JOURNAL OF FUNCTIONAL ANALYSIS 18, 429–454 (1975)

One-Sided Maximal Functions and H^p

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Communicated by Paul Malliavin

Received May, 1974

Let N be the nontangential maximal function of a function u harmonic in the Euclidean half-space $\mathbf{R}^n \times (0, \infty)$ and let N^- be the nontangential maximal function of its negative part. If $u(0, y) = o(y^{-n})$ as $y \to \infty$, then $\| N \|_p \leqslant c_p \| N^- \|_p$, $0 < p < 1$, and more. The basic inequality of the paper (Theor. 2.1) can be used not only to derive such global results but also may be used to study the behavior of u near the boundary. Similar results hold for martingales with continuous sample functions. In addition, Theorem 1.3 contains information about the zeros of u. For example, if u belongs to H^p for some $0 < p < 1$, then every thick cone in the half-space must contain a zero of u.

1. Introduction

We study here to what extent a one-sided maximal function can control the ordinary nontangential maximal function. Our results have applications to H^p spaces and the boundary behavior of harmonic functions. Similar results hold for martingales with continuous paths. In fact, this is the simplest setting for such problems and our study of the martingale case motivated the present work. We also obtain some results about the zeros of harmonic functions.

Let u be harmonic in the Euclidean half-space

$$\mathbf{R}_+^{n+1} = \{(x, y) : x \in \mathbf{R}^n, y > 0\},$$

and let $N = N_a(u)$ denote the nontangential maximal function of u defined on $\mathbf{R}^n$ by

$$N(x) = \sup\{| u(s, y)| : (s, y) \in \Gamma_a(x)\}, \tag{1.1}$$

where $\Gamma_a(x) = \{(s, y) \in \mathbf{R}_+^{n+1}: | x - s | < ay\}$ and a is a positive

* This research was supported in part by the National Science Foundation under Grant GP 28154.

real number. Let $N^- = N_a^-(u)$ be the nontangential maximal function of $u^- = (-u) \vee 0$:

$$N^-(x) = \sup\{u^-(s, y) : (s, y) \in \Gamma_a(x)\}. \tag{1.2}$$

We call N^- the one-sided nontangential maximal function of u.

Let Φ be a nondecreasing continuous function on $[0, \infty]$ such that $\Phi(0) = 0$ and

$$\Phi(\beta\lambda) \leqslant \gamma\Phi(\lambda), \tag{1.3}$$

for some $\beta > \gamma > 1$ and all $\lambda > 0$. For example, if $0 < p < 1$, then $\Phi(\lambda) = \lambda^p$ defines such a function.

The following theorem, proved in Section 2, describes one way the ordinary nontangential maximal function N is controlled by its one-sided version N^-.

THEOREM 1.1. *If the harmonic function u satisfies*

$$u(0, y) = o(y^{-n}) \tag{1.4}$$

as $y \to \infty$, then

$$\int_{\mathbf{R}^n} \Phi(N)\, dx \leqslant c \int_{\mathbf{R}^n} \Phi(N^-)\, dx \tag{1.5}$$

and the choice of c depends only on β, γ, n, and a.

In particular, if (1.4) holds and $0 < p < 1$, then

$$\| N \|_p \leqslant c \, \| N^- \|_p \tag{1.6}$$

and the choice of c depends only on p, n, and a.

Throughout the paper c denotes a positive real number not necessarily the same number from one use to the next.

Analogous results are true for functions u harmonic in the unit ball of $\mathbf{R}^n$. In this case the normalization (1.4) takes the simple form $u(0) = 0$.

Note that (1.4) cannot be replaced by $u(0, y) = O(y^{-n})$. Consider the example

$$u(x, y) = y(|\, x\,|^2 + y^2)^{-(n+1)/2}.$$

Here $u(0, y) = y^{-n}$ but N^- vanishes. Therefore, for nonconstant Φ, the left-hand side of (1.5) is positive but the right-hand side is zero.

Also, it is not hard to show that (1.6) does not hold in general if $p \geqslant 1$. See Example 6.2.

The inequality (1.6) can be improved. Call N_0^-, defined by

$$N_0^-(x) = \sup_{y>0} u^-(x, y),$$

the one-sided radial maximal function of u. Using a method of Fefferman and Stein [11], we can show that, for $0 < p < \infty$,

$$\| N^- \|_p \leqslant c \, \| N_0^- \|_p \tag{1.7}$$

with the choice of c depending only on p, n, and a. Combining (1.6) and (1.7), we obtain the following theorem.

THEOREM 1.2. *If the harmonic function u satisfies $u(0, y) = o(y^{-n})$ as $y \to \infty$, then, for $0 < p < 1$,*

$$\| N \|_p \leqslant c \, \| N_0^- \|_p$$

and the choice of c depends only on p, n, and a.

The corresponding Φ-inequality does not hold in the radial case; that is, if N^- in Theorem 1.1 were replaced by N_0^- or even by the two-sided radial maximal function, then the statement of Theorem 1.1 would no longer be true. This is most easily seen for the analogous case of u harmonic in the unit ball. Consult [4, Remark (a), p. 152].

If $0 < p < \infty$, let H^p be the set of all harmonic functions $u\colon \mathbf{R}_+^{n+1} \to \mathbf{R}$ such that the nontangential maximal function of u belongs to L^p. It follows from the recent work of Fefferman and Stein [11] and from [4] in the case $n = 1$, that this is one natural way to define H^p. Theorem 1.2 shows that given (1.4) and $0 < p < 1$, a sufficient condition for u to belong to H^p is that its one-sided radial maximal function belongs to L^p. The condition is obviously necessary and, moreover, if u does belong to H^p for some $0 < p < 1$, then (1.4) must hold; see Lemma 3 of [11].

The strategy for proving Theorem 1.1 derives from [2]. First a distribution function inequality (Theor. 2.1) is proved. The Φ-inequality of Theorem 1.1 then follows rather easily. One auxiliary result is required which we state now and prove in Section 3.

THEOREM 1.3. *Let $0 < p < 1$ and suppose the harmonic function u satisfies $u(0, y) = o(y^{-n})$ and*

$$\int_{\mathbf{R}^n} [u^-(x, y)]^p \, dx = O(1)$$

as $y \to \infty$. *Then, for some* $a > 0$ *and all* $t > 0$, *the translated cone*

$$\Gamma_a^{\,t}(0) = \{(x, y) \in \mathbf{R}_+^{n+1} : |\,x\,| < a(y - t)\}$$

with vertex at $(0, t)$ *contains a zero of* u. *The choice of* a *depends only on* n *and* p.

In particular, if u belongs to H^p for some $0 < p < 1$, then $\{u = 0\} \cap \Gamma_a(0)$ is unbounded for some $a > 0$. By way of contrast, if $p > 1$ and $u \in H^p$, the zero set $\{u = 0\}$ can be empty. The space H^1 is intermediate. If $u \in H^1$, then $\{u = 0\}$ is nonempty. (See [15, p. 48] or note that $|\,u(0, |\,x\,|)| \leqslant N(ax)$, $0 \neq x \in \mathbf{R}^n$, so $\int_0^\infty |\,u(0, y)|\, y^{n-1}\, dy$ is finite and (3.3) cannot hold.) However, there does exist a $u \in H^1$ such that $\{u = 0\} \cap \Gamma_a(0)$ is bounded for all $a > 0$; see Example 6.4.

The condition $u(0, y) = o(y^{-n})$ is not enough by itself to imply the conclusion of Theorem 1.3. This follows directly from Example 6.4 or from the fact that if $u \in H^1$, then $u(0, y) = o(y^{-n})$ as $y \to \infty$; see the remark after Example 6.4.

We now describe the martingale analogs of Theorem 1.1 and the distribution function inequality (Theor. 2.1) leading to it. Let $Y = \{Y(t), 0 \leqslant t < \infty\}$ be a local martingale with continuous sample functions such that $Y(0) = 0$. Let $M = \sup_{t \geqslant 0} |\,Y(t)|$ and $M^- = \sup_{t \geqslant 0} Y^-(t)$ where $Y^-(t) = (-Y(t)) \vee 0$.

