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Properties of Certain Families of 2k-Cycle-Free Graphs*

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Let $v = v(G)$ and $e = e(G)$ denote the order and size of a simple graph G, respectively. Let $\mathscr{G} = \{G_i\}_{i \geq 1}$ be a family of simple graphs of magnitude $r > 1$ and constant $\lambda > 0$, i.e., $e(G_i) = (\lambda + o(1))\, v(G_i)^r$, $i \to \infty$. For any such family $\mathscr{G}$, whose members are bipartite and of girth at least $2k+2$, and every integer t, $2 \leqslant t \leqslant k-1$, we construct a family $\tilde{\mathscr{G}}_t$ of graphs of the same magnitude r, of constant greater than λ, and all of whose members contain each of the cycles $C_4, C_6, ..., C_{2t}$, but none of the cycles $C_{2t+2}, ..., C_{2k}$. We also prove that for every family of $2k$-cycle-free extremal graphs (i.e., graphs having the greatest size among all $2k$-cycle-free graphs of the same order), all but finitely many such graphs must be either non-bipartite or have girth at most $2k-2$. In particular, we show that the best known lower bound on the size of $2k$-cycle-free extremal graphs for $k = 3, 5$, namely $(2^{-(k+1)/k} + o(1))\, v^{(k+1)/k}$, can be improved to $((k-1) \cdot k^{-(k+1)/k} + o(1))\, v^{(k+1)/k}$.

1. INTRODUCTION

All graphs we consider are simple. Let $\mathscr{F}$ be a family of graphs, and let G be a graph which contains no subgraph isomorphic to a graph from $\mathscr{F}$. Then G is called $\mathscr{F}$-free. We consider the following problem from extremal

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graph theory: Find the greatest number of edges $\mathrm{ex}(v, \mathscr{F})$ of any graph on v vertices which is $\mathscr{F}$-free. In this context $\mathscr{F}$ is called the *family of forbidden graphs*.

Let k be a fixed but arbitrary integer, $k \geqslant 2$, and let C_{2k} denote the $2k$-cycle. By the *even circuit theorem* (see [2, 4, 6]) we have an upper bound for $\mathrm{ex}(v, \{C_{2k}\})$:

$$\mathrm{ex}(v, \{C_{2k}\}) = O(v^{1+1/k}).$$

A general lower bound for $\mathrm{ex}(v, \{C_{2k}\})$ is also available. In fact [11, 12],

$$\mathrm{ex}(v, \{C_{2k}\}) \geqslant \mathrm{ex}(v, \{C_3, C_4, ..., C_{2k+1}\}) = \Omega(v^{1+2/(3k+3)}).$$

For $3 \leqslant k \leqslant 8$, this bound can be improved [10] to

$$\mathrm{ex}(v, \{C_{2k}\}) \geqslant \mathrm{ex}(v, \{C_3, C_4, ..., C_{2k+1}\}) = \Omega(v^{1+1/(2k-3)}),$$

and for certain values of k the bound can be improved still further (see [14]). Most notably, $\mathrm{ex}(v, \{C_4\}) \sim \frac{1}{2} v^{3/2}$ [3, 5, 7, 8] and $\mathrm{ex}(v, \{C_{2k}\}) = \Omega(v^{1+1/k})$ for $k=3$ and 5 [1, 9, 13], but similar results have not been proved for any other values of k.

