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Isoperimetric functions for graph products[☆]

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Abstract

Let Γ be a finite graph, and for each vertex i let G_i be a finitely presented group. Let G be the graph product of the G_i. That is, G is the group obtained from the free product of the G_i by factoring out by the smallest normal subgroup containing all $[g, h]$ where $g \in G_i$ and $h \in G_j$ and there is an edge joining i and j. We show that G has an isoperimetric function of degree $k > 1$ (or an exponential isoperimetric function) if each vertex group has such an isoperimetric function.

1. Graph products

Let Γ be a finite graph; that is, Γ consists of a finite set of vertices and a finite set of edges, where each edge is an unordered pair of vertices. We call two vertices adjacent if there is an edge joining them. Let us be given a group G_i for each vertex i. Then the *graph product* G of the G_i is the group obtained from the free product of the G_i by factoring out by the smallest normal subgroup containing all $[g, h]$ where $g \in G_i$ and $h \in G_j$ and i and j are adjacent. Note that if G_i has presentation $\langle A_i; R_i \rangle$, where the A_i are disjoint, then G has presentation $\langle \bigcup A_i; \bigcup R_i \cup S \rangle$, where S is the set of commutators $[a, b]$ for $a \in A_i, b \in A_j$, where i and j are adjacent. The free product and the direct product are examples of graph products (corresponding to graphs with no edges and complete graphs, respectively). All groups considered will be finitely presented.

Gersten [3] defines an *isoperimetric function* for a finite presentation $\langle Y; S \rangle$ of a group H to be a function f with $f0 = 0$ such that if w is a word of length at most n in the free group on Y and w equals 1 in H then w is the product of at most $f(n)$ conjugates of elements of S and their inverses. Brick [2] shows that if we change to another finite presentation then there are positive constants a,b,c such that the new

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presentation has an isoperimetric function g given by $g(n) = af(bn) + cn$. (Gersten shows an equivalent result, but states it in a form which is at first sight weaker.)

Consequently, we say that $g \preceq f$ if there are positive constants a,b,c such that $g(n) \leq af(bn) + cn$ for all n, and we call g equivalent to f if we have both $g \preceq f$ and $f \preceq g$. This is Brick's definition, which is slightly different from Gersten's. I prefer this definition because it makes all polynomials of a given degree equivalent, and also makes all exponentials equivalent.

Gersten also defines the function f_1 to be an *isodiametric function* for the presentation if a word w of length at most n which equals 1 in the group is, as before, the product of conjugates of the relators and their inverses with the conjugating elements having length at most $f_1(n)$. We may also say that f and f_1 are simultaneous isoperimetric and isodiametric functions if each such w can be expressed a product of conjugates satisfying both conditions (as distinct from having two different expressions, each satisfying one condition).

When the free monoid Y^* maps onto H (rather than just the free group on Y mapping onto H) we say that Y is a set of *monoid generators* of H. It is particularly useful if Y has the property that to each $y \in Y$ there is $\bar{y} \in Y$ such that $y\bar{y}$ equals 1 in H; such a Y will be called a set of monoid generators in the stronger sense. When this happens, it is easy to see that we can find a set of defining relators containing all the elements $y\bar{y}$ and lying in Y^*. It is also easy to check that any finite presentation can be changed to a finite presentation of this sort, and that, in looking for an isoperimetric or isodiametric function, we need only consider elements of Y^* and not general elements of the free group on Y. In this paper we prove the following theorem.

Theorem. *If each vertex group has an isoperimetric function which is polynomial of degree $k > 1$ (or an exponential isoperimetric function) then so does their graph product. It each vertex group has an isodiametric function which is polynomial of degree $h > 0$ (or an exponential isodiametric function) then so does their graph product. If the functions are simultaneous isoperimetric and isodiametric functions for the vertex groups then the same holds for the graph product.*

The theorem will also hold for other classes of isoperimetric functions (this follows immediately from the proof), but the precise condition is messy and these two cases are the most important. One requirement is that the function is at least quadratic. When this holds, it is sufficient that the equivalence class contains a function f such that $f(m + n) \leq f(m) + f(n)$ for all m and n (Brick calls such a function *subnegative*, and he remarks, without giving an example, that there are equivalence classes which contain no subnegative function). Ol'shankii [9] has proved that groups whose isoperimetric function is subquadratic are hyperbolic and hence have linear isoperimetric function. Note that the graph product of groups with a linear isoperimetric function usually does not have a linear isoperimetric function.

