A characterization of projective-planar signed graphs

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Abstract

A signed graph has a plus or minus sign on each edge. A simple cycle is positive or negative depending on whether it contains an even or odd number of negative edges, respectively. We consider embeddings of a signed graph in the projective plane for which a simple cycle is essential if and only if it is negative. We characterize those signed graphs that have such a projective-planar embedding. Our characterization is in terms of a related signed graph formed by considering the theta subgraphs in the given graph.

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1. Introduction and our main result

We begin with the basic definitions. Our graphs are all 2-connected and are allowed to have multiple edges. We forbid loops and cut vertices, since they do not seem to add to our theory and serve to complicate the statement of some results. We divide each edge into two half-edges corresponding to the two ends of the edge.

A signature $\sigma$ on a graph is an assignment of a plus or minus sign on each edge. A signed graph $G^{\pm}$ is a graph $G$ together with a signature. A path or simple cycle in a signed graph is positive or negative according to whether it contains an even or odd number of negative edges, respectively. A vertex switch on a signed graph changes the sign of each edge incident

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with a vertex. A vertex switch does not change the sign of any simple cycle. Two signed graphs are switching equivalent if there is a sequence of vertex switches transforming one signature into the other. Any signature is switching equivalent to one where the edges in a fixed spanning tree $T$ are all positive. A signed graph is balanced if every simple cycle is positive. If the edges of $T$ are all positive in a balanced signed graph, then every co-tree edge is also positive. Hence a signed graph is balanced if and only if it is switching equivalent to one signed with every edge positive.

The projective plane is the quotient space formed by identifying each point on a sphere with its antipodal point. The projective plane is the simplest surface, except for the sphere, on which to embed a graph.

A cellular embedding of a graph $G$ in a surface can be described by a signature $\sigma$ and a rotation system $\{\rho(v) \mid v \in V(G)\}$, where each $\rho(v)$ is a cyclic permutation of the half-edges incident with $v$. We will call $\rho(v)$ the rotation at $v$. Basically, the rotation at $v$ records the cyclic order of the edges around $v$, while the signature $\sigma$ is used to keep track of whether that order is in a locally clockwise or locally anticlockwise sense. The surface is orientable if and only if $\sigma$ is a balanced signature on $G$. For more details we refer the reader to [4].

An embedding of a signed graph in a surface is an embedding of the underlying graph such that a simple cycle is orientation reversing if and only if it is negative. For the projective plane, this is equivalent to embedding the underlying graph such that positive simple cycles are contractible and negative simple cycles are essential.

In this paper we will give a characterization of which signed graphs embed in the projective plane. Before describing our result we need the following definitions.

A claw in a (signed or unsigned) graph is a set of three pairwise adjacent half-edges. We say the claw is rooted at the common incident vertex. In the literature, a claw sometimes means an induced $K_{1,3}$; here we only take the root and the three incident half-edges and do not care about adjacencies between the other vertices. A claw rotation system on a set $C$ of claws selects one of the two cyclic permutations on the three half-edges in a claw $c$ for each $c \in C$. We will call each such cyclic permutation a rotation on the claw. We emphasize that a rotation system on $G$ has, for each vertex, a cyclic rotation on all half-edges incident with $v$. A claw rotation system assigns a cyclic rotation on each subset of incident edges of cardinality three.

A theta graph is a graph homeomorphic to $K_{2,3}$, or equivalently, is a pair of distinct vertices joined by three pairwise internally disjoint paths. A theta graph contains exactly two claws, rooted at the two degree-three vertices. We say that the theta graph is of type $i$, for $i = 0, 1, 2, 3$, if there are exactly $i$ negative paths joining the two degree-three vertices.

Each rotation on a claw in a theta graph gives a cyclic permutation of the three paths. A rotation on one claw in a theta graph agrees with a rotation on the other claw if these permutations are the same. They disagree otherwise.