THEOREM 1.4. *Suppose that* Φ *is a nondecreasing continuous function on* $[0, \infty]$ *such that* $\Phi(0) = 0$ *and the growth condition* (1.3) *holds with* $\beta > \gamma > 1$. *Then*

$$E\Phi(M) \leqslant cE\Phi(M^-) \tag{1.8}$$

and the choice of c *depends only on* β *and* γ.

As usual, E denotes integration over the whole space with respect to the probability measure P.

In particular, for $0 < p < 1$,

$$\|\,M\,\|_p \leqslant c_p \|\,M^-\,\|_p\,. \tag{1.9}$$

This inequality does not hold in general for $p \geqslant 1$; see Example 6.3.

THEOREM 1.5. *Let* $\beta > 1$ *and* $0 < \delta < \beta$. *Then, for all* $\lambda > 0$,

$$P(M > \beta\lambda, M^- \leqslant \delta\lambda) \leqslant ((1 + \delta)/(\beta + \delta))\, P(M > \lambda). \tag{1.10}$$

We prove these two theorems in Section 5. Here we note that by letting $\beta \to \infty$ in (1.10), we get

$$P(M = \infty, M^- \leqslant \delta\lambda) = 0.$$

If we now let $\lambda \to \infty$, we obtain

$$\{M^- < \infty\} \subset_{\text{a.e.}} \{M < \infty\}.$$

A similar calculation with the inequality of Theorem 2.1 shows how Carleson's theorem on the boundary behavior of harmonic functions can be derived from Calderón's theorem. We return to these questions near the end of Section 2.

2. The Basic Inequality and Its Applications

The close relationship between the one-sided nontangential maximal function of u and its ordinary nontangential maximal function is best described by a distribution function inequality. Here we give the inequality for the case of truncated cones. The nontruncated version also holds (see Sect. 6) but is not as useful in the applications.

Let $\Gamma_{a,h}(x)$ denote the truncated cone

$$\{(s, y) : | x - s | < ay, 0 < y < h\},$$

and let $N_{a,h}$ and $N_{a,h}^-$ be defined by (1.1) and (1.2) with $\Gamma_{a,h}(x)$ replacing $\Gamma_a(x)$.

If Q is a measurable subset of $\mathbf{R}^n$, let m_Q be the measure defined by $m_Q(E) = m(E \cap Q)$ where E is any measurable subset of $\mathbf{R}^n$ and m is Lebesgue measure.

Theorem 2.1. *Let $0 < a < b$, $0 < h < k$, $0 < \delta < \beta$, $\beta > 1$, and suppose that $Q \subset \mathbf{R}^n$ is a cube with center q and diameter $2ah$. Then*

$$m_Q(N_{a,h} > \beta\lambda, N_{b,k}^- \leqslant \delta\lambda) \leqslant \epsilon m_Q(N_{a,h} > \lambda) \tag{2.1}$$

for all positive λ satisfying $\lambda \geqslant u(q, h)$ where

$$\epsilon = c((1 + \delta)/(\beta + \delta)) + (1/b) + (h/k)) \tag{2.2}$$

and the choice of c depends only on n and a.

The proof is given in Section 4.

To prove Theorem 1.1 with this inequality we need the following lemma from [1].

LEMMA 2.1. *Let Φ be a continuous nondecreasing function on $[0, \infty]$ with $\Phi(0) = 0$. Let f and g be nonnegative measurable functions on a finite measure space $(\Omega, \mathscr{A}, \mu)$ and $\beta > 1$, $\gamma > 1$, $\epsilon > 0$ real numbers such that $\gamma\epsilon < 1$, $\Phi(\beta\lambda) \leqslant \gamma\Phi(\lambda)$, and*

$$\mu(g > \beta\lambda, f \leqslant \lambda) \leqslant \epsilon\mu(g > \lambda)$$

for all $\lambda > 0$. Then

$$\int_\Omega \Phi(g)\, d\mu \leqslant \gamma(1 - \gamma\epsilon)^{-1} \int_\Omega \Phi(f)\, d\mu.$$

Proof of Theorem 1.1. Suppose that a, h, q, and Q are as in Theorem 2.1 and $u(q, h) = 0$. Let $\beta > \gamma > 1$ be the constants of the growth condition (1.3). By induction, $\Phi(\beta^j\lambda) \leqslant \gamma^j\Phi(\lambda)$, $j \geqslant 1$. Also, $\beta^j\gamma^{-j} \to \infty$ as $j \to \infty$. Fix j just large enough so that $\beta^j > a$ and $8c_{(2.2)} < \beta^j\gamma^{-j}$ where $c_{(2.2)}$ denotes the number c in (2.2). Now let $b = \beta^j$ and $k = bh$. Then, by Theorem 2.1,

$$\begin{aligned} m_Q(N_{a,h} > b\lambda, N_{b,k}^- \leqslant \lambda) &\leqslant c_{(2.2)}(4/b)\, m_Q(N_{a,h} > \lambda) \\ &\leqslant (1/2)\, \gamma^{-j} m_Q(N_{a,h} > \lambda) \end{aligned}$$

for all $\lambda > 0$. Lemma 2.1 now gives

$$\int_{\mathbf{R}^n} \Phi(N_{a,h})\, dm_Q(x) \leqslant 2\gamma^j \int_{\mathbf{R}^n} \Phi(N_{b,k}^-)\, dm_Q(x) \leqslant 2\gamma^j \int_{\mathbf{R}^n} \Phi(N_b^-)\, dx.$$

By the proof of Lemma 2 in [3],

$$m(N_b^- > \lambda) \leqslant cm(N_a^- > \lambda), \qquad \lambda > 0, \tag{2.3}$$

where the choice of c depends only on n and the ratio a/b. Therefore,

$$\int_{\mathbf{R}^n} \Phi(N_b^-)\, dx = \int_0^\infty m(N_b^- > \lambda)\, d\Phi(\lambda) \leqslant c \int_{\mathbf{R}^n} \Phi(N_a^-)\, dx, \tag{2.4}$$

and we have

$$\int_Q \Phi(N_{a,h})\, dx \leqslant c \int_{\mathbf{R}^n} \Phi(N_a^-)\, dx \tag{2.5}$$

for u satisfying $u(q, h) = 0$ with the choice of c depending only on β, γ, n, and a.

To complete the proof of Theorem 1.1, we note that if the right-hand side of (1.5) is infinite or if Φ is identically zero, then (1.5) is trivially true. Therefore, we may assume that the right-hand side of (1.5) is finite and, without loss of generality, that $\Phi(1) = 1$. By the growth condition (1.3), $\Phi(1) \leqslant \gamma^i \Phi(\beta^{-i})$ so $\gamma^{-i} \leqslant \Phi(\beta^{-i})$, $i \geqslant 0$. Let p satisfy $\beta^p = \gamma$. Then, for $\beta^{-i-1} < \lambda \leqslant \beta^{-i}$,

$$\begin{aligned} \lambda^p &\leqslant (\beta^{-i})^p = \gamma^{-i} \leqslant \Phi(\beta^{-i}) \\ &\leqslant \gamma\Phi(\beta^{-i-1}) \leqslant \gamma\Phi(\lambda). \end{aligned}$$

Hence,

$$\lambda^p \leqslant \gamma\Phi(\lambda), \qquad 0 \leqslant \lambda \leqslant 1. \tag{2.6}$$

If $u^-(x, y) > 1$ for a point (x, y) in $\mathbf{R}_+^{n+1}$, then the set $\{N^- > 1\}$ contains the ball in $\mathbf{R}^n$ with center x and radius ay. But

$$m(N^- > 1) \leqslant \int_{\mathbf{R}^n} \Phi(N^-)\, dx < \infty.$$

Accordingly, there is a number $y_0 > 0$ such that if $(x, y) \in \mathbf{R}_+^{n+1}$ and $y > y_0$, then $u^-(x, y) \leqslant 1$. By (2.6), for all $y > y_0$,

$$\begin{aligned} \int_{\mathbf{R}^n} [u^-(x, y)]^p\, dx &\leqslant \gamma \int_{\mathbf{R}^n} \Phi(u^-(x, y))\, dx \\ &\leqslant \gamma \int_{\mathbf{R}^n} \Phi(N^-(x))\, dx < \infty. \end{aligned}$$

By this inequality and (1.4), the conditions of Theorem 1.3 are satisfied. Therefore, there is a number $a > 0$ such that the translated cone $\Gamma_a{}^t(0)$ contains a zero of u for all $t > 0$. The choice of a depends only on p, defined above, and n, and hence only on β, γ, and n.

