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Multiplicativity of acyclic digraphs

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Abstract

A homomorphism of a digraph to another digraph is an edge preserving vertex mapping. A digraph W is said to be multiplicative if the set of digraphs which cannot be homomorphically mapped to W is closed under categorical product. We discuss the necessary conditions for a digraph to be multiplicative. Our main result is that almost all acyclic digraphs which have a Hamiltonian path are nonmultiplicative. We conjecture that almost all digraphs are nonmultiplicative.

1. Introduction

The notion of multiplicativity was first introduced in [4], and analyzed in detail in [18]. This notion can be traced back to the conjecture of Hedetniemi [7] which states that the chromatic number of the categorical product (product for brief) of two n-chromatic graphs is n. An equivalent statement of the conjecture is that the class of graphs which are not homomorphic to K_n is closed under taking the product, i.e., K_n is multiplicative. See [18] for details. For undirected graphs, some results can be found in [1–5,7,12,15]. The principal results are that K_3 is multiplicative, and that all odd cycles are multiplicative. (The multiplicativity of K_1 and any bipartite graph is obvious). For directed graphs, some results can be found in [4,8,10,12,17–20,22,23]. The main results are as follows. All transitive tournaments are multiplicative [4,12]. An oriented path are multiplicative if and only if it is hom-equivalent to a directed path (the research on this was initiated in [4,12] and was completed in [19]). A directed cycle C of length n is multiplicative if and only if n is a prime power [4,17,23]. An oriented cycle C is multiplicative if and only if C is a $\mathscr{C}$-cycle (research on this was initiated in [20] and was completed in [8,9]).

We only consider digraphs without loops and multiple edges in this paper. A *homomorphism* of a digraph G to a digraph H is a mapping $f: V(G) \rightarrow V(H)$ such that $gg' \in E(G)$ implies $f(g)f(g') \in E(H)$. If such a homomorphism exists, we say G is homo-

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morphic to H and write $G \to H$. Otherwise we write $G \not\to H$. Two digraphs, G and H, are *hom-equivalent* if $G \to H$ and $H \to G$. The *product* $G \times H$ of two digraphs G and H has the vertex set $V(G) \times V(H)$ and has the (directed) edges $(g,h)(g',h')$ if $gg' \in E(G)$ and $hh' \in E(H)$. A digraph D is *multiplicative* if for any two digraphs G and H, $G \not\to D$ and $H \not\to D$ implies that $G \times H \not\to D$.

An *oriented walk* $W = [w_0, w_1, \ldots, w_n]$ in a digraph G is a sequence of vertices $w_0, w_1, \ldots, w_n$, where the vertices may be repeated and there are edges $w_i w_{i+1}$ or $w_{i+1} w_i$ for each $i = 0, 1, \ldots, n-1$. The edge $w_i w_{i+1}$ is called a forward edge. The edge $w_{i+1} w_i$ is called a backward edge. The *order of traversal* starts at w_0, continues with $w_1, w_2, \ldots, w_{n-1}$, and ends at w_n. The order of traversal can also be specified by saying that w_0 is the initial vertex of W, or w_n is the terminal vertex of W. The subwalk of W, induced by the vertices $w_i, w_{i+1}, \ldots, w_j$, is denoted by $[w_i, \ldots, w_j]$, or simply by $[w_i, w_j]$. The *algebraic length* $\mathrm{al}(W)$ of W is the difference between the number of forward edges and the number of backward edges. The *net length* $\mathrm{nl}(W) = |\mathrm{al}(W)|$. The *level* of a vertex w_i in the walk W (with respect to the vertex w_0) is $\mathrm{al}([w_0, w_i])$. W is called a *minimal* oriented walk if the net length of any subwalk of W is not greater than the net length of W. An *oriented path* P in a digraph G is an oriented walk without repetition of vertices. An oriented cycle is a closed walk without repetition of vertices except the first and the last vertex. A directed cycle is an oriented cycle with all the edges directed in one direction. An oriented path $P = [p_0, p_1, \ldots, p_n]$ is called a *forward directed path* if all the edges are forward along the order of traversal, a *backward directed path* if all the edges are backward along the order of traversal. Both forward and backward directed paths are called directed paths. The net length of a directed path (or cycle) is also called the *length*. A digraph G is *balanced* if for any two vertices x and y of G the net length of any oriented path from x to y only depends on x and y. Obviously any tree is a balanced graph. The *net length of a balanced digraph G* is defined by

$$\mathrm{nl}(G) = \max\{\mathrm{nl}(P_{xy}) \mid x, y \in V(G),\ P_{xy} \text{ is the oriented path from } x \text{ to } y\}.$$

A digraph G is said to be acyclic if it contains no directed cycles. A graph G is said to have a *Hamiltonian path* if there exists a directed path containing all the vertices of G. In Section 2 we consider the nonmultiplicativity of acyclic digraphs whose maximum directed paths have some special property. Our main result is Theorem 9 which states that almost all acyclic digraphs which have a Hamiltonian path are nonmultiplicative.