The signs of each of the three paths in a theta graph and the rotations on each claw determine whether or not the theta graph embeds in the projective plane (possibly non-cellularly), where the clockwise/anticlockwise sense at each vertex is the given rotation on the claws and the signature records these changes in this local sense. If a theta graph is of type 0 or 1, it embeds in the projective plane if and only if its rotations on the two claws disagree. If a theta graph is of type 2 or 3, it embeds in the projective plane if and only if the rotations agree. In Fig. 1 we show four theta graphs $\theta_i$ of type $i$ embedded in
the projective plane formed by identifying pairs of antipodal points on the boundary of the
disk. The sign on a path in a theta graph is negative if and only if it crosses the boundary of
the disk.

Given a graph $G$, form the **triple graph** $T = T(G)$, the simple graph whose vertex set is
all claws of $G$ and in which an edge joins two claws whenever they lie in a common theta
subgraph of $G$.

Let $G^\pm$ be a signed graph and let $R$ be a claw rotation system on $G^\pm$. Form the **signed
triple graph** $T^\pm(G^\pm, R)$ as follows. The vertex set of $T^\pm(G^\pm, R)$ is again the set of claws of
$G^\pm$. Join two claws by a positive edge if they lie in a common theta graph and the rotations
on these claws embed this theta graph in the projective plane (possibly non-cellularly). Join
two claws by a negative edge if they lie in a common non-projective-planar theta. We allow
multiple edges in $T^\pm(G^\pm, R)$. See Table 1 for a summary of how signs are assigned to
edges in a signed triple graph.

The signature on $T^\pm(G^\pm, R)$ depends on the claw rotation system $R$. However, reversing
a rotation on a claw switches the sign of each edge incident with the corresponding vertex
in $T^\pm$, so the two signatures on $T^\pm$ are switching equivalent. In particular, the property of
$T^\pm(G^\pm, R)$ being balanced is independent of $R$.

Next, we observe that the graph $T^\pm(G^\pm, R)$ depends, up to switching equivalence, only
on which simple cycles of $G^\pm$ are positive or negative. If we switch the signs on each
edge incident with a vertex $v$ of $G^\pm$, then every theta graph of type $i$ with a claw rooted
at $v$ changes to a theta graph of type $3 - i$. Thus the signs are switched on all edges of $T$.


Table 1
Relating the type and rotations of a theta to its sign in $T^\pm$

<table>
<thead>
<tr>
<th>Type of the $\theta$</th>
<th>Rotations</th>
<th>Sign in $T^\pm$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Agree</td>
<td>$-$</td>
</tr>
<tr>
<td>0</td>
<td>Disagree</td>
<td>$+$</td>
</tr>
<tr>
<td>1</td>
<td>Agree</td>
<td>$-$</td>
</tr>
<tr>
<td>1</td>
<td>Disagree</td>
<td>$+$</td>
</tr>
<tr>
<td>2</td>
<td>Agree</td>
<td>$+$</td>
</tr>
<tr>
<td>2</td>
<td>Disagree</td>
<td>$-$</td>
</tr>
<tr>
<td>3</td>
<td>Agree</td>
<td>$+$</td>
</tr>
<tr>
<td>3</td>
<td>Disagree</td>
<td>$-$</td>
</tr>
</tbody>
</table>

incident with vertices corresponding to claws on $v$. The resulting signature on $T$ is switching equivalent to the original signature.

Summarizing the preceding two paragraphs, if $G^\pm_1$ and $G^\pm_2$ are the same underlying graph with different, switching-equivalent signatures, then for any two claw rotation systems $R_1$ and $R_2$, $T^\pm(G^\pm_1, R_1)$ is switching equivalent to $T^\pm(G^\pm_2, R_2)$. Whether or not $T^\pm(G^\pm, R)$ is balanced depends only on the switching equivalence class of $G^\pm$.

A claw rotation system $R$ is totally positive if $T^\pm(G^\pm, R)$ has every edge positive. If $T^\pm(G^\pm, R')$ is balanced for some claw rotation system $R'$, then it has a totally positive claw rotation system $R$. In fact, it has exactly $2^k$ such systems, where $k$ is the number of components of $T^\pm$.