We now show that (1.5) holds for this particular number a. This is enough to prove Theorem 1.1 since (1.5) holding for one value of a implies that it holds for all a by (2.4) and the analogous inequality, [3, (15)], for the ordinary nontangential maximal function.

Choose (q_i, h_i) in $\mathbf{R}_+^{n+1}$ so that $u(q_i, h_i) = 0$, $h_1 \leqslant \cdots \leqslant h_i \to \infty$, and the corresponding cubes Q_i satisfy $Q_1 \subset \cdots \subset Q_i \to \mathbf{R}^n$ as $i \to \infty$. This is possible by Theorem 1.3. Applying (2.5) to $Q = Q_i$ and $h = h_i$ and using the monotone convergence theorem, we obtain (1.5) and the proof of Theorem 1.1 is complete.

Theorem 1.2 now follows rather easily with the aid of the following lemma.

LEMMA 2.2 (Hardy–Littlewood–Fefferman–Stein). *Let v be a nonnegative subharmonic function defined on a ball $B \subset \mathbf{R}^n$. If x is the center of B, then, for $0 < p < \infty$,*

$$v^p(x) \leqslant c_p \int_B v^p(s)\, ds / | B |.$$

For a proof in the case that v is the modulus of a harmonic function, see [11]; the proof for the nonnegative subharmonic case is the same.

Fefferman and Stein [11] use the lemma to show that the radial maximal function N_0 of a function u harmonic in $\mathbf{R}_+^{n+1}$ dominates the nontangential maximal function in the L^p sense:

$$\| N \|_p \leqslant c_p \| N_0 \|_p, \qquad 0 < p < \infty. \tag{2.7}$$

The same argument applied to the subharmonic function $v = (-u) \vee 0 = u^-$ proves (1.7). As a consequence, Theorem 1.2 follows from Theorem 1.1.

We now consider the implications of the distribution function inequality of Theorem 2.1 for the boundary behavior of harmonic functions. In [5], Calderón showed that if the harmonic function u is nontangentially bounded at every point of a measurable set E, then u has a nontangential limit at almost every point of E. Later, Carleson [7] showed that the same conclusion holds if u is merely assumed to be nontangentially bounded from below.

Here let E be the set of all points at which u is nontangentially bounded from below. That is,

$$E = \{x : N_{b,k}^-(x) < \infty \text{ for some pair } (b, k)\}.$$

A familiar point of density argument [6] shows that

$$E =_{\text{a.e.}} \{N_{b,k}^- < \infty\}, \qquad b > 0, \quad k > 0. \tag{2.8}$$

Using Theorem 2.1, we now show that

$$\{N_{b,k}^- < \infty\} \subset_{\text{a.e.}} \{N_{a,h} < \infty\}. \tag{2.9}$$

Thus u is nontangentially bounded at almost every point of E and Carleson's theorem follows from Calderón's theorem.

To prove (2.9), let $\beta \to \infty$ in (2.1) or (4.1) to obtain

$$m_Q(N_{a,h} = \infty, N^-_{b,k} \leqslant \delta\lambda) \leqslant c(b^{-1} + hk^{-1}) \mid Q \mid$$

for all positive $\lambda \geqslant u(q, h)$. Now let $\lambda \to \infty$ to obtain

$$m_Q(N_{a,h} = \infty, N^-_{b,k} < \infty) \leqslant c(b^{-1} + hk^{-1}) \mid Q \mid. \tag{2.10}$$

By (2.8), the left-hand side of (2.10) has the same value for all b, k but the right-hand side converges to zero as $b, k \to \infty$. Therefore,

$$m_Q(N_{a,h} = \infty, N^-_{b,k} < \infty) = 0$$

for all Q as in Theorem 2.1. This implies (2.9), which can now be seen to hold for all a, b, h, and k.

3. Zeros of Harmonic Functions

In this section we prove Theorem 1.3 using an argument of the Phragmén–Lindelöf type and the following lemmas.

Lemma 3.1. *Under the conditions of Theorem* 1.3,

$$u(0, y) = O(y^{-n/p}), \qquad y \to \infty. \tag{3.1}$$

Lemma 3.2. *Let α be a real number. There is a function w harmonic in $\mathbf{R}^{n+1}_+$ such that*

$$w(x, y) = y^\alpha g(\mid x \mid / y) \tag{3.2}$$

for some g: $[0, \infty) \to \mathbf{R}$ with $g(0) = 1$.

This means that $w(0, y) = y^\alpha$ and $w(x, y) = y^\alpha g(a)$ for every (x, y) in $\mathbf{R}^{n+1}_+$ on the boundary of the cone $\Gamma_a(0)$.

Another fact that we shall need about harmonic functions, and this is well known, is that if u is everywhere positive on $\mathbf{R}^{n+1}_+$, then

$$\liminf_{y\to\infty} y^n u(0, y) > 0. \tag{3.3}$$

Before proving these lemmas, we shall show how Theorem 1.3 follows from them.

Proof of Theorem 1.3. Lemma 3.1 implies that

$$u(0, y) = o(y^{-n-\epsilon}), \qquad y \to \infty, \tag{3.4}$$

for some $\epsilon > 0$. Let w be the harmonic function of Lemma 3.2 with $\alpha = -n - \epsilon$. Then $y^n w(0, y) = y^{-\epsilon}$ and, in view of (3.3), w has at least one zero, say (x_0, y_0). Let $a = |x_0|/y_0$. Clearly, $a > 0$ and $w(x, y) = 0$ if $(x, y) \in \partial\Gamma_a(0)$ and $y > 0$.

Now suppose that u does not satisfy the conclusion of Theorem 1.3 for this particular a. Then there is a number $t > 0$ such that u is always positive or always negative on the closure of $\Gamma_a{}^t(0)$. We may suppose the former so the harmonic function u_t defined by $u_t(x, y) = u(x, y + t)$ is positive on the closure of $\Gamma_a(0)$. We now compare u_t and w in the region $R_k = \{(x, y) \in \Gamma_a(0): 1 < y < k\}$ and in $R_\infty = \bigcup_{k>1} R_k$. Let $M = \sup_{r \leqslant a} g(r)$. By choosing $\beta > 0$ large enough, we can satisfy $\beta u_t \geqslant M \geqslant w$ on the lower part of the boundary of R_k. Also, $w \leqslant Mk^{-n-\epsilon}$ on the upper part of ∂R_k. Since $w = 0$ on the middle part,

$$\beta u_t + Mk^{-n-\epsilon} \geqslant w$$

on all of ∂R_k and, by the maximum principle, on all of R_k. Letting $k \to \infty$, we have $\beta u_t \geqslant w$ on R_∞. Therefore, $\beta u_t(0, y) \geqslant w(0, y) = y^{-n-\epsilon}$ for $y > 1$. But this contradicts (3.4) and the theorem is proved.

In the proof of Lemma 3.1, we shall need the fact that if u is bounded from below on $\mathbf{R}_+^{n+1}$ and $u(0, y) = o(y)$ as $y \to \infty$, then, for all $y > 0$ and $t > 0$,

$$u(0, y + t) = \int_{\mathbf{R}^n} u(x, y)\, p_t(x)\, dx \tag{3.5}$$

where p_t is the Poisson kernel

$$p_t(x) = \frac{c_n t}{(|x|^2 + t^2)^{(n+1)/2}}, \qquad \int_{\mathbf{R}^n} p_t(x)\, dx = 1. \tag{3.6}$$

To show this, we may assume that u is nonnegative. Then, as is well known,

$$u(\cdot, y) = p_y * \mu + ay \geqslant ay$$

for some nonnegative measure μ and nonnegative number a. (For example, see [15, p. 235].) But $u(0, y) = o(y)$ implies that $a = 0$. Therefore,

$$u(\cdot, y + t) = p_{y+t} * \mu = p_t * (p_y * \mu) = p_t * u(\cdot, y),$$

and this implies (3.5).