For an isodiametric function f_1 the only restriction is that f_1 is of order of magnitude at least n.

The inspiration for this paper came from work on graph products by Hermiller and Meier [5]. Their discussion of normal forms in graph products, and a similar discussion by Laurence [7], led me to the approach given here. The same theorem has been proved by Meier [8] by different methods.

In proving the theorem we may take any finite presentations of the vertex groups. It will be convenient to take the A_i to be disjoint finite sets which are monoid generators (in the stronger sense) of G_i, so that there is a homomorphism $\pi_i : A_i^* \to G_i$.

A non-trivial element of A_i^* will be called an i-word. To each i-word u we take a symbol $[u]$. Let X be the set of all such symbols. Then there is a homomorphism ρ from X^* onto G which sends $[u]$ to $\pi_i u$ when u is an i-word. An element of X^* will just be called a *word*. We say that $[u]$ is *in the link of* i if u is a j-word where i and j are adjacent. We say that the word W is in the link of i if W is $[u_1] \cdots [u_m]$ with each of $[u_1], \dots, [u_m]$ in the link of i.

A sequence of words $W_1, \dots, W_n = \varepsilon$, where ε is the empty word, will be called a *reduction sequence* if, for all $m < n$, W_{m+1} is obtained from W_m by one of the following moves:

(1) replace $P[u][v]Q$ by $P[uv]Q$, for any words P,Q and, for any i, any i-words u and v;

(2) replace $P[u]Q[v]T$ by $P[uv]QT$ for any words P, T, any i-words u,v, any i, and any word Q in the link of i;

(3) replace $P[u]Q$ by PQ for any words P,Q, any i, and any i-word u such that $\pi_i u = 1$.

We refer to *i-moves* if there is a need to mention i explicitly.

The following lemma will be proved in Section 2.

Lemma. *If $\rho W = 1$ then there is a reduction sequence starting with W.*

Let $W = W_1, \dots, W_n = \varepsilon$ be a reduction sequence. We show how to replace it by another reduction sequence with nice properties.

Since the sequence ends with ε, a move of type 3 must be used at some point. Let the first such move be an i-move, going from W_m to W_{m+1}. Since all earlier moves are of types 1 and 2, it is easy to check that, in the sequence $W_1, \dots, W_{m+1}$, a j-move followed by an i-move can be replaced by an i-move followed by a j-move. Thus we may assume that each of the first m moves is an i-move.

We can now see easily (by induction, looking at the reduction sequence beginning with W_2) that W must be of the form $P[u_1]Q_1[u_2] \dots Q_{r-1}[u_r]P'$ for some r, where $u_1, \dots, u_r$ are i-words, $Q_1, \dots, Q_{r-1}$ are (possibly empty) words in the link of i, and $\pi_i(u_1 \dots u_r) = 1$.

For an arbitrary word $V = [v_1] \cdots [v_s]$, define $\| V \|$ to be $|v_1| + \cdots + |v_s|$, where $|v|$ is the length of v. We define the *weight* of a move of type 1 to be 0, the weight of a move of type 2 to be $\| Q \| \cdot |v|$, and the weight of a move of type 3 to be $f(|u|)$, where f is an isoperimetric function for all of the groups G_i. We define the weight of a reduction sequence to be the sum of the weights of its moves, and we define the weight of a word

W for which $\rho W = 1$ to be the minimum weight of the reduction sequences beginning with W.

Let $g(n)$ be the maximum of $f(n_1) + \cdots + f(n_s)$ over all s and all positive integers $n_1, \ldots, n_s$ whose sum is n. Note that if f is polynomial of degree k (or exponential) then so is g. Brick calls g the subnegative closure of f; it is easy to see that it is the smallest subnegative function which is an upper bound for f.