We need one more concept. Let $R$ be a claw rotation system for the set of claws rooted at a fixed vertex $v$. Then $R$ has a twist if there are three claws on half-edges $a, b, c$ and $x$ with local rotations $xab, xbc$ and $xca$ ($xab$ means the cyclic permutation that sends $x$ to $a$, $a$ to $b$, and $b$ to $x$). The claw rotation system is twist-free if no such twist exists. Observe that a cyclic permutation of all half-edges incident with $v$ induces a twist-free rotation on the set of all claws rooted at $v$. Archdeacon et al. [2] proved the following lemma.

**Lemma 1.1.** Let $R$ be a claw rotation system on the set of all claws in a graph $G$. Then $R$ is twist-free if and only if it is induced by a rotation system of $G$.

Suppose that every simple cycle of $G^\pm$ is positive. By the preceding comments, $G^\pm$ is switching equivalent to a graph $G$ with all positive signs. We form the triple graph $T^\pm(G)$ as before; now all theta graphs are of type 0. In this context Archdeacon and Širáň [3] have shown the following.

**Theorem 1.1.** $G$ is planar if and only if $T^\pm(G)$ is balanced.

We seek the analog of Theorem 1.1 for embedding signed graphs in the projective plane. In Theorem 1.1 it is only necessary that $T^\pm(G)$ is balanced. This implies that there is a totally positive claw rotation system. For the projective plane we need more: the totally positive claw rotation system must be twist-free.
Theorem 1.2 (The main result). A loopless 2-connected signed graph $G^\pm$ is projective planar if and only if $G^\pm$ has a totally positive twist-free claw rotation system.

Section 2 proves this main result and Section 3 gives some concluding remarks.

2. The proof

In this section we prove our main result. One implication is easy.

Consider a projective-planar graph $G^\pm$. The rotation system on $G^\pm$ induces a claw rotation system $R$ on all claws in $G^\pm$. By Lemma 1.1, this system is twist-free. The restriction of the embedding of $G^\pm$ to an embedding of a theta subgraph embeds that theta in the projective plane. Hence every edge in $T^\pm(G^\pm, R)$ is positive, i.e. the claw rotation system is totally positive.

Before proving the reverse implication we need some background. Let $G_2$ be formed from a signed graph $G_1$ by deleting an edge $uv$ and adding in a path of length two, $uwv$, where $w \notin V(G_1)$, and putting signs on the two new edges so that the sign of $uwv$ equals the sign of $uv$. We call $G_2$ an elementary subdivision of $G_1$. Call $G_2$ a subdivision of $G_1$ if it can be formed by a sequence of elementary subdivisions. Two signed graphs are homeomorphic if they have a common subdivision. Finally, $H$ is a topological subgraph of a signed graph $G$ if it is homeomorphic to a subgraph of $G$.

We offer the following two simple observations, the proofs of which are left to the reader.

Lemma 2.1. (1) If $H^\pm$ is a signed subgraph of $G^\pm$, then $T^\pm(H^\pm)$ is a subgraph of $T^\pm(G^\pm)$.

(2) If $H^\pm$ is homeomorphic to $K^\pm$, then $T^\pm(H^\pm)$ is isomorphic to $T^\pm(K^\pm)$.

Kuratowski’s Theorem [5] states that a graph is non-planar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or to $K_5$. Zaslavsky [6] gave an analogous theorem for signed graphs in the projective plane. His result gives the complete set of $8$ topologically excluded signed graphs, up to switching equivalence. These are $8$ of the $9$ graphs of Fig. 2 (all but $\bar{V}_2^\circ$); their names are from [6]. In this figure the positive edges are solid lines and the negative ones are dashed.

Theorem 2.1 (Zaslavsky). A signed graph is projective-planar if and only if it has no topological subgraph switching equivalent to one of: $+K_5$, $+K_{3,3}$, $K_5(e)$, $K_{3,3}(e)$, $V_2^o$, $-W_4$, $\Phi_4$, $\Psi_5$.