If $u > 0$ everywhere on $\mathbf{R}_+^{n+1}$, then, by (3.5), (3.6), and Fatou's lemma,

$$\liminf_{t\to\infty} t^n u(0, y + t) \geqslant \int_{\mathbf{R}^n} u(x, y) \liminf_{t\to\infty} t^n p_t(x)\, dx$$

$$= c_n \int_{\mathbf{R}^n} u(x, y)\, dx > 0,$$

which gives (3.3).

Proof of Lemma 3.1. We may assume in the proof that

$$\sup_{y>0} \int_{\mathbf{R}^n} [u^-(x, y)]^p\, dx = K < \infty;$$

otherwise replace u by u_t where $u_t(x, y) = u(x, y + t)$. Applying the Hardy–Littlewood–Fefferman–Stein lemma (Lemma 2.2) to the subharmonic function u^- and the ball $B \subset \mathbf{R}_+^{n+1}$ with center (x, y) and radius y, we obtain

$$\begin{aligned}[u^-(x, y)]^p &\leqslant cy^{-n-1} \iint_B [u^-(s, t)]^p\, ds\, dt \\ &\leqslant cy^{-n-1} \int_0^{2y} \int_{\mathbf{R}^n} [u^-(s, t)]^p\, ds\, dt \\ &\leqslant cy^{-n-1} \int_0^{2y} K\, dt \\ &= cy^{-n},\end{aligned}$$

so that $u^-(x, y) \leqslant cy^{-n/p}$. Therefore,

$$\begin{aligned}\int_{\mathbf{R}^n} u^-(x, y)\, dx &= \int_{\mathbf{R}^n} [u^-(x, y)]^p [u^-(x, y)]^{1-p}\, dx \\ &\leqslant K(cy^{-n/p})^{1-p} \\ &= cy^{-n(1-p)/p}. \end{aligned} \tag{3.7}$$

These facts imply that $u^-(\cdot, y)$ is integrable and u_t defined by $u_t(x, y) = u(x, y + t)$ is bounded from below on $\mathbf{R}_+^{n+1}$, $t > 0$. Applying (3.5) to u_t, we easily see that (3.5) also holds for u. Also,

$$-c_n u^-(x, y) \leqslant u(x, y)\, t^n p_t(x) \to c_n u(x, y)$$

as $t \to \infty$, where the left-hand side is an integrable function of x.

Therefore, using Fatou's lemma, (3.5) applied to our function u, and the assumption $u(0, y) = o(y^{-n})$, we have

$$c_n \int_{\mathbf{R}^n} u(x, y)\, dy \leqslant \liminf_{t\to\infty} \int_{\mathbf{R}^n} u(x, y)\, t^n p_t(x)\, dx$$
$$= \liminf_{t\to\infty} t^n u(0, y + t) = 0.$$

Therefore, $\int u^+(x, y)\, dx \leqslant \int u^-(x, y)\, dx$ so that, by (3.7),

$$\int_{\mathbf{R}^n} |\, u(x, y)|\, dx \leqslant 2 \int_{\mathbf{R}^n} u^-(x, y)\, dx \leqslant cy^{-n(1-p)/p}. \tag{3.8}$$

Now let $B \subset \mathbf{R}_+^{n+1}$ be the ball with center $(0, 2y)$ and radius y. Then

$$|\, u(0, 2y)| \leqslant cy^{-n-1} \int_B \int |\, u(s, t)|\, ds\, dt$$
$$\leqslant cy^{-n-1} \int_y^{3y} \int_{\mathbf{R}^n} |\, u(s, t)|\, ds\, dt.$$

Using (3.8), we obtain

$$|\, u(0, 2y)| \leqslant cy^{-n-1} \int_y^{3y} cy^{-n(1-p)/p}\, dt = cy^{-n/p}.$$

This completes the proof of Lemma 3.1.

Remark 3.1. In the proof of Lemma 3.1, the assumption $u(0, y) = o(y^{-n})$ is not used in its full strength. It can be replaced by the slightly weaker assumption

$$\liminf_{y\to\infty} y^n u(0, y) \leqslant 0. \tag{3.9}$$

This is true throughout; in particular, (1.4) can be replaced by (3.9).

Proof of Lemma 3.2. Suppose that w is harmonic in $\mathbf{R}_+^{n+1}$ and has the form

$$w(x, y) = y^\alpha G(|\, x\, |^2/y^2). \tag{3.10}$$

Then $G(t) = w((t^{1/2}, 0,..., 0), 1)$ and the harmonicity of w implies that G is infinitely differentiable on $(0, \infty)$. Furthermore, Laplace's equation $\Delta w = 0$ implies that, for $t > 0$,

$$4t(1 + t)\, G''(t) + 2[n - (2\alpha - 3)t]\, G'(t) + \alpha(\alpha - 1)\, G(t) = 0. \tag{3.11}$$

On the other hand, if we start with this differential equation, we easily see that there is a solution G on $(-1, \infty)$ such that $G(0) = 1$.

(Note that $t = 0$ is a regular singular point with corresponding indicial equation $r(r-1) + \frac{1}{2}nr = 0$. Therefore, a solution G on $(-1, 1)$ exists with the power series expansion $1 + \sum_{k=1}^{\infty} a_k t^k$. Since every point of $(0, \infty)$ is an ordinary point of the differential equation, G has an extension to $(-1, \infty)$ and this is the desired solution.)

It is easy to check by differentiation that if G is a solution of (3.11) on $(-1, \infty)$ and $G(0) = 1$, then w defined by (3.10) is harmonic and satisfies the conclusion of Lemma 3.2. This concludes the proof.

Remark 3.2. It is not difficult to give a constructive proof of this lemma. For example, if $\alpha = -n - \epsilon$ with $0 < \epsilon < 1$, the only case for which we need the lemma, and $W(x, y) = (\partial/\partial y)\, p_y(x)$, the derivative of the Poisson kernel defined in (3.6), then it is not hard to see that

$$w(x, y) = \int_0^\infty W(x, y + t)\, t^{-\epsilon}\, dt$$

is harmonic and, apart from a normalizing constant, has the desired form of Lemma 3.2.

Note also that if w has the form (3.2), then $\partial w/\partial y$ has the form (3.2) with α replaced by $\alpha - 1$.

4. Proof of the Basic Inequality

The key difference between the following preliminary result and Theorem 2.1 is the difference between $|Q|$ in the right-hand side of (4.1) and $m_Q(N_{a,h} > \lambda)$ in (2.1).

LEMMA 4.1. *Let* $0 < a < b$, $0 < h < k$, $0 < \delta < \beta$, $\beta > 1$, $\xi > 0$, *and suppose that* $Q \subset \mathbf{R}^n$ *is a cube with center* q *and diameter* $2ah$. *Then*

$$m_Q(N_{a,h} > \beta\lambda,\ N_{b,k}^- \leqslant \delta\lambda) \leqslant \epsilon\, |Q| \tag{4.1}$$

for all positive λ *satisfying* $u(q, h) \leqslant \xi\lambda$ *where*

$$\epsilon = c\left(\frac{\xi + \delta}{\beta + \delta} + \frac{1}{b} + \frac{h}{k}\right) \tag{4.2}$$

and the choice of c *depends only on* n *and* a.

Before proving this lemma, we show how Theorem 2.1 follows from it.

Proof of Theorem 2.1. The problem may be transformed into an equivalent one in which $Q = [0, 1] \times \cdots \times [0, 1]$. Then $q = (1/2,..., 1/2)$ and $h = n^{1/2}/(2a)$ with a, b and the ratio h/k unchanged.