We next show that the weight of a word W with $\rho W = 1$ is at most $\| W \|^2 + g(\| W \|)$. As already remarked, we can write W as $P[u_1]Q_1[u_2] \ldots Q_{r-1}[u_r]P'$, where $u_1, \ldots, u_r$ are i-words, $Q_1, \ldots, Q_{r-1}$are (possibly empty) words in the link of i, and $\pi_i(u_1 \ldots u_r) = 1$. Then there is a reduction sequence beginning with

$$W,\ P[u_1u_2]Q_1Q_2[u_3] \ldots [u_r]P', \ldots, P[u_1 \ldots u_r]Q_1 \ldots Q_{r-1},\ W',$$

where W' is $PQ_1 \cdots Q_{r-1}P'$. Since the sum of the weights of the moves from W to W' is at most $\| W \| \cdot (|u_1| + \cdots + |u_r|) + f(|u_1 \cdots u_r|)$, the required result holds by induction.

Finally, we use this result to obtain an isoperimetric function for G. We use the set of monoid generators $\bigcup A_i$. There are homomorphisms $\pi{:}(\bigcup A_i)^* \to G$, $\alpha{:}(\bigcup A_i)^* \to X^*$, and $\beta{:}X^* \to (\bigcup A_i)^*$, defined by $\pi a = \pi_i a$ for $a \in A_i$, $\alpha a = [a]$, and $\beta[u] = u$. Plainly, for any $w \in (\bigcup A_i)^*$, we have $\beta\alpha w = w$ and $\rho\alpha w = \pi w$. Also $|w| = \| \alpha w \|$.

It is easy to see that if W' is obtained from W by a move of weight k then βW is the product of $\beta W'$ and k conjugates of the defining relators (and their inverses) of the finite presentation of G. By induction on the length of the reduction sequence, if $\rho W = 1$ then βW is the product of at most weight (W) conjugates of the definig relators and their inverses.

Applying this to αw, where $\pi w = 1$, and using the formula for the weight, we see that $g(n) + n^2$ is an isoperimetric function for our presentation of G, proving the theorem for the isoperimetric case. The result for the isodiametric case is similar but simpler, and the details will be left to the reader.

2. Semi-Thue-systems

Let X be an arbitrary set. A *semi-Thue system*, or *rewriting system* on X is a subset S of $X^* \times X^*$. Such a system induces an equivalence relation on X^*; namely, the smallest equivalence relation such that $ulv \equiv urv$ for all words u, v and all pairs (l, r) in S. The quotient of X^* by this equivalence is called *the monoid presented by* $\langle X; S \rangle$.

When we look for normal forms for the equivalence classes, there are two ways to proceed. One treats all members of S alike, and compares the equivalence relation with the non-symmetric relation in which we can replace ulv by urv but not vice versa. We then endeavour to see if this terminates, and whether it provides a unique normal form.

The other approach, which is more convenient in our situation, begins by assuming that S consists only of pairs for which $|l| \geq |r|$ and that, for any (l, r) in S with $|l| = |r|$ we also have (r, l) in S. This can be done without loss of generality, since we get the same equivalence relation if we replace a pair (l, r) by (r, l) and also if we add pairs (r, l) for which (l, r) are already in S.

If we do this, then, when we consider replacing ulv by urv but not viceversa, if $|l| = |r|$ we can use the further pair (r, l) to return from urv to ulv. Consequently, we treat such pairs differently from those pairs for which $|l| > |r|$. It is quite common in computer science to distinguish between the two approaches by using the phrase 'rewriting system' for the first one and the phrase 'semi-Thue system' for the second.

We write $ulv \to urv$ for a pair (l, r) with $|l| > |r|$, and $ulv \sim urv$ for a pair with $|l| = |r|$. We let $\xrightarrow{*}$ and $\overset{*}{\sim}$ be the transitive reflexive closures of these.

We say that the pair u, v is *almost confluent* if there are u_1, v_1 such that $u \xrightarrow{*} u_1$, $v \xrightarrow{*} v_1$, and $u_1 \overset{*}{\sim} v_1$. Plainly, almost confluent words are equivalent, and we call S almost confluent if every pair of equivalent words is almost confluent. In searching for nice representatives of the equivalence classes, it is particularly helpful if S is almost confluent. Clearly, when this holds, if u is equivalent to ε then $u \xrightarrow{*} \varepsilon$.