2. Main results

Theorem 1. *Let W be an acyclic digraph with a Hamiltonian path $w_0, w_1, \ldots, w_n$. If w_0 has outdegree 1 and w_n has no in-edge from the first vertex on W of outdegree greater than 1, then W is nonmultiplicative.*

Proof. Let w_s be the first vertex on W with outdegree greater than 1, let w_t the last vertex on W such that there is an edge from w_s to w_t. By our assumption, $0<s<t<n$, $t-s\geqslant 2$, there is an edge w_sw_t, and there is neither an edge of the form w_iw_j with $i<s$ and $i+1<j$ nor an edge of the form w_sw_j with $t<j$.

Let G be a forward directed path of length $n+1$. Let $H=P_1\cdot P_2\cdot P_3$ where P_1 has vertices $p_{10},p_{11},\dots,p_{1t},p_{1(t+1)}=a$, and edges $p_{1i}p_{1(i+1)}$ for $i=0,1,\dots,t$; P_2 has vertices $a=p_{2(s+1)},p_{2s},p_{2(s-1)},\dots,p_{21},p_{20}=b$, and edges $p_{2i}p_{2(i+1)}$ for $i=0,1,\dots,s$; and P_3 has vertices $p_{30}=b,p_{31},\dots,p_{3n}$, and edges $p_{3i}p_{3(i+1)}$ for $i=0,1,\dots,n-1$.

Obviously $G\not\rightarrow W$. We can also prove that $H\not\rightarrow W$. Assume, otherwise, that $H\rightarrow W$. Let $\Phi:H\rightarrow W$. Since the homomorphic image of a directed path P_3 of length n must also be a directed path of length n in W, it follows that $\Phi(p_{3n})=w_n$ and $\Phi(b)=w_0$ which forces $\Phi(p_{2s})=w_s$ since w_s is the first vertex which can have edges to w_j with $j>s+1$. In order for P_1 to be homomorphically mapped to W, the subscript of $\Phi(a)$ should be greater than t. But $\Phi(p_{2s})$ must be w_s, it follows that t is the largest choice for the subscript of $\Phi(a)$. A contradiction. Therefore, $H\not\rightarrow W$.

Now we claim that any proper subpath of H can be homomorphically mapped to W. It suffices to show that $H-\{p_{10}\}\rightarrow W$ and $H-\{p_{3n}\}\rightarrow W$. In fact, the following mapping Φ is a homomorphism of $H-\{p_{10}\}$ to W,

$$\Phi(p_{3i})=w_i \quad \text{for } i=0,1,2,\dots,n,$$

$$\Phi(p_{2i})=w_i \quad \text{for } i=0,1,\dots,s,$$

$$\Phi(a)=w_t,$$

$$\Phi(p_{1i})=w_{i-1} \quad \text{for } i=1,2,\dots,t;$$

the following mapping Ψ is a homomorphism of $H-\{p_{3n}\}$ to W,

$$\Psi(p_{1i})=w_i \quad \text{for } i=0,1,\dots,t+1,$$

$$\Psi(p_{2s})=w_t,$$

$$\Psi(p_{2i})=w_{i+1} \quad \text{for } i=0,1,2,\dots,s-1,$$

$$\Psi(p_{3i})=w_{i+1} \quad \text{for } i=0,1,\dots,n-1.$$

In the product $G\times H$, each component is an oriented path. Since $\mathrm{nl}(G)=n+1$, the maximum net length of subpaths of each component is at most $n+1$. Each component of $G\times H$ can be homomorphically mapped to H by a natural projection. Since $\mathrm{nl}(H)=n+t-s\geqslant n+2$, the homomorphic image of each component of $G\times H$ in H under this projection is a proper subpath of H, which can be homomorphically mapped to W. Therefore, each component of $G\times H$ can be homomorphically mapped to W, i.e., $G\times H\rightarrow W$. □

Remark. (1) In our construction the condition $n>t$ is critical. If $n=t$, then we cannot define Ψ, since $\Psi(p_{1t})=w_{t+1}$. The inequality $s>0$ is also critical. Only if there exists an edge $w_{s+1}w_{t+1}$, our construction will work for the condition $s=0$. Otherwise we cannot find homomorphism $\Psi: H - \{p_{3n}\} \to W$.