Consider the open set $G_\lambda = \{N_{a,h} > \lambda\} \cap Q^0$ where Q^0 denotes the interior of Q. If $G_\lambda = Q^0$, then $m_Q(N_{a,h} > \lambda) = |Q|$ and (2.1) follows from (4.1). If $G_\lambda \cap Q^0$ is empty, then both sides of (2.1) are zero. Therefore, assume from now on that G_λ is a nonempty proper subset of Q^0. Let F be the closure of $\{N_{a,h} \leqslant \lambda\} \cap Q^0$ and G the complement of F relative to $\mathbf{R}^n$. Note that $G_\lambda = G \cap Q^0$ and F need not contain all of the boundary of Q. Now decompose G into dyadic Whitney cubes $Q_1, Q_2, ...$ so that $G = \bigcup_{j=1}^\infty Q_j$, their interiors are disjoint, and

$$\text{diameter } Q_j \leqslant \text{distance } (Q_j, F) \leqslant 4 \text{ diameter } Q_j. \tag{4.3}$$

(See [15, p. 167].) One feature of this decomposition is that every Q_j^0 intersecting the initial cube $Q = [0, 1] \times \cdots \times [0, 1]$ satisfies $Q_j^0 \subset Q^0$. Let $J = \{j: Q_j^0 \cap Q \neq \varnothing\}$. Then

$$\bigcup_{j \in J} Q_j^0 = G_\lambda \subset \bigcup_{j \in J} Q_j$$

so that

$$\sum_{j \in J} |Q_j| = m_Q(N_{a,h} > \lambda). \tag{4.4}$$

Moreover,

$$m_Q(N_{a,h} > \beta\lambda, N_{b,k}^- \leqslant \delta\lambda) = \sum_{j \in J} |E_j| \tag{4.5}$$

where $E_j = \{N_{a,h} > \beta\lambda, N_{b,k}^- \leqslant \delta\lambda\} \cap Q_j$.

Now let $\theta = 3^{n(40a+1)}$ and $\xi = \theta(1 + \delta)$. We may assume that

$$\beta > \xi, b > 20a + 1, k > 2h \tag{4.6}$$

for we may always choose the number c in (2.2) to be at least as large as 2θ so that if one or more of the inequalities in (4.6) is not satisfied, then $\epsilon > 1$ and (2.1) holds trivially.

We now estimate $|E_j|$ in (4.5) for all j in J for which E_j is nonempty. Let q_j be the center of Q_j, d_j the diameter, and $h_j = d_j/(2a)$. We show that if $x \in E_j$ and $(s, y) \in \Gamma_{a,h}(x)$ with $y \geqslant h_j$, then

$$u(s, y) \leqslant \xi\lambda. \tag{4.7}$$

By (4.6), this implies that

$$E_j = \{N_{a,h_j} > \beta\lambda, N^-_{b,k} \leqslant \delta\lambda\} \cap Q_j$$

and $u(q_j, h_j) \leqslant \xi\lambda$. Therefore, by Lemma 4.1,

$$| E_j | \leqslant c_{(4.1)} \left(\frac{\xi + \delta}{\beta + \delta} + \frac{1}{b} + \frac{h}{k}\right) | Q_j |$$

$$\leqslant 2\theta c_{(4.1)} \left(\frac{1 + \delta}{\beta + \delta} + \frac{1}{b} + \frac{h}{k}\right) | Q_j |.$$

Comt ning this with (4.5) and (4.4), we obtain the desired inequality (2.1).

We now prove (4.7). Let $x \in E_j$. Then, by (4.3), there is an $x_0 \in F$ such that

$$\begin{aligned} | x - x_0 | &= \text{distance } (x, F) \\ &\leqslant d_j + \text{distance } (Q_j, F) \\ &\leqslant 5d_j, \end{aligned}$$

and necessarily $N_{a,h}(x_0) \leqslant \lambda$. Let $(s, y) \in \Gamma_{a,h}(x)$ with $y \geqslant h_j$. Then (s, y) is within distance $5d_j$ of a point (s_1, y_1) in $\Gamma_{a,h}(x_0)$ with $y_1 = y$ and $| u(s_1, y_1)| \leqslant \lambda$. Since $b > 20a + 1$ and $k > 2h$, there is a positive integer $\nu \leqslant 40a + 1$ and a family of ν balls $B_i \subset \Gamma_{b,k}(x)$ such that the radius of B_i is $r_j = \frac{1}{2}h_j$, the center is (s_i, y_i) with $y_i = y$, and $| s_{i+1} - s_i | \leqslant \frac{1}{2}r_j$, $i = 1, \dots, \nu$, with $(s_{\nu+1}, y_{\nu+1}) = (s, y)$. Applying Harnack's inequality to the nonnegative function $u + \delta\lambda$ in B_1, we get

$$u(s_2, y_2) + \delta\lambda \leqslant 3^n(u(s_1, y_1) + \delta\lambda) \leqslant 3^n(\lambda + \delta\lambda).$$

Repeating the argument for $B_2, \dots, B_\nu$, we obtain

$$\begin{aligned} u(s, y) \leqslant u(s, y) + \delta\lambda &\leqslant 3^{n\nu}(\lambda + \delta\lambda) \\ &\leqslant \theta(1 + \delta)\lambda = \xi\lambda \end{aligned}$$

and the proof is complete.

Proof of Lemma 4.1. We may assume here that

$$Q = [-1/2, 1/2] \times \cdots \times [-1/2, 1/2]$$

so $q = 0$ and $h = n^{1/2}/(2a)$. Also, we assume that $\delta = 0$ since the problem remains the same if u, δ, β, and ξ, are replaced by $u + \delta\lambda$, 0,

$\beta + \delta$, and $\xi + \delta$. Finally, we assume that $b > 2a + 1$ and $k > 2h$, for if either of these inequalities does not hold, then, provided we let c in (4.2) be at least as large as $2a + 2$, the number ϵ in (4.2) satisfies $\epsilon > 1$ and (4.1) holds trivially.

Let $E = \{N_{a,h} > \beta\lambda,\ N_{b,k}^{-} = 0\} \cap Q$. The problem is to show that $| E | \leqslant \epsilon$ where ϵ has the form (4.2) with $\delta = 0$. But this is equivalent to a problem about harmonic measure. Consider the harmonic measure of E with respect to $\mathbf{R}_{+}^{n+1}$ at a point $z = (x, y)$ in $\mathbf{R}_{+}^{n+1}$:

$$p(z, E) = \int_E \frac{c_n y}{(| x - s |^2 + y^2)^{(n+1)/2}}\, ds.$$

Let $z_0 = (q, h) = (0, n^{1/2}/(2a))$. Then

$$p(z_0, E) \geqslant | E |\, c_n h(n + h^2)^{-(n+1)/2} = c_{n,a} | E |,$$

so it is enough to show that

$$p(z_0, E) \leqslant \epsilon \tag{4.8}$$

where ϵ is of the same form.

We may suppose that E is nonempty. Let

$$W = \bigcup_{x \in E} \Gamma_{b,k}(x)$$

and note that $z_0 \in W$ and u is nonnegative in W. Let $Z = (X, Y)$ be Brownian motion in $\mathbf{R}^{n+1}$ starting at z_0. Let

$$\begin{aligned} \tau &= \inf\{t > 0: Z(t) \in \mathbf{R}^n\} \\ &= \inf\{t > 0: Y(t) = 0\}, \\ \nu &= \inf\{t > 0: Z(t) \in \partial W\}. \end{aligned}$$

These are stopping times of Z such that $\nu \leqslant \tau$ and, with probability one, $\tau < \infty$, $Z(\tau) \in \mathbf{R}^n$, and $Z(\nu) \in \partial W$. (A convenient reference for Brownian motion and stopping times is [12]. A basic reference for applications to harmonic functions is [9]. For other applications similar to this one, see [4].)