This situation is very familiar to computer scientists. A sufficient condition for the property to hold can be given in terms of the behaviour of certain *critical pairs* of words, which arise from certain words in which two of the elements of S may be used. The situation is less well- known to group theorists, but results sometimes referred to as 'Peak Reduction Lemmas' are essentially of this form.

Huet [6] showed that S is almost confluent whenever almost confluence holds for all pairs u, v such that there is some w with $w \to u$ and either $w \to v$ or $w \sim v$. It is not difficult to prove this directly using peak reduction arguments. (If readers want to look at [6], they should note that Huet's $\sim$ is our $\overset{*}{\sim}$.)

Huet also showed that we do not even have to consider all such pairs. It is enough to consider those w of form abc for some words a,b,c such that S either has elements (ab, r_1) and (bc, r_2) or has elements (abc, r_1) and (b, r_2), with u and v (or v and u) obtained from w by applying these two elements; we refer to the first case as an *overlap ambiguity* and to the second case as an *inclusion ambiguity* (for an overlap ambiguity we require a,b,c to be non-empty; for an inclusion ambiguity, either a or c or both may be empty, but r_1 and r_2 must be distinct if a and c are both empty). These pairs u, v of words are called the *critical pairs*. Huet's proof applies in much more generality, and it is quite easy to prove this directly in our situation. A proof can also be found in [1] section 3.6.

We now return to graph products, with the set X as in Section 1. We shall prove the lemma by applying this theory of semi-Thue systems. The set S will consist of the following pairs:

(1) $([u][v], [uv])$, where u and v are i-words for some i;
(2) $([u]P[v], [uv]P)$, where u and v are i-words for some i and P is in the link of i;
(3) $([u], \varepsilon)$, where u is an i-word such that $\pi_i u = 1$;
(4) $([u],[v])$, where u and v are i-words for some i and $\pi_i u = \pi_i v$;
(5) $([u][v], [v][u])$, where u is an i-word, v is a j-word, and i and j are adjacent.

It is clear that the set of i-words for a given i, together with the corresponding pairs of types 1,3, and 4, form a monoid presentation for G_i; this is just a variant of the multiplication table presentation. If we take all pairs of types 1,3,4, and 5 we then clearly obtain a monoid presentation for G. We can then add the pairs of type 2 and still get a monoid presentation for G, since the two elements of a pair of type 2 clearly give the same element of G.

To prove the lemma, we need only show that the criterion mentioned above is satisfied.

First look at the inclusion ambiguities. The simplest cases occur when u is an i-word such that $\pi_i u = 1$, and w is an i-word such that $\pi_i w = \pi_i u$. Then $[u] \to \varepsilon$ and $[u] \sim [w]$, and we also have $[w] \to \varepsilon$. Further, let v be a j-word, where i and j are adjacent. Then we have $[u][v] \sim [v][u]$ and also $[u][v] \sim [v]$. Here we have $[v][u] \sim [v]$. A similar argument holds when we have $\pi_j v = 1$ instead of $\pi_i u = 1$.

For the remaining cases, let u and v be i-words for some i, and let P be a (possibly empty) word in the link of i. Thus $[u]P[v] \sim [uv]P$.

Suppose that some pair can be applied to P, giving P'. For notational convenience, suppose $P \sim P'$ (the case where $P \to P'$ is obtained by replacing $\sim$ by $\to$ throughout). Then we also have $[uP[v] \sim [u]P'[v]$. It is easy to check that P' is still in the link of i, and so we have both $[uv]P \sim [uv]P'$ and $[u]P'[v] \to [uv]P'$.

Let w be an i-word such that $\pi_i w = \pi_i u$. Then $[u]P[v] \to [uv]P$ and $[u]P[v] \sim [w]P\{v]$. We then have $[w]P[v] \to [wv]P$ and $[uv]P \sim [wv]P$. A similar argument works when, instead of $\pi_i w = \pi_i u$, we have $\pi_i w = \pi_i v$.

