5+8} : (S_3 \times 3^2 : 4)$.

Let Δ be the 2-local geometry studied in [2, Sections 7.16 and 8.4] for the Fischer group Fi_{22}. The vertex stabilizers are the following maximal 2-local subgroups of Fi_{22}:

$$H_1 = \left(2 \times 2_+^{1+8} : U_4(2)\right) : 2, \qquad H_2 = 2^{5+8} : (S_3 \times A_6),$$
$$H_3 = 2^6 : Sp_6(2), \qquad H_4 = 2^{10} : M_{22}.$$

The flag stabilizers are listed below:

$$H_{1,2} = 2^{5+8} : (S_3 \times S_4), \qquad H_{1,2,3} = 2^6 : 2^5 : (2 \times S_4),$$
$$H_{1,3} = 2^6 : 2^5 : S_6, \qquad H_{1,2,4} = 2^{5+8} : (2 \times S_4),$$
$$H_{1,4} = 2^{10} : 2^4 : S_5, \qquad H_{1,3,4} = 2^6 : 2^5 : (2 \times S_4),$$
$$H_{2,3} = 2^6 : 2^{1+6} : (S_3 \times S_3), \qquad H_{2,3,4} = 2^6 : 2^{1+6} : (S_3 \times 2),$$
$$H_{2,4} = 2^{5+8} : (2 \times A_6), \qquad H_{1,2,3,4} = 2^6 : 2^{1+6} : 2^2,$$
$$H_{3,4} = 2^6 : 2^6 : L_3(2).$$

The geometry Δ is G-homotopy equivalent to $\mathcal{B}_2^{\text{cen}}(Fi_{22})$, and since Fi_{22} has parabolic characteristic 2 and using Proposition 3.7(b), this is equal to the distinguished collection $\widehat{\mathcal{B}}_2(Fi_{22})$; for details we refer the reader to Benson and Smith [2, Sections 7.16 and 8.4].

We shall use the notation from the Modular Atlas homepage [7], where φ_i denotes an irreducible module of Fi_{22} and $P_{Fi_{22}}(\varphi_i)$ is its corresponding projective cover.

Proposition 5.2. *Let Δ be the 2-local geometry for the Fischer group Fi_{22}.*

(a) *The reduced Lefschetz module, as an element of the Green ring, is*

$$\widetilde{L}_{Fi_{22}}(\Delta) = -P_{Fi_{22}}(\varphi_{12}) - P_{Fi_{22}}(\varphi_{13}) - 6\varphi_{15} - 12P_{Fi_{22}}(\varphi_{16}) - \varphi_{16}.$$

(b) *The fixed point sets Δ^{2B} and Δ^{2C} are contractible.*
(c) *The fixed point set Δ^{2A} is equivariantly homotopy equivalent to the building for the Lie group $U_6(2)$.*
(d) *There is precisely one nonprojective summand of the reduced Lefschetz module, it has vertex $\langle 2A \rangle$ and lies in a block with the same group as defect group.*

Proof. (a) The alternating sum of the induced characters was computed using GAP [6], and the character corresponding to the formula for $\widetilde{L}_{Fi_{22}}(\Delta)$ given above was obtained. The eight terms lying in $H_3 = 2^6 : Sp_6(2)$ can be combined to yield the character of the inflation of the Steinberg module for the symplectic group.

The first two terms of the reduced Lefschetz module, the projective covers of φ_{12} and φ_{13}, lie in the principal block. Also φ_{15} is projective and lies in block 2, with defect zero. Next, φ_{16} is not projective, and lies in block 3 with defect one. This formula will be shown to be valid at the level of the Green ring of virtual modules, and not just the Grothendieck ring of characters.

(b) Recall that $\Delta \simeq \widehat{\mathcal{B}}_2(Fi_{22})$, and by Proposition 3.9 the latter is equivariantly homotopy equivalent to $\widehat{\mathcal{S}}_2(Fi_{22})$. Thus we may apply to Δ the results 3.11, 4.7 and 4.14 about fixed points of $\widehat{\mathcal{S}}_2(Fi_{22})$. Proposition 3.11 (also see Remark 3.13) tells us that the fixed point set Δ^{2B} is contractible. The contractibility of Δ^{2B} implies that Δ^{Q} is mod 2 acyclic for any 2-group Q containing an involution of type $2B$ (by Smith theory), and thus the reduced Lefschetz module $\widetilde{L}_{N_G(Q)}(\Delta^Q) = 0$. Lemma 5.1 implies that the vertices of the indecomposable summands of $\widetilde{L}_{Fi_{22}}(\Delta)$ do not contain any 2-central involutions.