(2) As a special case of Theorem 1, the portion between w_s and w_t can be a transitive tournament, the portions between w_0 and w_s, and between w_t and w_n are directed paths, there is no edge from $\{w_0,\dots,w_{s-1}\}$ to $\{w_s,\dots,w_n\}$ except the edge $w_{s-1}w_s$, and there is no edge from $\{w_0,\dots,w_t\}$ to $\{w_{t+1},\dots,w_n\}$ except the edge w_tw_{t+1}.

(3) Dually we can have the following result. Let W be an acyclic digraph with a Hamiltonian path $w_0,w_1,\dots,w_n$. If w_n has indegree 1 and w_0 has no out-edge to the last vertex on W of indegree greater than 1, then W is nonmultiplicative. The proof of this dual theorem is very similar to the proof of Theorem 1. Let w_t be the last vertex on W with indegree greater than 1, let w_s the first vertex on W such that there is an edge from w_s to w_t. By our assumption, $0<s<t<n$, $t-s\geqslant 2$, there is an edge w_sw_t, and there is neither an edge of the form w_iw_j with $t<j$ and $i+1<j$ nor an edge of the form w_iw_t with $i<s$. Construct G and H as follows. G is the same as in the proof of Theorem 1. $H=P_1\cdot P_2\cdot P_3$ where P_1 is a forward directed path of length n, P_2 is a backward directed path of length $n-t+1$, and P_3 is a forward directed path of length $n-s+1$. Then $G \not\to W$, $H \not\to W$, and $G\times H \to W$. We should also have a similar note: $s>0$ is critical. $t<n$ is also critical with the exception that there exists an edge $w_{s-1}w_{t-1}$. If there is such an edge, then our construction will work for $t=n$.

In the remaining parts of the paper, we shall omit all the dual results. Based on Theorem 1 and its dual result we have the following corollary.

Corollary 2. *Let W be an acyclic digraph with a Hamiltonian path $w_0,w_1,\dots,w_n$. If w_0 has outdegree* 1 *and w_n has indegree* 1, *then W is nonmultiplicative* (*unless W is a directed path*).

An oriented walk P is called a $(a-b-c)$-shaped walk if $P=P_1\cdot P_2\cdot P_3$ where P_1 is a forward directed walk of length a, P_2 is a backward directed walk of length b and P_3 is a forward directed walk of length c. If G has no directed cycle, then any directed walk in G must also be a directed path, but any oriented walk may not necessarily be an oriented path.

Theorem 3. *Let W be an acyclic digraph. Let $W_i=[w_{i0},w_{i1},\dots,w_{in}]$, $(i=1,2,\dots,k)$, be all directed paths of W with maximum length n. For $0\leqslant s$ and $s+1<t\leqslant n$, let $X=\{w_{is}\mid i=1,2,\dots,k\}$ and $Y=\{w_{it}\mid i=1,2,\dots,k\}$. Let a,b and c be such that $n\geqslant b>a$, $b>c$, $t-s-1\geqslant a$, and $t-s-1\geqslant c$. If there are forward directed paths of length a and c from X to Y and there is no $(a-b-c)$-shaped walk from X to Y, then W is nonmultiplicative.*

Proof. Let G be a forward directed path of length $n+1$. Let P_1 and P_2 be two forward directed paths of length n, and P be an $(a-b-c)$-shaped path. Let H be a digraph obtained from P_1, P and P_2 by identifying the first vertex of P with the $(s+1)$th vertex

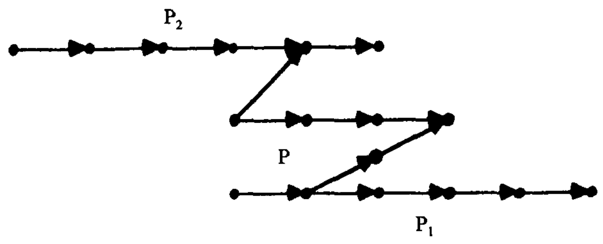


Fig. 1.

of P_1, and the last vertex of P with the $(t+1)$th vertex of P_2. See Fig. 1 for the illustration of an example of H, where $n=5$, $s=1$, $t=4$, both P_1 and P_2 are forward directed paths of length 5, and P is a $(2-3-1)$-shaped path.