Next, suppose that $\pi_i v = 1$. Then $[u]P[v] \to [uv]P$ and also $[u]P[v] \to [u]P$. Since $\pi_i(uv) = \pi_i u$, we have $[uv]P \sim [u]P$, using a pair of type 4. If we have $\pi_i u = 1$ instead of $\pi_i v = 1$, then $[u]P[v] \to [uv]P$ and $[u]P[v] \to P[v]$. Here we have $[uv]P \sim [v]P$, using a pair of type 4, and $[v]P \sim P[v]$, using pairs of type 5 (since P is in the link of i).

The remaining inclusion ambiguity occurs by writing P as $Q[w]$ or as $[w]Q$, and applying a pair of type 5 to $[w][v]$ or to $[u][w]$. This case is almost the same as the next one, so the details will be left to the reader.

Our first case of an overlap ambiguity is when w is a j-word, where i and j are adjacent. Then $[u]P[v][w] \to [uv]P[w]$ and $[u]P[v][w] \sim [u]P[w][v]$. Since $P[w]$ is in the link of i, we have $[u]P[w][v] \to [uv]P[w]$. Also, we have $[w][u]P[v] \to [w][uv]P$ and $[w][u]P[v] \sim [u][w]P[v]$. We then have $[u][w]P[v] \to [uv][w]P$ and $[w][uv]P \sim [uv][w]P$.

Next, look at $[u]P[v]Q[w]$, where u, v, and w are i-words for some i, and P and Q are (possibly empty) words in the link of i. We have $[u]P[v]Q[w] \to [uv]PQ[w]$ and also $[u]P[v]Q[w] \to [u]P[vw]Q$. Here we find that $[uv]PQ[w] \to [uvw]PQ$ and also $[u]P[vw]Q \to [uvw]PQ$.

The final case is notationally the most complicated. Consider $[u]P[z]Q[v]R[t]$, where u and v are i-words for some i, z and t are j-words for some j adjacent to i, P is in the link of i, R is in the link of j, and Q is in both links. We then have $[u]P[z]Q[v]R[t] \to [uv]P[z]QR[t]$ and $[u]P[z]Q[v]R[t] \to [u]P[zt]Q[v]R$. We find, as required, that $[uv]P[z]QR[t] \to [uv]P[zt]QR$ and $[u]P[zt]Q[v]R \to [uv]P[zt]QR$.

We have now shown that all critical pairs satisfy the required criterion, and the lemma is proved.

Green, in her thesis [4], proves a normal form theorem for graph products. An easier proof can be obtained by the current methods. (This is not surprsing. The easiest proof of the normal form theorem for free products uses the similar results for length-reducing semi-Thue systems.)

The free product $*G_i$ of the vertex groups maps onto G. An element of the free product is in normal form if it is $g_1 \dots g_n$, where each g_r is in a vertex group, $g_r \neq 1$ for $r \leq n$ and g_r, g_{r+1} in different vertex groups for $r < n$. An element in normal form is called *graphically reduced* if for any p, q with $p < q$ such that g_p and g_q are in the same vertex group G_i there is some r with $p < r < q$ such that g_r is in a vertex group G_j with j not adjacent to i.

If, for some s, g_s and g_{s+1} are in vertex groups at adjacent vertices, we say that $g_1 \dots g_{s-1} g_{s+1} g_s g_{s+2} \dots g_n$ is a *shuffle* of $g_1 \dots g_n$. It is easy to see that a shuffle of a graphically reduced element is graphically reduced, but an element in normal form which is not graphically reduced will be transformed to an element not in normal form by a suitable sequence of shuffles.

There is a homomorphism θ from X^* onto $*G_i$ defined by $\theta[u] = \pi_i u$ for $u \in G_i$. Plainly, θU is in normal form iff no rules of types 1 and 3 can be applied to U, and θU will be graphically reduced iff no rules of types 1, 2, or 3 can be applied to U.

If V comes from U by application of a rule of type 4 then $\theta V = \theta U$, while if V comes from U by application of a rule of types 5 then θV is a shuffle of θU. Since our semi-Thue system presents the graph product, we immediately get the following theorem of Green from the almost-confluence property by using θ.

Normal form theorem for graph products. *Every element of the graph product is the image of a graphically reduced element of the free product. Two graphically reduced element of the free product give the same element of the graph product iff one comes from the other by a sequence of shuffles.*

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