In particular, we shall use Kakutani's fundamental observation (see [9]) that

$$p(z_0, E) = P(Z(\tau) \in E). \tag{4.9}$$

Let $x \in E$. Then $N_{a,h}(x) > \beta\lambda$ so that $\Gamma_{a,h}(x)$ contains a point (s, y) satisfying $u(s, y) > \beta\lambda$. Let B be the ball with center (s, y)

and radius $r = \frac{1}{2}y$ and B_0 the ball with the same center but with radius $\frac{1}{2}r$. Since $B \subset \Gamma_{2a+1,2h}(x) \subset \Gamma_{b,k}(x)$, the harmonic function u is nonnegative in B. By Harnack's inequality, u is bounded from below by $3^{-n}\beta\lambda$ in B_0. Let

$$u_\tau{}^* = \sup_{0<t<\tau} |u(Z(t))|, \qquad u_\nu{}^* = \sup_{0<t<\nu} u(Z(t)).$$

Given that $Z(\tau) = x$, the conditional probability that Z hits B_0 before $\mathbf{R}^n$ is bounded away from zero by a number $c = c_{n,a}$. (For information about such conditioned Brownian motion in a half-space, see [10].) If Z hits B_0 before $\mathbf{R}^n$, then $u_\tau{}^* > 3^{-n}\beta\lambda$. Therefore,

$$P(u_\tau{}^* > 3^{-n}\beta\lambda \mid Z(\tau)) \geqslant c$$

on the set $\{Z(\tau) \in E\}$. Let $I(Z(\tau) \in E)$ denote the indicator function of this set. Then

$$cI(Z(\tau) \in E) \leqslant P(u_\nu{}^* > 3^{-n}\beta\lambda \mid Z(\tau)) + I(Z(\tau) \in E)\, P(\nu < \tau \mid Z(\tau)).$$

Taking expectations of both sides, we obtain

$$cP(Z(\tau) \in E) \leqslant P(u_\nu{}^* > 3^{-n}\beta\lambda) + P(Z(\tau) \in E, \nu < \tau) \tag{4.10}$$

Now $P(u_\nu{}^* > 3^{-n}\beta\lambda) = \lim_{j\to\infty} P(u_{\nu_j}^* > 3^{-n}\beta\lambda)$ where

$$\nu_j = \inf\{t > 0 \colon \text{distance } (Z(t), \partial W) \leqslant 2^{-j}\}$$

and $P(u_{\nu_j}^* > 3^{-n}\beta\lambda)$ is the probability that the nonnegative martingale $\{u(Z(\nu_j \wedge t)), t \geqslant 0\}$ starting at $u(z_0) \leqslant \xi\lambda$ ever exceeds $3^{-n}\beta\lambda$. By an inequality of Doob [8, p. 353], this probability never exceeds $u(z_0)/(3^{-n}\beta\lambda)$ so that

$$P(u_\nu{}^* > 3^{-n}\beta\lambda) \leqslant 3^n\xi/\beta. \tag{4.11}$$

The other probability on the right-hand side of (4.10) satisfies

$$P(Z(\tau) \in E, \nu < \tau) \leqslant P(Z(\tau) \in Z, \nu < \tau, Y(\nu) < k) + P(Y(\nu) = k). \tag{4.12}$$

Since $P(Y(\nu) = k)$ is less than the probability that the nonnegative martingale $\{Y(\tau \wedge t), t \geqslant 0\}$ hits k,

$$P(Y(\nu) = k) \leqslant Y(0)/k = h/k. \tag{4.13}$$

Let $\partial_0 W = \{(x, y) \in \partial W \colon 0 < y < k\}$. Then

$$\{\nu < \tau, Y(\nu) < k\} = \{Z(\nu) \in \partial_0 W\}$$

and, by the strong Markov property,

$$P(Z(\tau) \in E, \nu < \tau, Y(\nu) < k) = E[I(Z(\nu) \in \partial_0 W) p(Z(\nu), E)]. \quad (4.14)$$

But if $z = (x, y) \in \partial_0 W$, then

$$E \subset \{s \in \mathbf{R}^n : | x - s | \geqslant by\}.$$

Denoting the latter set by F_z, we have

$$\begin{aligned} p(z, E) &\leqslant p(z, F_z) \\ &\leqslant \int_{F_z} c_n y \, | x - s |^{-n-1} \, ds \\ &= c \int_{by}^{\infty} y r^{-n-1} r^{n-1} \, dr \\ &= c/b. \end{aligned}$$

Therefore, by (4.14),

$$P(Z(\tau) \in E, \nu < \tau, Y(\nu) < k) \leqslant c/b.$$

Combining this with the inequalities from (4.9) to (4.13), we obtain

$$\begin{aligned} p(z_0, E) &\leqslant c[(3^n \xi/\beta) + (c/b) + (h/k)] \\ &\leqslant c[(\xi/b) + (1/b) + (h/k)], \end{aligned}$$

and the proof of (4.8) and the lemma is complete.

5. Martingale Analogs

Let $Y = \{Y(t), t \geqslant 0\}$ be a local martingale with continuous sample functions and constant initial position $Y(0)$. These conditions assure that if

$$\tau_n = \inf\{t > 0 : | Y(t) - Y(0) | = n\},$$

then $Y^{\tau_n} = \{Y(\tau_n \wedge t), t \geqslant 0\}$ is a uniformly bounded martingale. Let M be the ordinary maximal function of Y and M^- the one-sided maximal function as defined in Section 1.

Lemma 5.1. *If Y is nonnegative, then*

$$P(M > \beta\lambda) \leqslant (1/\beta) \, P(M > \lambda)$$

for all $\beta > 1$ and all positive $\lambda \geqslant Y(0)$.

Proof. We may assume that Y is uniformly bounded; otherwise, replace Y by Y^{τ_n}, defined above, for some $n > \beta\lambda$. Let

$$\mu = \inf\{t > 0: Y(t) > \lambda\},$$
$$\nu = \inf\{t > 0: Y(t) > \beta\lambda\}.$$

These are stopping times of Y and, by the sample-function continuity, $Y(\mu) = \lambda$ on the set $\{\mu < \infty\}$ and $Y(\nu) = \beta\lambda$ on the set $\{\nu < \infty\}$. Since $Y(\infty) = \lim_{t\to\infty} Y(t)$ exists with probability one, $Y(\mu)$ and $Y(\nu)$ are defined also on the sets where μ and ν are infinite. By Doob's optional sampling theorem (see [13, p. 98]), $\{Y(\mu), Y(\nu)\}$ is a martingale, and, in fact,

$$\{I(\mu < \infty)\, Y(\mu), I(\mu < \infty)\, Y(\nu)\}$$

is also a martingale. Since each term in a martingale has the same expectation,

$$\begin{aligned}
\beta\lambda P(M > \beta\lambda) &\leqslant \beta\lambda P(\mu < \infty, Y(\nu) = \beta\lambda) \\
&\leqslant E[I(\mu < \infty)\, Y(\nu)] \\
&= E[I(\mu < \infty)\, Y(\mu)] \\
&= \lambda P(\mu < \infty) \\
&= \lambda P(M > \lambda).
\end{aligned}$$

Now let $M^+ = \sup_{t\geqslant 0} Y^+(t)$.

THEOREM 5.1. *Let* $\beta > 1$ *and* $\delta > 0$. *Then*

$$P(M^+ > \beta\lambda, M^- \leqslant \delta\lambda) \leqslant [(1 + \delta)/(\beta + \delta)]\, P(M^+ > \lambda)$$

for all positive $\lambda \geqslant Y(0)$.