It is obvious that $G \not\to W$, and $H \not\to W$.

The digraph H is balanced. Now it is easy to see that any subgraph of H with length at most $n+1$ can be homomorphically mapped to W. Any component of $G \times H$ is isomorphic to a subgraph of H (by the natural projection). The subpaths of any component has net length at most $n+1$ since G has length $n+1$. Therefore, any component of $G \times H$ can be homomorphically mapped to W. So does $G \times H$. □

Corollary 4. *Let W be an acyclic digraph with a Hamiltonian path $w_0, w_1, \ldots, w_n$. Assume that $w_0w_n \in E(W)$ but there is no $(1-2-1)$-shaped oriented walk from w_0 to w_n. This means that for each three tuple i,j,k $(0 \leqslant i < j < k \leqslant n)$, at least one edge among w_0w_k, w_iw_j, w_jw_k, and w_iw_n does not exist. Then W is nonmultiplicative.*

Corollary 5. *Let W be an acyclic digraph. Let $W_i = [w_{i0}, w_{i1}, \ldots, w_{in}]$, $(i = 1,2,\ldots,k)$, be all directed paths of W with maximum length n. Let s and t $(s+1<t)$ be two indices such that there is a directed path of length a $(a \leqslant t-s-1, 2a \leqslant n)$ from $X = \{w_{is} \mid i = 1,2,\ldots,k\}$ to $Y = \{w_{it} \mid i = 1,2,\ldots,k\}$, but there is no $(a-2a-a)$-shaped oriented walk from X to Y. Then W is nonmultiplicative.*

Corollary 6. *Let w_sw_t be the only edge added to the Hamiltonian path $W = w_0w_1 \ldots w_n$. Then W is multiplicative if and only if either $n=2$ or if $n>2$, then either $s+2=t=n$ or $0=s=t-2$.*

Proof (*Necessity*). Prove by contradiction. Assume that $n>2$. If $s+2<t$, then there is no $(1-2-1)$-shaped oriented walk from w_s to w_t. Therefore, W is nonmultiplicative by applying Corollary 5. If $t=s+2$, and if neither $s=0$ nor $t=n$, then W is nonmultiplicative by applying Theorem 1.

(*Sufficiency*) If $n=2$, then $s=0$ and $t=n$ and hence W is multiplicative by [4]. If $n>2$, either $s+2=t=n$ or $0=s=t-2$, then it is multiplicative by [21]. In fact we have proved in general [21] that a connected acyclic local tournment is multiplicative if and only if it is a digraph obtained by concatenating a directed path with a transitive tournament. □

Theorem 7. *Let W be an acyclic digraph. Let $W_i=[w_{i0},w_{i1},\dots,w_{in}]$, $(i=1,\dots,k)$, be all of the directed paths of W with maximum length n. Assume that there exists two indices s and t $(t-s>2)$, such that for any i $(i=1,2,\dots,k)$, $w_{is}w_{it}\notin E(W)$. Assume further that for some i_1 and i_2, $w_{i_1s}w_{i_2(t-1)}\in E(W)$ and for some i_3 and i_4, $w_{i_3j}w_{i_4t}\in E(W)$ with $s<j<t-1$. Then W is nonmultiplicative.*

Proof. Let $G=[g_0,g_1,\dots,g_{n+1}]$ be a forward directed path of length $n+1$. Let H be a digraph obtained by attaching one edge h_sh_t to a forward directed path $[h_0,h_1,\dots,h_n]$, i.e., $E(H)=\{h_ih_{i+1}\mid i=0,1,\dots,n-1\}\cup\{h_sh_t\}$. Obviously, $G\not\to W$ and $H\not\to W$. If we can prove that $G\times H\to W$, then W is nonmultiplicative.