The graph $V_2^o$ is awkward in our context as it has no theta subgraphs. Since we are restricting our attention to 2-connected graphs, we may use the following corollary that replaces $V_2^o$ with $V_2^o$ (see Fig. 2).

Corollary 2.1. A loopless 2-connected signed graph is projective-planar if and only if it has no topological subgraph switching equivalent to one of: $+K_5$, $+K_{3,3}$, $K_5(e)$, $K_{3,3}(e)$, $\bar{V}_2^o$, $-W_4$, $\Phi_4$, $\Psi_5$. 
Proof. Suppose that $G$ is 2-connected and contains $V_2^0$. Then, because there are no loops, there are two vertex-disjoint paths from one cycle to the other. We can assume that both paths are signed positive by using the switching equivalent signature that is positive on the spanning tree with paths corresponding to edges 2, 5 and 6. The cycle with paths corresponding to edges 3 and 4 is negative, so without loss of generality 3 is negative and 4 is positive.

We now prove the remaining implication in our main result. By Lemma 2.1 it suffices to show that none of the 8 graphs of Corollary 2.1 have totally positive twist-free claw rotation systems. We deal with each graph in turn.

Cases $+K_5$, $K_5(e)$, $+K_{3,3}$, and $K_{3,3}(e)$: Observe that there is only one negative edge in $K_5(e)$, so every theta is of type 0 or 1. Consulting Table 1, $+K_5$ and $K_5(e)$ have identical signed triple graphs. Similarly, $+K_{3,3}$ and $K_{3,3}(e)$ have identical signed triple graphs. So, it suffices to deal with the cases $+K_5$ and $+K_{3,3}$. However, [3] shows that $T^{\pm}(K_5)$ and $T^{\pm}(K_{3,3})$ are not balanced, thereby concluding these four cases.
Three of the remaining four cases have balanced triple graphs, but every totally positive claw rotation system has a twist. Before examining those four cases we establish some notation.

A theta consists of three edge-disjoint paths between two vertices. Denote this theta by an ordered triple of these paths, directed so that each of the three paths begin at the same vertex. For example, \((6, 13, 45)\) is a theta graph in the graph \(\Phi_4\) of Fig. 2. It is easy to recover the two degree-3 vertices as those incident with edges 6, 1, 4 and with 6, 3, 5, respectively. It is also easy to read the type of the theta graph by the parities of the numbers of negative edges in the three paths.

Case \(V_2^o\): Consider the two theta graphs \((1, 2, 536)\) and \((1, 2, 546)\). These have the same claws. The first is of type 2, the second is of type 1. Regardless of the claw rotation system \(R\), one theta graph will correspond to a positive edge in \(T^\pm(V_2^o, R)\), the other to a negative edge. This makes a negative cycle of length two in \(T^\pm(V_2^o, R)\), and so it is not balanced.

Case \(-W_4\): Label the edges of \(-W_4\) as shown in Fig. 2. Consider a totally positive claw rotation system that includes the claw rotation \(xab\).

Consider the type 1 theta graph \((x1, a, b2)\). Since the corresponding edge is positive in \(T^\pm\), we must have claw rotation \(2a1\). Next, the theta graph \((23, ac, 14)\) is of type 1. Since this edge is positive in \(T^\pm\), claw rotation \(2a1\) forces claw rotation \(4c3\). The type 1 theta graph \((4x, c, 3b)\) shows that rotation \(4c3\) forces \(bcx\). The type 2 theta graph \((234, ax, 1)\) shows that rotation \(2a1\) forces \(4x1\). Finally, the type 1 theta graph \((4c, x, 1a)\) shows that rotation \(4x1\) forces \(axc\).

We have shown that any totally positive claw rotation system with \(xab\) must also have \(xbc\) and \(xca\). But these three claw rotations form a twist, and our result follows.