Let $\mathscr{F}$ be a family of graphs, and let G be an $\mathscr{F}$-free graph of order $v = v(G)$ and size $e = e(G)$ for which $e = \mathrm{ex}(v, \mathscr{F})$. We call such graphs G *extremal* (or *$\mathscr{F}$-extremal* when we wish to emphasize $\mathscr{F}$). Let $\mathscr{H} = \{H_i\}_{i \geqslant 1}$ be an arbitrary family of $\mathscr{F}$-free graphs such that $\{v_i = v(H_i)\}_{i \geqslant 1}$ is an increasing sequence of integers. We denote by $r(\mathscr{H})$ the least number r such that $e(H_i) = O(v_i^r)$, should such an r exist, and we call it the *magnitude* of $\mathscr{H}$. For a family $\mathscr{H}$ of magnitude r, let $\lambda(\mathscr{H})$ denote the greatest λ such that $e(H_i) = (\lambda + o(1))\, v_i^r$, $i \to \infty$, should such a λ exist, and call it the *constant* of $\mathscr{H}$. Let $\mathscr{G} = \{G_i\}_{i \geqslant 1}$ be any family of $\mathscr{F}$-extremal graphs with magnitude $r(\mathscr{G})$ and constant $\lambda(\mathscr{G})$, and $v_i = v(G_i)$, $i \geqslant 1$. Then $\mathrm{ex}(v_i, \mathscr{F}) = (\lambda(\mathscr{G}) + o(1))\, v_i^{r(\mathscr{G})}$. Note that for fixed $\mathscr{F}$, all families of extremal graphs have the same magnitude and constant, so we may denote them by $r_{\mathscr{F}}$ and $\lambda_{\mathscr{F}}$. Let us call $\mathscr{H}$ *magnitude extremal* (or *$\mathscr{F}$-magnitude extremal*) if $r(\mathscr{H}) = r_{\mathscr{F}}$. In practice, investigation of the constant is limited to magnitude extremal graphs.

As an example, let $\mathscr{F} = \{C_4\}$. A family $\mathscr{H}$ of magnitude extremal graphs consists of bipartite point–line incidence graphs of projective planes of order q (thick generalized triangles). For this family we have $r(\mathscr{H}) = \frac{3}{2}$ and $\lambda(\mathscr{H}) = 2^{-3/2}$. But since $\lambda_{\mathscr{F}} = \frac{1}{2}$ (see any of [3, 5, 7, 8]) the graphs of $\mathscr{H}$ are not extremal.

For the remainder of this note we restrict ourselves to the case $\mathscr{F} = \{C_{2k}\}$. We also write r_k and λ_k, should they exist, in place of the more cumbersome $r_{\{C_{2k}\}}$ and $\lambda_{\{C_{2k}\}}$. Note that in this case $1 < r_k \leqslant 1 + 1/k$.

We now mention an interesting phenomenon concerning certain families of $\{C_{2k}\}$-magnitude extremal graphs. Namely, in every case in which r_k is known (viz. $r_2=\frac{3}{2}$, $r_3=\frac{4}{3}$, $r_5=\frac{6}{5}$) there exists a family of $\{C_{2k}\}$-magnitude extremal graphs whose members are bipartite graphs of high girth g_k (viz. $g_2=6$, $g_3=8$, $g_5=12$) [1, 9, 13]. We think this is quite remarkable since the only requirement of such graphs is that they do not contain $2k$-cycles, and it would seem (if not for the aforementioned examples which illustrate otherwise) that the magnitude could always be increased by selectively adding edges to form smaller cycles. In this note we show that while $\{C_{2k}\}$-magnitude extremality can be achieved with families of bipartite graphs of high girth, ordinary extremality cannot! In particular, we give a general and simple constructive procedure for producing from any family of high girth bipartite $\{C_{2k}\}$-magnitude extremal graphs a family of $\{C_{2k}\}$-magnitude extremal graphs with greater constant. More precisely, we prove the following.

THEOREM. *Let $k\geqslant 3$ and let $\mathscr{G}$ be a family of $2k$-cycle-free graphs with magnitude $r>1$ and constant $\lambda>0$, the members of which are bipartite graphs of girth at least $2k+2$. Then, for any t, $2\leqslant t\leqslant k-1$, there exists a family $\tilde{\mathscr{G}}_t$ of $2k$-cycle-free graphs with magnitude r and constant $\tilde{\lambda}\geqslant t(2/(t+1))^r\lambda>\lambda$, all of whose members are bipartite and contain each of the cycles $C_4, C_6, ..., C_{2t}$, and none of the cycles $C_{2t+2}, ..., C_{2k}$. Consequently, any family of $\{C_{2k}\}$-extremal graphs must consist (with finitely many exceptions) either of graphs that are non-bipartite or have girth at most $2k-2$.*

2. The Construction

Let G be an arbitrary bipartite graph, the partitions of which we denote by P (points) and L (lines) for convenience. For any fixed integer $t\geqslant 2$, construct the bipartite graph $\tilde{G}=\tilde{G}(t)$ as follows: Let $P^1, P^2, ..., P^t$ denote t disjoint copies of the (labeled) points of P, i.e., $P^i=\{p^i \mid p\in P\}$, $i=1, 2, ..., t$. The vertex and edge sets are now defined by $V(\tilde{G})=L\cup P^1\cup P^2\cup\cdots\cup P^t$ and $E(\tilde{G})=\{\{p^i, l\}\mid\{p, l\}\in E(G), i=1, 2, ..., t\}$.