In $G\times H$, there are two isolated vertices (g_0,h_n) and (g_{n+1},h_0), $n-t+s$ directed paths of length at most $\max\{n-t,s\}$, and $t-s-1$ other components. Therefore, we only need to show that each component of those $t-s-1$ components can be homomorphically mapped to W. Each component is composed of several directed paths and a few edges joining these paths. Let C_m be the component containing the vertex (g_m,h_0) $(m=0,1,\dots,t-s-2)$. The component C_0 has two main directed paths:

$$P_{01}:(g_0,h_0),(g_1,h_1),\dots,(g_i,h_i),\dots,(g_n,h_n),$$

$$P_{02}:(g_{t-s-1},h_0),\dots,(g_{t-s-1+i},h_i),\dots,(g_{n+1},h_{n-t+s+2})$$

with edges forward in this order, and an edge $(g_{t-1},h_s)(g_t,h_t)$. Let B_0 be the subgraph of C_0 induced by the vertices of P_{01} and P_{02}.

For any vertex $v=(g_i,h_j)\in V(C_0)$ with $j\geqslant i$, define

$$\Phi(v)=(g_i,h_i).$$

For any vertex $v=(g_{t-s-1+i},h_j)\in V(C_0)$ $(0\leqslant i\leqslant n-t+s+2)$ with $j\leqslant i$, define

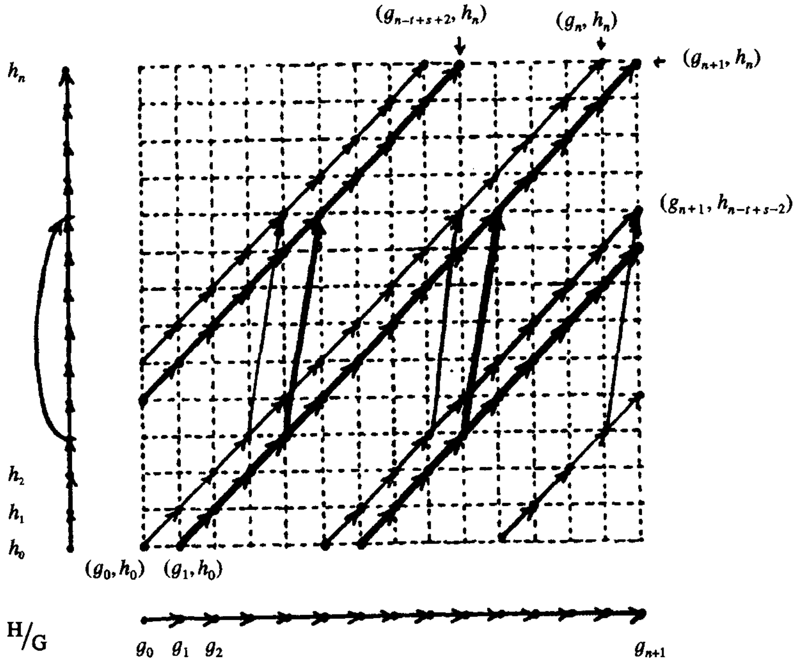
$$\Phi(v)=(g_{t-s-1+i},h_i).$$

In other words, every vertex of C_0 above P_{01} is mapped to the corresponding vertex of P_{01} with the same first coordinate, every vertex of C_0 below P_{02} is mapped to the corresponding vertex of P_{02} with the same first coordinate. We draw the product $G\times H$ in the first quadrant of the xy-plane with integer grid point (i,j) to represent the vertex (g_i,h_j).

Obviously, Φ is a homomorphic mapping of C_0 to B_0 and we keep the vertices in B_0 fixed.

See Fig. 2 for the structure of B_0.

Since there exists an edge $w_{i_3j}w_{i_4t}$ in W for some $j, s<j<t-1$, it follows that $B_0\to W$ by mapping (g_t,h_t) to w_{i_4t} and (g_{t-1},h_s) to w_{i_3j} (the images of the other vertices of B_0 are also determined in a natural way). Therefore, $C_0\to W$.



$n=13, \quad t=9, \quad s=3$

Fig. 2.

Each component C_m $(m = 1, 2, \ldots, t-s-2)$, which contains (g_m, h_0), has two main directed paths:

$$P_{m1} : (g_0, h_{t-s-1-m}), (g_1, h_{t-s-m}), \ldots, (g_k, h_{t-s-m+k-1}), \ldots, (g_{n-t+s+1+m}, h_n),$$

$$P_{m2} : (g_m, h_0), (g_{m+1}, h_1), \ldots, (g_{m+k}, h_k), \ldots, (g_{n+1}, h_{n+1-m}),$$

and an edge $(g_{m+s}, h_s)(g_{m+s+1}, h_t)$.