Proposition 4.7 applies to the element $2C$ since the subgroup $O_C = O_2(C_{Fi_{22}}(2C))$ contains 2-central elements. The elementary abelian $2^5 \leqslant 2^{5+8}$ is of the type $2^5 = 2A_6B_{15}C_{10}$, containing 6 involutions in the conjugacy class A, 15 involutions of type B, and 10 involutions of type C. There is a purely 2-central $2^4 \leqslant 2^5$. Note that Δ^{2C} being contractible implies that no vertex of a summand of $\widetilde{L}_{Fi_{22}}(\Delta)$ contains an involution of type $2C$.

(c) Theorem 4.14 applies to the fixed point set Δ^{2A}, since the quotient group of the centralizer $C_{Fi_{22}}(2A)/O_2(C_{Fi_{22}}(2A)) = U_6(2)$ is a Lie group defined over a field of characteristic 2. Therefore Δ^{2A} is homotopy equivalent to the building for $U_6(2)$ (recall 3.13). It is a consequence of the Solomon–Tits theorem that the reduced Lefschetz module associated to this action on the building is the irreducible Steinberg module. The vertex under the action of $2U_6(2)$ is $\langle 2A\rangle$. Thus there exists one indecomposable summand of the Lefschetz module with vertex $2 = \langle 2A\rangle$, lying in a block with the same group as defect group (block 3 with defect one). Note that since this block has cyclic defect group of order two, there is only one nonprojective indecomposable module lying in this block, namely the irreducible module φ_{16}. Also note that since Δ^{2A} is homotopy equivalent to a building, Δ^Q will be contractible for any 2-group Q (again recall 3.13) of order at least four which contains an involution of type $2A$. This implies that such a group Q cannot be a vertex of a summand of $\widetilde{L}_{Fi_{22}}(\Delta)$.

(d) It follows from the previous steps that there is precisely one nonprojective summand of the reduced Lefschetz module; it has vertex $\langle 2A\rangle$ and it lies in a block with the same group as defect group. The remaining summands are projective and are determined by their characters, and so the formula for $\widetilde{L}_{Fi_{22}}(\Delta)$ is valid at the level of the Green ring of virtual modules, and not just the Grothendieck ring of characters. $\square$

The Conway group Co_3 and $p = 3$

The group Co_3 has parabolic characteristic 3, and it has three conjugacy classes of elements of order three, denoted $3A$, $3B$ and $3C$, with the 3-central elements being those of type $3A$. The normalizers are $N_{Co_3}(\langle 3A\rangle) = 3_+^{1+4} : 4S_6$, $N_{Co_3}(\langle 3B\rangle) = 3^5(2 \times S_5)$ and $N_{Co_3}(\langle 3C\rangle) = S_3 \times L_2(8) : 3$. Note that $L_2(8) : 3 = Ree(3) = {}^2G_2(3)$ is a twisted group of Lie type having local characteristic 3.

Proposition 5.3. *Let Δ be the subgroup complex associated to the distinguished 3-radical collection $\widehat{\mathcal{B}}_3(Co_3)$.*

(a) *The fixed point sets Δ^{3A} and Δ^{3B} are contractible, and Δ^{3C} is equivariantly homotopy equivalent to the building for ${}^2G_2(3)$.*
(b) *The reduced Lefschetz module $\widetilde{L}_{Co_3}(\Delta)$ has precisely one nonprojective irreducible summand, which has vertex $\langle 3C\rangle$ and lies in a block with the same group as defect group.*

Proof. As in the previous example, by Proposition 3.9 we may apply to $\Delta = \widehat{\mathcal{B}}_3(Co_3)$ the fixed point results in 3.11, 4.7 and 4.14 stated for $\widehat{\mathcal{S}}_3(Co_3)$. Proposition 3.11 (and Remark 3.13) implies that the fixed point set Δ^{3A} of the 3-central element is contractible. Proposition 4.7 implies that Δ^{3B} is contractible since $O_C = O_3(C_{Co_3}(3B)) = 3^5 = 3A_{55}B_{66}$ contains 3-central elements. The elementary abelian group $3A_{55}B_{66}$ contains 55 cyclic subgroups of order three with both nonidentity elements of type A, and 66 cyclic subgroups with both nonidentity elements of type B. The reduced Lefschetz module $\widetilde{L}_{Co_3}(\Delta)$ has no summands with a vertex containing either an element of type $3A$ or an element of type $3B$, by a similar argument to the one in part (b) of the previous proposition.

