Thin Bases of Order Two

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A set \( A \subseteq \mathbb{N}_0 \) is called a basis of order two if \( A + A := \{ a + a' \mid a, a' \in A \} = \mathbb{N}_0 \). If \( n \in \mathbb{N} \) then \( A(n) \) denotes the number of \( a \in A \) with \( 1 \leq a \leq n \). In this paper bases \( A, B, C \) of order two are given such that

\[
\lim \frac{A(n)}{\sqrt{n}} = 2 \sqrt{\frac{5}{3}}, \quad \lim \frac{B(n)}{\sqrt{n}} = \sqrt{\frac{7}{3}}, \quad \text{and} \quad \lim \frac{C(n)}{\sqrt{n}} = 10 \sqrt{\frac{5}{3}}.
\]

For \( A, B \subseteq \mathbb{N}_0 \) let

\[ A + B := \{ a + b \mid a \in A, b \in B \}. \]

A set \( A \) is a basis of order two if \( A + A = \mathbb{N}_0 \).

\[ A(n) := 2 \{ a \in A \mid 1 \leq a \leq n \}. \]

A simple combinatorial argument (Rohrbach [7]) shows

\[
\frac{A(n)}{\sqrt{n}} \geq \sqrt{2} - \frac{2}{\sqrt{n}}
\]

for each basis \( A \) (of order two)

so that

\[
\lim \frac{A(n)}{\sqrt{n}} \geq \sqrt{2} = 1.414 \ldots \text{ for each basis } A.
\]
This was improved by Klotz [4]:

$$\lim \frac{A(n)}{\sqrt{n}} \geq \frac{2}{\sqrt{1 - 0.0369}} = 1.441 \ldots$$ for each basis $A$.

Using a combinatorial argument Cassels [2] proves

$$\lim \frac{A(n)}{\sqrt{n}} \geq \frac{8}{\sqrt{n}} = 1.595 \ldots$$ for each basis $A$.

On the other side there is a basis $A$ such that (Stöhr [9], [10], Raikov [6])

$$\lim \frac{A(n)}{\sqrt{n}} = 2 \quad \text{and} \quad \lim \frac{A(n)}{\sqrt{n}} = \frac{3}{2} = 2.598 \ldots$$

In addition Cassels [2] gave a basis $A$ such that

$$\lim \frac{A(n)}{\sqrt{n}} = 3 \sqrt{3} = 5.196 \ldots$$

In this paper we modify the basis of Stöhr and Raikov and get a basis $\tilde{A}$ such that

$$\lim \frac{\tilde{A}(n)}{\sqrt{n}} = 2 \quad \text{and} \quad \lim \frac{\tilde{A}(n)}{\sqrt{n}} = 2 \sqrt{\frac{5}{3}} = 2.581 \ldots$$

Furthermore we give a basis $B$, such that

$$\lim \frac{B(n)}{\sqrt{n}} = \frac{7}{2} = 1.807 \ldots \quad \text{(but $\lim \frac{B(n)}{\sqrt{n}} = \infty$)}$$

and for any $\varepsilon > 0$ a basis $C_\varepsilon$ with

$$\lim \frac{C_\varepsilon(n)}{\sqrt{n}} = (1 + \varepsilon) \frac{7}{5} = (1 + \varepsilon) 1.897 \ldots \quad \text{(and $\lim \frac{C_\varepsilon(n)}{\sqrt{n}} < \infty$)}.$$  

Finally a basis $D$ is constructed such that

$$\lim \frac{D(n)}{\sqrt{n}} = 10 \frac{\sqrt{5}}{\sqrt{6} \sqrt{3}} = 4.638 \ldots$$
A basis $A$ of order two is called

(a) partially thin if $\lim_{n \to \infty} \frac{A(n)}{\sqrt{n}} < \infty$,

(b) thin if $\lim_{n \to \infty} \frac{A(n)}{\sqrt{n}} < \infty$,

(c) uniformly thin if $\lim_{n \to \infty} \frac{A(n)}{\sqrt{n}} < \infty$.

1. STÖHR’S AND RAIKOV’S THIN BASIS

If $I, J \subseteq \mathbb{N}_0$ then $I \sim J$ means that both sets differ only by a finite number of elements. Otherwise we write $I \varpropto J$. We denote by $I := \mathbb{N}_0 \setminus I$ the complement of $I$ and by $E := \{0, 2, 4, \ldots\}$ the even numbers.