Let B_m be the subgraph of C_m $(m = 1, 2, \ldots, t-s-2)$ induced by the vertices of P_{m1} and P_{m2}. We can define $\Phi : V(C_m) \to V(B_m)$ similarly as above, i.e., every vertex of C_m above P_{m1} is mapped to the corresponding vertex of P_{m1} with the same first coordinate, every vertex of C_m below P_{m2} is mapped to the corresponding vertex of P_{m2} with the same first coordinate. The mapping Φ is a homomorphism of C_m to B_m keeping the vertices of B_m fixed. See Fig. 2 for the structure of B_m where $m = 1$.

Note that m can only have one of the values $1, 2, \ldots, t-s-2$. Note also that there is an edge $w_{i_1 s} w_{i_2 (t-1)}$ in the digraph W. Therefore, $B_m \to W$ by mapping (g_{m+s}, h_s) to $w_{i_1 s}$ and (g_{m+s+1}, h_t) to $w_{i_2 (t-1)}$ (the images of the other vertices of B_m are also determined in a natural way). □

The following corollary deals with a digraph W having a Hamiltonian path. It can be directly derived from the above theorem.

Corollary 8. *Let W be an acyclic digraph with a Hamiltonian path $w_0, w_1, \dots, w_n$. If there exist two vertices w_s and w_t $(s+2<t)$, such that $w_s w_t \notin E(W)$, $w_s w_{t-1} \in E(W)$, and $w_j w_t \in E(W)$ for some j $(s<j<t-1)$, then W is nonmultiplicative.*

Let $\mathscr{F}_n$ be the set of digraphs of n vertices which have property $\mathscr{P}_1$. Let $\mathscr{Q}_n$ be the set of digraphs of n vertices which have both property $\mathscr{P}_1$ and $\mathscr{P}_2$. We say that almost all digraphs which have property $\mathscr{P}_1$ have property $\mathscr{P}_2$ if

$$\lim_{n\to\infty} \frac{|\mathscr{Q}_n|}{|\mathscr{F}_n|} = 1.$$

From Corollary 8, we have the following interesting theorem.

Theorem 9. *Almost all acyclic digraphs which have a Hamiltonian path are nonmultiplicative.*

Proof. Let $\mathscr{F}_{n+1}$ be the set of acyclic digraphs of $n+1$ vertices which have a Hamiltonian path. Let $\mathscr{Q}_{n+1}$ be the subset of digraphs of $\mathscr{F}_{n+1}$, which are nonmultiplicative. Consider a directed path P of length n with $V(P)=\{p_0, p_1, \dots, p_n\}$ and $E(P)=\{p_i p_{i+1} \mid i=0,1,\dots,n-1\}$. Any digraph in $\mathscr{F}_{n+1}$ can be obtained from P by choosing some pairs $(i,j) \in (n+1)\times(n+1)$ with $i+1<j$ and there is an edge from p_i to p_j. We assume that the probability of having an edge $p_i p_j$ is q. Then the probability of having no edge $p_i p_j$ is $1-q$. By Corollary 8, if there exists a pair (s,t), $(0 \leqslant s < t \leqslant n, s+2<t)$, such that there is no edge $p_s p_t$, there is an edge $p_s p_{t-1}$ and edge $p_j p_t$ for $s<j<t-1$, (we shall call such a directed graph an (s,t)-structure in brief afterwards), then we have a nonmultiplicative digraph.

Consider only the subfamily of (s,t)-structures for $s=0,2,4,\dots,$ and $t=s+3$, and denote by A_s the event that the $(s,s+3)$-structure is not present. Then events $A_0, A_2, A_4, \dots$ are independent since disjoint sets of edges are responsible for them. The probability that none of them is present is $[1-(1-q)qq]^{n/2}$. This number tends to zero exponentially, i.e., the probability that there exists a $(s,s+3)$-structure somewhere tends to 1, as $n \to \infty$. If there exists an $(s,s+3)$-structure, then the digraph is nonmultiplicative. Therefore, almost all acyclic digraphs which have a Hamiltonian path are nonmultiplicative. □

It is not hard to prove that almost all digraphs which have a Hamiltonian path have a directed cycle. Therefore, we must investigate the property of multiplicativity for digraphs with directed cycles. It turns out to be much more difficult. By the evidence that we provided in this paper we may conclude our paper with the following conjecture.

Conjecture 10. Almost all digraphs are nonmultiplicative.

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