We will need the following two lemmas.

LEMMA 1. *Let $\Delta>1$ be the maximum degree among all points of bipartite graph G. Then $\tilde{G}$ contains each of $C_4, C_6, ..., C_{2m}$ as subgraphs, where $m=\min\{t, \Delta\}$.*

Proof. Let $p\in P$ have degree Δ and let the neighbors of p be $l_1, l_2, ..., l_\Delta\in L$. Then $l_1p^1l_2p^2l_3p^3\cdots l_ip^il_1$ is clearly a $2i$-cycle, $2\leqslant i\leqslant m$.

Let $\{G_i\}_{i\geqslant 1}$ be a family of bipartite graphs with magnitude $r>1$ and constant $\lambda>0$. Without loss of generality we may assume $|L|\geqslant|P|$ for each G_i. Then, for any $t\geqslant 2$, family $\{\tilde{G}_i\}_{i\geqslant 1}$ has magnitude $\tilde{r}=r$ but the constant $\tilde{\lambda}$ need not exist. In fact, the existence of $\tilde{\lambda}$ depends on the behavior of the sequence $\{\mu_i\}$, where μ_i is defined to be the ratio of the number $|P|$ of points in G_i to the number $v=(G_i)$ of vertices. Note that $0<\mu_i\leqslant\frac{1}{2}$. If $\mu_i\rightarrow\mu$, $i\rightarrow\infty$, then, as we shall show in Lemma 2, $\tilde{\lambda}$ exists; in fact $\tilde{\lambda}=t[1+(t-1)\mu]^{-r}\lambda$. In any case, we can always find a subsequence of $\{\mu_i\}$ which converges to some μ, $0\leqslant\mu\leqslant\frac{1}{2}$, and hence a subfamily $\{\tilde{G}_{i(\mu)}\}$ of $\{\tilde{G}_i\}_{i\geqslant 1}$ for which $\tilde{\lambda}$ exists and is described as above. Therefore, without loss of generality, we may assume that $\mu_i\rightarrow\mu$ for the original family $\{\tilde{G}_i\}$.

LEMMA 2. *$\{\tilde{G}_i\}$ has magnitude $\tilde{r}=r$ and constant $\tilde{\lambda}=t[1+(t-1)\mu]^{-r}\lambda\geqslant t(2/(t+1))^r\lambda$. Moreover, $\tilde{\lambda}>\lambda$ if $2\leqslant t\leqslant(r-1)^{-1}$.*

Proof. Clearly, graph $\tilde{G}_i$ has parameters $\tilde{v}=v+(t-1)|P|$ and $\tilde{e}=te$. Then $\tilde{e}\tilde{v}^{-r}=te[v+(t-1)|P|]^{-r}=tev^{-r}[1+(t-1)\mu_i]^{-r}\rightarrow t[1+(t-1)\mu]^{-r}\lambda:=\tilde{\lambda}$. Since $1+(t-1)\mu\leqslant\frac{1}{2}(t+1)$, then $\tilde{\lambda}\geqslant t(2/(t+1))^r\lambda$. The last statement of the lemma follows from the fact that the function $f(t)=t(2/(t+1))^r$ is increasing on the interval $[1,(r-1)^{-1}]$.