Proof. Let $\sigma = \inf\{t > 0: Y(t) < -\delta\lambda\}$. If $Y(0) \geqslant -\delta\lambda$, then

$$Z = \{Y(\sigma \wedge t) + \delta\lambda, t \geqslant 0\}$$

is a nonnegative local martingale with initial position $Y(0) + \delta\lambda \leqslant \lambda + \delta\lambda$. Therefore, by Lemma 5.1,

$$\begin{aligned}
P(M^+ > \beta\lambda, M^- \leqslant \delta\lambda) &= P(M^+ > \beta\lambda, \sigma = \infty) \\
&\leqslant P(M(Z) > \beta\lambda + \delta\lambda) \\
&\leqslant [(1 + \delta)/(\beta + \delta)]\, P(M(Z) > \lambda + \delta\lambda) \\
&\leqslant [(1 + \delta)/(\beta + \delta)]\, P(M^+ > \lambda).
\end{aligned}$$

If $\delta < \beta$, then the left-hand side of this inequality is equal to $P(M > \beta\lambda, M^- \leqslant \delta\lambda)$ so Theorem 1.5 follows.

Proof of Theorem 1.4. Replace β by β^j in (1.10) and let $\delta = 1$ to obtain

$$P(M > \beta^j\lambda, M^- \leqslant \lambda) \leqslant 2\beta^{-j}P(M > \lambda)$$

for all $\lambda > 0$. By the growth condition, $\Phi(\beta^j\lambda) \leqslant \gamma^j\Phi(\lambda)$. Choose j just large enough so that $2\gamma^j\beta^{-j} \leqslant \frac{1}{2}$. This can be done since $\gamma < \beta$. Then, by Lemma 2.1,

$$E\Phi(M) \leqslant 2\gamma^j E\Phi(M^-).$$

One-sided random boundaries for Brownian motion. Here is a simple application of Theorem 1.4. Let $X = \{X(t), t \geqslant 0\}$ be real Brownian motion starting at zero. Let $A = \{A(t), t \geqslant 0\}$ be another stochastic process on the same underlying probability space such that: (i) every sample path of A starts at zero and is continuous and nondecreasing, (ii) $A(t)$ is measurable with respect to the σ-field generated by $\{X(s), s \leqslant t\}$, and (iii) if $0 < p < 1$, there are positive real numbers c_p and C_p such that, for all stopping times τ of X,

$$c_p \| \tau^{1/2} \|_p \leqslant \| A(\tau)\|_p \leqslant C_p \| \tau^{1/2} \|_p . \tag{5.1}$$

(Define $A(\tau) = \lim_{t\to\infty} A(t)$ on the set $\{\tau = \infty\}$.)

It is trivial that $A(t) = t^{1/2}$ satisfies these conditions. Another example is $M(t) = \sup_{s\leqslant t} | X(s)|$; by Theorem 7.1 of [2], this process satisfies (5.1) for $0 < p < \infty$. Therefore, by Theorem 1.4, $M^-(t) = \sup_{s\leqslant t} X^-(s)$ satisfies (5.1) for $0 < p < 1$ and is still another example.

Now consider the particular stopping time

$$\tau = \tau(a, b) = \inf\{t : X(t) = a + bA(t)\}$$

where $a > 0$ and $b > 0$.

THEOREM 5.2. *Let $0 < p < 1$. There is a positive real number β_p such that $\| \tau^{1/2} \|_p$ is finite if $0 < b < \beta_p$, but is infinite if $b > \beta_p$. The choice of β_p is independent of a.*

Novikov [14] obtains this result for the case $A(t) = t^{1/2}$ and gives a precise description of β_p. Our method is quite different and somewhat more general but does not give much information about β_p. It would be interesting to know β_p for the case $A(t) = M^-(t)$.

Proof. Fix $a > 0$. Since $\tau = \tau(a, b)$ is nondecreasing in b, the existence of $\beta_p = \beta_{a,p}$ will follow if we can show that $\| \tau^{1/2} \|_p$ is infinite for large b and finite for small b.

Suppose that $\| \tau^{1/2} \|_p$ is finite. Then with probability one, τ is finite and $X(\tau) = a + bA(\tau)$. Also, by (5.1), $\| A(\tau)\|_p$ is finite and

$$\begin{aligned} b \| A(\tau)\|_p < \| a + bA(\tau)\|_p &= \| X(\tau)\|_p \\ &\leqslant \| M(\tau)\|_p \leqslant c \| \tau^{1/2} \|_p \\ &\leqslant c \| A(\tau)\|_p . \end{aligned}$$

Dividing by $\| A(\tau)\|_p$, which is both finite and positive, we see that b must be less than a number c the choice of which depends only on p. Accordingly, if b is larger than this number, $\| \tau^{1/2} \|_p$ must be infinite.

We now show that $\| \tau^{1/2} \|_p$ is finite if b is small. Let n be a positive integer. Then $X(\tau \wedge n \wedge t) \leqslant a + bA(\tau \wedge n \wedge t)$ so that $M^+(\tau \wedge n) \leqslant a + bA(\tau \wedge n)$. Therefore, by the analog of Theorem 4.1 for M^+,

$$\begin{aligned} \|(\tau \wedge n)^{1/2} \|_p^p \leqslant c \| M(\tau \wedge n)\|_p^p &\leqslant c \| M^+(\tau \wedge n)\|_p^p \\ &\leqslant c \| a + bA(\tau \wedge n)\|_p^p \\ &\leqslant ca^p + cb^p \| A(\tau \wedge n)\|_p^p \\ &\leqslant ca^p + cb^p \|(\tau \wedge n)^{1/2} \|_p^p . \end{aligned}$$

Therefore, for b suitably small,

$$\| \tau \wedge n)^{1/2} \|_p^p \leqslant ca^p/(1 - cb^p), \qquad n \geqslant 1,$$

and, by the monotone convergence theorem, $\| \tau^{1/2} \|_p$ is finite.

Now suppose that $\beta_p = \beta_{a,p}$ does depend on a. Then there are positive numbers a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and

$$\beta_{a_2,p} < b_2 < b_1 < \beta_{a_1,p}$$

so that $\| \tau^{1/2}(a_1, b_1)\|_p$ is finite but $\| \tau^{1/2}(a_2, b_2)\|_p$ is infinite. Let

$$\mu = \inf\{t : X(t) = (b_1 a_2 - a_1 b_2)/(b_1 - b_2)\}.$$

Then it is not hard to see (by drawing a picture) that

$$\tau(a_2, b_2) \leqslant \mu \vee \tau(a_1, b_1).$$

Therefore,

$$\| \tau^{1/2}(a_2, b_2)\|_p^p \leqslant \| \mu^{1/2} \|_p^p + \| \tau^{1/2}(a_1, b_1)\|_p^p$$
$$\leqslant c \| M^+(\mu)\|_p^p + \| \tau^{1/2}(a_1, b_1)\|_p^p < \infty$$

giving a contradiction and completing the proof.

6. Examples, Questions, and Remarks

Although the inequality (2.1) for harmonic functions is remarkably, similar to the corresponding inequality (1.10) for martingales, the analogy is not exact. Consider the nontruncated version of (2.1): If $\beta > 1$, $0 < \delta < \beta$, and $\lambda > 0$, then

$$m(N_a > \beta\lambda, N_b^- \leqslant \delta\lambda) \leqslant \epsilon m(N_a > \lambda) \tag{6.1}$$

where

$$\epsilon = c((1 + \delta)/(\beta + \delta)) + (1/b)) \tag{6.2}$$

and the choice of c depends only on n and a. The martingale inequality (1.10) suggests that the optimum value of ϵ for (6.1) might satisfy the inequality

$$\epsilon \leqslant c((1 + \delta)/(\beta + \delta)). \tag{6.3}$$

Even if $b \geqslant 2a$, which we assume from now on, this cannot hold for a real number c whose choice depends only on n and a; see Example 6.1. *Question*: Does

$$\epsilon \leqslant c((1 + \delta)/(\beta + \delta))^r \tag{6.4}$$

with the choice of c and $0 < r < 1$ depending only on n and a? It is quite likely, we believe, that (6.4) does hold. Moreover, it seems likely that r can be near 1 if a is chosen sufficiently large.