Furthermore let

$$A_I^{(1)} := \left\{ \sum_{i=0}^{k} d_i 4^i \mid k \geq 0, \quad d_i \in \{0, 1\} \text{ for } i \in I, \quad d_i \in \{0, 2\} \text{ for } i \notin I \right\},$$

$$A_I^{(2)} := \left\{ \sum_{i=0}^{k} e_i 4^i \mid k \geq 0, \quad e_i \in \{0, 2\} \text{ for } i \in I, \quad e_i \in \{0, 1\} \text{ for } i \notin I \right\}.$$

For each $I \subseteq \mathbb{N}_0$ we have

$$A_I^{(1)} + A_I^{(2)} = \mathbb{N}_0$$

so that

$$A_I := A_I^{(1)} \cup A_I^{(2)}$$

is a basis with $\lim_{n \to \infty} \frac{A_I(n)}{\sqrt{n}} = 2$.

Stöhr [10] and Raikov [6] considered the case $I = \mathbb{N}_0$ or $I = \emptyset$ and received $\lim_{n \to \infty} \frac{A_I(n)}{\sqrt{n}} = \frac{\sqrt{5}}{2} = 2.598 \ldots$ and obviously the same holds for $I \sim \mathbb{N}_0$.

Theorem 1. (a) If $I \sim E$ or $I \sim \bar{E}$ we have

$$\lim_{n \to \infty} \frac{A_I(n)}{\sqrt{n}} = \frac{\sqrt{5}}{3} = 2.581 \ldots.$$

(b) If $I \not\subseteq \mathbb{N}_0$, $\mathbb{N}_0$, $E$, $\bar{E}$ we have

$$\lim_{n \to \infty} \frac{A_I(n)}{\sqrt{n}} \geq \frac{\sqrt{3}}{\sqrt{113}} = 2.607 \ldots.$$
For the proof we need a lemma.

**Lemma 1.** Let $J \subseteq \mathbb{N}_0$ be finite with at least two elements, let $j_0$ and $j_k$ be the minimal and maximal element in $J$, let

$$n = \sum_{i \in J} x_i 4^i, \quad x_i \in \{1, 2\}$$

$$n_0 = \sum_{i \in J \setminus \{j_0\}} x_i 4^i.$$

Then with $A := A_J$

$$\frac{A(n)}{\sqrt{n}} \geq \frac{A(n_0)}{\sqrt{n_0}}.$$

**Proof:** We observe that

$$A(n) \leq \sum_{i \in J} 2^i + x_{j_0} 2^i = 1$$

and

$$A(n_0) x_{j_0} 2^{j_0} = x_{j_0} \sum_{i \in J \setminus \{j_0\}} 2^{i+j_0} + x_{j_0} x_{j_0} 2^{i+j_0} - x_{j_0} 2^{j_0}$$

$$< x_{j_0} (1 + x_{j_0}) 2^{j_0-1} + x_{j_0} \sum_{i \in J \setminus \{j_0\}} 2^{j_0-1}$$

$$\leq 2 x_{j_0} 4^j + 2 \sum_{i \in J \setminus \{j_0\}} x_i 4^i = 2 n_0.$$

Therefore, we have

$$\frac{A(n)^2}{n} - \frac{A(n_0)^2}{n_0} = \frac{1}{n n_0} \left[ (A(n_0) + 2^j) (n_0 + A(n_0)^2 (n_0 + x_{j_0} 2^j)) \right]$$

$$= \frac{1}{n n_0} \left[ 2 A(n_0) 2^{h_0} n_0 + 4^h n_0 - A(n_0)^2 x_{j_0} 2^j \right]$$

$$= \frac{1}{n n_0} \left[ A(n_0) 2^{j_0} (2n_0 - A(n_0) x_{j_0} 2^j) + 4^h n_0 \right]$$

and the lemma follows.
This lemma shows that it is enough for \( \lim_{n \to \infty} A_f(n)/\sqrt{n} \) to consider elements of the form

\[ n_j = \sum_{i=0}^{j} x_i 4^i \in A_f \text{ with } x_j \in \{1, 2\}, \]

so (a) follows by a simple calculation.

To show (b) we simply observe that there must be an infinite set \( J \) with

\[ n_j = 2 \cdot 4^j + 1 \cdot 4^{j-1} + 1 \cdot 4^{j-2} + \sum_{i=0}^{j-3} x_i 4^i \in A_f \quad \text{for all } j \in J \]

\[ \leq 2 \cdot 4^j + 1 \cdot 4^{j-1} + 1 \cdot 4^{j-2} + 2 \cdot 4^{j-3} + \cdots + 2 \cdot 4^0 \]

\[ = 2 \cdot \frac{4^{j+1} - 1}{4 - 1} - 4^{j-1} - 4^{j-2} \leq \frac{113}{48} \]

so that

\[ \lim_{j \to \infty} \frac{A_f(n_j)}{\sqrt{n_j}} \geq \lim_{j \to \infty} \frac{2 (2^{j+1} - 1) \sqrt{48}}{2^j \sqrt{113}} = 16 \sqrt{\frac{3}{113}} = 2.607 \ldots \]

2. PARTIALLY THIN BASES

For the next theorem we need two results concerning the postage stamp problem.