Proof of the Theorem. Let $k\geqslant 3$ and let $\mathcal{G}$ be a family of graphs as in the theorem statement. Thus $G\in\mathcal{G}$ is bipartite and has girth at least $2k+2$. For fixed t, $2\leqslant t\leqslant k-1$, form the family $\tilde{\mathcal{G}}_t=\{\tilde{G}\mid\Delta(G)\geqslant t\}$. (Note that $\Delta\geqslant 2e/v\sim 2\lambda v^{r-1}\rightarrow\infty$ as $v\rightarrow\infty$, so that $\Delta\geqslant t$ for v sufficiently large.) By Lemma 1, $\tilde{G}$ contains each of the cycles $C_4, C_6, ..., C_{2t}$. Suppose $\tilde{G}$ contains a $2s$-cycle for some s, $(t+1)\leqslant s\leqslant k$, which we describe by its sequence of consecutive vertices

$$a_1b_1a_2b_2\ldots a_sb_s, \tag{1}$$

where $a_i\in P^1\cup\cdots\cup P^t$, $b_i\in L$, $1\leqslant i\leqslant s$. We consider the closed walk in G,

$$\eta(a_1)\,b_1\eta(a_2)\,b_2\ldots\eta(a_s)\,b_s, \tag{2}$$

which is the image of (1) under the map $\eta\colon V(\tilde{G})\rightarrow V(G)$ defined by $\eta(p^i)=p$ for all $p^i\in P^i$, $1\leqslant i\leqslant t$, and $\eta(l)=l$ for all $l\in L$. Note that while the b_i are certainly distinct, this need not be the case for the $\eta(a_i)$.

Sequence (2) defines an eulerian multigraph in which every edge has multiplicity at most two and each line b_i has degree exactly two. Delete from this multigraph all edges of multiplicity two (all two-cycles). Then, in the resulting simple graph, each connected component is eulerian. But if there exists such a component which is not an isolated vertex, then G contains a cycle of length at most $2s$, a contradiction. Thus, every component is an isolated vertex; i.e., every edge in the multigraph has multiplicity two. This implies that all $\eta(a_i)$ are equal, so (1) has the form

$$p^1 b_1 p^2 b_2 \cdots p^s b_s$$

for some $p \in P$, an impossibility as $\tilde{G}$ has only t copies of P and $t \leqslant s-1$. We conclude that $\tilde{G}$ is free of all cycles $C_{2t+2}, ..., C_{2k}$. Finally, by Lemma 2, $\tilde{\mathscr{G}}_t$ has the same magnitude as $\mathscr{G}$ with constant as in the statement of the theorem. It remains only to show that $\tilde{\lambda} > \lambda$. But since $r \leqslant 1 + 1/k$, this follows immediately from Lemma 2.

COROLLARY. $\lambda_3 \geqslant 2/3^{4/3}$, $\lambda_5 \geqslant 4/5^{6/5}$.

Proof. Apply our construction to the known families of magnitude extremal graphs [1, 9, 13] which have magnitudes $\frac{4}{3}$ and $\frac{6}{5}$ and constants $2^{-4/3}$ and $2^{-6/5}$, respectively.

3. CONCLUDING REMARKS

It would seem that the constructive procedure described in Section 2 could be applied to graphs with a forbidden family different from $\{C_{2k}\}$. While this is probably the case, the situation might be a bit subtle. For example, consider $\mathscr{F} = \{K_{3,3}\}$, where $K_{3,3}$ is the complete bipartite graph on $3+3$ vertices. It is easy to see that if G is a $\{K_{3,3}\}$-free bipartite graph which just happens to have no $K_{2,3}$ subgraphs, then $\tilde{G}(2)$ will be $\{K_{3,3}\}$-free and with larger constant. This result shows that any family of $\{K_{3,3}\}$-extremal graphs must consist either of non-bipartite graphs or graphs which contain a copy of $K_{2,3}$. The point is that, as can easily be shown from a result of Brown [3], *every* family of $\{K_{3,3}\}$-magnitude extremal graphs must consist of graphs which contain a $K_{2,3}$ subgraph, so that, in this case, there are no graphs to which our construction applies.

Note added in proof. Recently the authors proved that $\mathrm{ex}(v, \{C_3, C_4, ..., C_{2k+1}\}) = \Omega(v^{1+2/(3k-3+\varepsilon)})$, where $\varepsilon = 0$ if k is odd and $\varepsilon = 1$ if k is even. To our knowledge this is the best known asymptotic lower bound for all k, $k \geqslant 2$, $k \neq 5$. (The result will appear elsewhere.)

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