Theorem 4.14 applies to Δ^{3C}. The reduced Lefschetz module for the ${}^2G_2(3)$ building is the irreducible Steinberg module (recall 3.13), and under the action of $N_{Co_3}(\langle 3C\rangle)$, the vertex is $\langle 3C\rangle$. Also, since Δ^{3C} is equivariantly homotopy equivalent to a building, Δ^Q will be contractible for any 3-group Q of order at least nine which contains an element of type $3C$. This implies that such a group Q cannot be a vertex of a summand of $\widetilde{L}_{Co_3}(\Delta)$. Lemma 5.1 implies now the result of part (b). $\square$

The Harada–Norton group HN and $p = 5$

The group HN has parabolic characteristic 5 and has five conjugacy classes of elements of order 5, denoted $5A$, $5B$, $5C$, $5D$ and $5E$; see [4]. The elements of type $5B$ are 5-central. An element of type $5D$ is the square of an element of type $5C$, so either one generates the same group of order 5 denoted $5CD$. The centralizers are $C_{HN}(5A) = 5 \times U_3(5)$, $C_{HN}(5B) = 5^{1+4}.2^{1+4}.5$, $C_{HN}(5CD) = 5^3.SL_2(5)$ and $C_{HN}(5E) = 5 \times 5^{1+2} : 2^2$.

Proposition 5.4. *Let Δ be the subgroup complex associated to the distinguished 5-radical collection $\widehat{\mathcal{B}}_5(HN)$.*

(a) *The fixed point sets Δ^{5B}, Δ^{5CD} and Δ^{5E} are contractible, and Δ^{5A} is equivariantly homotopy equivalent to the building for $U_3(5)$.*
(b) *The reduced Lefschetz module $\widetilde{L}_{HN}(\Delta)$ has precisely one nonprojective summand, which has vertex $\langle 5A \rangle$ and lies in a block with the same group as defect group.*

Proof. By Proposition 3.9 we may apply 3.11, 4.7 and 4.14 to Δ; by 3.13, the collection in 4.14 translates to the building of $\overline{C}$. Proposition 3.11 (and Remark 3.13) applies to the 5-central elements of type $5B$, so that the fixed point set Δ^{5B} is contractible. The two groups $5^3 = B_6(CD)_{25}$ and $5 \times 5^{1+2}$ contain 5-central elements; the center $Z(5^{1+2}) = \langle 5B \rangle$, and $B_6(CD)_{25}$ contains 6 cyclic subgroups of order five with all nonidentity elements of type B (and 25 cyclic subgroups with two elements of type C and two elements of type D). According to Proposition 4.7 the fixed point sets Δ^{5CD} and Δ^{5E} are both contractible. Thus the reduced Lefschetz module $\widetilde{L}_{HN}(\Delta)$ has no summands with a vertex containing elements of type $5B$, $5C$, $5D$ or $5E$.

Theorem 4.14 applies to the elements of type $5A$, so that Δ^{5A} is equivariantly homotopy equivalent to the building for $U_3(5)$. The reduced Lefschetz module for this building is the irreducible Steinberg module, and under the action of $N_{HN}(\langle 5A \rangle)$, the vertex is $\langle 5A \rangle$. Thus $\widetilde{L}_{HN}(\Delta)$ has one summand with vertex $\langle 5A \rangle$ lying in a block with the same group as defect group. Since Δ^{5A} is homotopy equivalent to a building, Δ^Q will be contractible for any 5-group Q of order at least 25 which contains an element of type $5A$. This implies that such a group Q cannot be a vertex of a summand of $\widetilde{L}_{HN}(\Delta)$. □

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