Case \(\Phi_4\): Once more label the edges as shown in Fig. 2. Assume by way of contradiction that the triple graph \(T^\pm\) has a totally positive twist-free claw rotation assignment that includes the claw rotation 365.

We note that \(T^\pm\) has 4 components. One with 14 vertices corresponds to claws that contain no pairs of edges of the form \(ii\). There are 3 other components \(C_i\); their underlying graphs are \(K_{2,4}\) and every theta corresponding to an edge in the \(i\)th component contains the parallel edges \(ii\). We begin with the big component.

The theta graph \((32, 64, 5)\) is of type 0. Since the corresponding edge in \(T^\pm\) is positive, the fixed rotation 365 implies that we have the claw rotation 254. Likewise, theta graphs \((3, 61, 52)\) and \((3, 61, 52)\) and rotation 365 force rotations \(123\) and \(132\), respectively. The type 1 theta graph \((2, 53, 41)\) and the type 2 theta graph \((2, 53, 41)\) together with the rotation 254 force rotations \(132\) and \(123\), respectively.

By the above, we have claw rotations \(231\) and \(213\). If we had \(211\) we would have a twist with 2 playing the role of x. Hence we must have \(211\). Similarly, to avoid a twist with rotations \(213\) and \(231\) we have to have rotation \(211\).

The rotation \(211\) forces \(114\) using the type 2 theta graph \((1, 1, 24)\). The rotation \(211\) forces \(114\) using the type 1 theta graph \((1, 1, 24)\). But these latest two rotations are not the same, giving our desired contradiction.

Case \(\Psi_5\): Label the edges as shown in Fig. 2. Assume by way of contradiction that the triple graph \(T^\pm\) has a totally positive twist-free claw rotation system that contains the claw rotation \(114\). Note that \(T^\pm\) has 4 components; a big one with underlying graph \(K_{1,8}\).
and three smaller ones with underlying graphs $K_{1,4}$. Our argument uses the three smaller components.

One component has the claw $1\bar{1}4$. There are 4 thetas on this claw. Since these edges are all positive we get claw rotations $\bar{1}13$, $\bar{1}12$, $\bar{1}2\bar{1}$, and $\bar{1}3\bar{1}$.

The other two components can be found by applying the automorphism of $\Psi_5$ $(1, 2, 3)$ $(\bar{1}, 2, 3)(4, 5, 6)$ having three orbits of length three. This, for example, maps claw $1\bar{1}4$ to $\bar{2}25$, but we do not know the rotation on the latter’s claw. Mapping the theta graphs from the preceding paragraph we see that the claws’ rotations $\bar{2}25$, $\bar{2}23$, $\bar{2}21$, $\bar{2}11$, and $\bar{2}33$ are all related: either they all appear in the order shown, or they all appear in the reverse order. However, if they appear in the reverse order, then we have a twist $12\bar{1}$, $1\bar{1}2$, and $1\bar{2}2$. So they must appear in the order shown.

Applying the automorphism a second time we have rotations $3\bar{3}6$, $3\bar{3}2$, $3\bar{3}1$, $3\bar{1}3$, and $3\bar{3}2$. To avoid a twist as above, they must be in this order and not the reverse.

We now have our desired contradiction: $31\bar{3}$, $3\bar{1}1$, and $3\bar{1}1$ form a twist.

This completes the proof of our main result and this section. □

3. Conclusion

The proof of our main result relies heavily on Zaslavsky’s characterization of projective-planar signed graphs. Thus a generalization to other surfaces is unlikely, as the analog of Zaslavsky’s Theorem is unknown in these cases.

It would be nice to see if there is a simple characterization of unsigned projective-planar graphs in terms of their triple graphs. This would be more in the spirit of [3]. The corresponding set of unsigned obstructions is known, see [1].

The authors tried to give a proof of Kuratowski’s Theorem using theta graphs, but abandoned the project when it appeared to be at least as long as current proofs. Still, we hope that theta graphs can give new insights into graph embeddings of both signed and unsigned graphs.

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