Example 6.1. Let u be the harmonic function on $\mathbf{R}_+^2$ defined by

$$u(x, y) = [(y + 1)^2 - x^2]/[(y + 1)^2 + x^2]^2.$$

If $-1 < x < 1$, then $N_{1/2}(x) > u(0, 2) = 1/9$ and $N_1^-(x) = 0$. Therefore,

$$m(N_{1/2} > \beta\lambda, N_1^- \leqslant \lambda) \geqslant 2$$

provided $\beta\lambda \leqslant 1/9$. If $|x| > 1$, then

$$|u(s, y)| \leqslant [(y + 1)^2 + s^2]^{-1} \leqslant 2x^{-2}, (s, y) \in \Gamma_{1/2}(x),$$

so $N_{1/2}(x) \leqslant 2x^{-2}$ and $m(N_{1/2} > 2x^{-2}) \leqslant 2|x|$. Now let $\beta = x^2/18$ and $\lambda = 2x^{-2}$ with $x > 0$ chosen large enough to make $\beta > 1$. If (6.3) is satisfied, then

$$\begin{aligned} 2 &\leqslant m(N_{1/2} > \beta\lambda, N_1^- \leqslant \lambda) \\ &\leqslant \epsilon m(N_{1/2} > \lambda) \\ &\leqslant c(36/(x^2 + 18))2x. \end{aligned}$$

But the last expression is less than 2 for all large x so (6.3) cannot hold.

EXAMPLE 6.2. Here we show that (1.6) does not hold in general for $p \geqslant 1$. Let u be defined on $\mathbf{R}_+^2$ by

$$u(x, y) = \frac{y}{x^2 + y^2} - \frac{y + 1}{x^2 + (y + 1)^2}.$$

Clearing fractions, we see that

$$u(x, y) \geqslant -x^2[(x^2 + y^2)(x^2 + (y + 1)^2)]^{-1}.$$

Therefore, $N_0^-(x) \leqslant (1 + x^2)^{-1}$ and, by (1.7), $\| N^- \|_p$ is finite for $p \geqslant 1$. However, for $0 < x < 1/4$,

$$N(x) \geqslant u(x, x) \geqslant (2x)^{-1} - 1 \geqslant (4x)^{-1}$$

so that $\| N \|_p$ is infinite, $p \geqslant 1$.

EXAMPLE 6.3. Similarly, (1.9) does not hold in general for $p \geqslant 1$. Let X be real Brownian motion starting at 0 and τ the stopping time defined by

$$\tau = \inf\{t > 0 : X(t) = -1\}.$$

Let $Y(t) = X(\tau \wedge t)$, $t \geqslant 0$. Then $M^- \leqslant 1$ so that $\| M^- \|_p \leqslant 1$. However,

$$\| M \|_p^p = \int_0^\infty p\lambda^{p-1} P(M > \lambda)\, d\lambda \geqslant \int_0^\infty p\lambda^{p-1}(1 + \lambda)^{-1}\, d\lambda,$$

and the latter integral is infinite for $p \geqslant 1$.

EXAMPLE 6.4. Here we construct a function u in H^1 that does not satisfy the cone property of Theorem 1.3. In particular, for all $a > 0$, the set of zeros of u in $\Gamma_a(0)$ is bounded.

Let $0 < \epsilon < 1/2$ and define the harmonic function w_ϵ on $\mathbf{R}_+^2$ by

$$w_\epsilon(x, y) = \int_0^\infty \frac{(y+t)^2 - x^2}{[(y+t)^2 + x^2]^2} \frac{dt}{t^\epsilon}; \tag{6.5}$$

see Remark 3.2. Then $w_\epsilon(x, y) = y^{-1-\epsilon} g_\epsilon(|x|/y)$ where

$$g_\epsilon(r) = \int_0^\infty \frac{(1+s)^2 - r^2}{[(1+s)^2 + r^2]^2} \frac{ds}{s^\epsilon}, \qquad r \geqslant 0.$$

The absolute value of the integrand does not exceed

$$[(1+s)^2 + r^2]^{-1} s^{-\epsilon} \leqslant (1+s)^{-2} s^{-\epsilon}$$

so the real number $M = \int_0^1 (1+s)^{-2} s^{-1/2}\, ds + \int_1^\infty (1+s)^{-2}\, ds$ satisfies $|g_\epsilon(r)| \leqslant M$, $r \geqslant 0$. Moreover, by the Lebesgue dominated convergence theorem,

$$\lim_{\epsilon \to 0} g_\epsilon(r) = \int_0^\infty \frac{(1+s)^2 - r^2}{[(1+s)^2 + r^2]^2}\, ds = -\frac{1+s}{(1+s)^2 + r^2}\Big|_0^\infty = \frac{1}{1+r^2}. \tag{6.6}$$

Differentiating g_ϵ under the integral sign and bounding the derivative of the integrand as above, we obtain $|g_\epsilon'(r)| \leqslant 3M$, $r > 0$. Therefore, the family $\{g_\epsilon : 0 < \epsilon < 1/2\}$ is equicontinuous and the convergence in (6.6) is uniform for r in compact intervals. Accordingly, for each positive integer j, there exists a number $\epsilon_j < 1/2$ satisfying

$$g_{\epsilon_j}(r) > (1/2)(1 + j^2)^{-1}, \qquad 0 \leqslant r \leqslant j;$$

we may assume that $\epsilon_1 > \epsilon_2 > \cdots$. Now let

$$v(x, y) = \sum_{j=1}^\infty \lambda_j w_{\epsilon_j}(x, y)$$

where $\lambda_j = 2^{-j}[1 + \int_0^\infty (1+t)^{-1} t^{-\epsilon_j}\, dt]^{-1}$. Let $a > 0$ and suppose k is a positive integer greater than a. If $(x, y) \in \Gamma_k(0)$, then $r = |x|/y < k$ and

$$\begin{aligned} y^{1+\epsilon_k} v(x, y) &= \sum_{j=1}^\infty \lambda_j y^{\epsilon_k - \epsilon_j} g_{\epsilon_j}(r) \\ &> -M \sum_{j=1}^{k-1} \lambda_j y^{\epsilon_k - \epsilon_j} + \tfrac{1}{2}\lambda_k (1 + k^2)^{-1}. \end{aligned}$$

There is a number $y_k > 0$ such that the right-hand side is positive for $y > y_k$. Therefore, v has no zero in $\Gamma_k(0) \supset \Gamma_a(0)$ above y_k. Hence, $\{v = 0\} \cap \Gamma_a(0)$ is bounded. Now let $u(x, y) = v(x, y + 1)$. Clearly, $\{u = 0\} \cap \Gamma_a(0)$ is also bounded, $a > 0$, so u does not satisfy the cone property of Theorem 1.3.

We now show that u belongs to H^1. By (6.5),

$$| u(x, y)| \leqslant \sum_{j=1}^{\infty} \lambda_j \int_0^{\infty} \frac{1}{(1 + t)^2 + x^2} \frac{dt}{t^{\epsilon_j}}$$

so that

$$\| N_0 \|_1 \leqslant \pi \sum_{j=1}^{\infty} \lambda_j \int_0^{\infty} (1 + t)^{-1} t^{-\epsilon_j}\, dt$$

$$\leqslant \pi \sum_{j=1}^{\infty} 2^{-j} = \pi.$$

Using the Fefferman–Stein inequality (2.7), we obtain $\| N \|_1 < \infty$, so u belongs to H^1.

In this example $u(0, y) = o(y^{-1})$ as $y \to \infty$ since $y \mid v(0, y)| \leqslant M \sum_{j=1}^{\infty} \lambda_j y^{-\epsilon_j}$. In general, if $u \in H^1$ on $\mathbf{R}_+^{n+1}$, then $u(0, y) = o(y^{-n})$ as $y \to \infty$ (and as $y \to 0$): Let $f(| x |) = \sup_{y > |x|} u(0, y)$. Then $f(| x |) \leqslant N(ax)$, $x \in \mathbf{R}^n$, so the integral $\int_0^{\infty} f(y) y^{n-1}\, dy$ is finite. Therefore,

$$y^n \mid u(0, 2y)| \leqslant y^n f(2y) \leqslant \int_y^{2y} f(t)\, t^{n-1}\, dt$$

and the last expression converges to zero as $y \to \infty$ (and as $y \to 0$).

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