If \( A_k = \{a_0, \ldots, a_k\}, \ a_0 = 0 < a_1 = 1 < \cdots < a_k \) is a set with \( k \) positive elements then \( n(2, A_k) \) denotes the greatest integer \( r \) such that

\[ n \in A_k + A_k \quad \text{for all } n = 0, 1, \ldots, r. \]

For \( k = 7t + 2, \ t \in \mathbb{N} \) Mrose [5] constructs a set \( A_k \) as follows:

Use the notation \( (a, b \in \mathbb{Z}, a < b) \)

\[ [a, (m), b] := \left\{ a + lm \mid l = 0, \ldots, \left\lfloor \frac{b-a}{m} \right\rfloor \right\} \]

and choose

\[ A^{(1)} := [0, (1), t], \]
\[ A^{(2)} := [2t, (t), 3t^2 + t], \]
\[ A^{(3)} := [3t^2 + 2t, (t+1), 4t^2 + 2t - 1], \]
\[ A^{(4)} := [6t^2 + 4t, (1), 6t^2 + 5t], \]
\[ A^{(5)} := [10t^2 + 7t, (1), 10t^2 + 8t]. \]
Then with \( A_k := A^{(1)} \cup \cdots \cup A^{(5)} \) he obtains
\[
\begin{align*}
&n(2, A_k) = 7k^2 + 0(k).
\end{align*}
\]
This gives
\[
\begin{align*}
k &\sqrt{n(2, A_k)} = k \sqrt{\frac{2}{7} k^2 + 0(k)} = \sqrt{\frac{7}{2} (1 + 0 \left( \frac{1}{k} \right))}.
\end{align*}
\]
In this situation it makes no sense letting \( k \) go against infinity, for \( A^{(1)} \) and therefore \( A_k \) goes against \( \mathbb{N}_0 \). However, using a similar idea as Atkin [1] used in connection with pseudosquares we start with
\[
A^{[0]} := A_k, \quad t_0 := t, \quad a_0 := 0, \quad k_0 := k
\]
and continue inductively by choosing
\[
\begin{align*}
t_i &:= a_{i-1} + 4t_{i-1}^2 + 2t_{i-1}, \quad k_i := k_{i-1} + 7t_i + 3, \\
a_i &:= n(2, A^{(i-1)}) + 1 = 2a_{i-1} + 14t_{i-1}^2 + 10t_{i-1}, \\
A^{(i,1)} &:= [a_i, (1), a_i + t_i], \\
A^{(i,2)} &:= [a_i + 2t_i, (t_i), a_i + 3t_i^2 + t_i], \\
A^{(i,3)} &:= [a_i + 3t_i^2 + 2t_i, (t_i + 1), a_i + 4t_i^2 + 2t_i - 1], \\
A^{(i,4)} &:= [a_i + 6t_i^2 + 4t_i, (1), a_i + 6t_i^2 + 5t_i], \\
A^{(i,5)} &:= [a_i + 10t_i^2 + 7t_i, (1), a_i + 10t_i^2 + 8t_i], \\
A^{[i]} &:= \bigcup_{j=1}^{5} A^{(i,j)}.
\end{align*}
\]
Then \( B := \bigcup_{i=0}^{\infty} A^{[i]} \) is a basis with
\[
\lim_{i \to \infty} \frac{B(n)}{\sqrt{n}} = \lim_{i \to \infty} \frac{k_0 + \cdots + k_i}{\sqrt{a_{i+1} - 1}} = \lim_{i \to \infty} \frac{7t_i}{\sqrt{14t_i^2}} = \sqrt{\frac{7}{2}}.
\]
But, choosing \( n_i = a_i + t_i \), we find
\[
\lim_{i \to \infty} \frac{B(n_i)}{\sqrt{n_i}} \geq \lim_{i \to \infty} \frac{B(n_i)}{\sqrt{n_i}} = \lim_{i \to \infty} \sqrt{t_i} = \infty.
\]
Instead of starting with Mrose's set $A_k$ we may use the set $A_k$ of Hämmerer and Hofmeister [3]:

\[
A^{(1)} := [a_i, (1), a_i + 2t_i],
\]

\[
A^{(2)} := [a_i + 4t_i, (1), a_i + 5t_i],
\]

\[
A^{(3)} := [a_i + 6t_i, (1), a_i + 7t_i],
\]

\[
A^{(4)} := [a_i + 2t_i, (m_i + 1), a_i + 2t_i + (t_i - 1)(m_i + 1)],
\]

\[
A^{(5)} := [a_i + 3t_i, (m_i + 1), a_i + 3t_i + (t_i - 1)(m_i + 1)],
\]

\[
A^{(6)} := [a_i + 5t_i, (m_i + 1), a_i + 5t_i + (t_i - 1)(m_i + 1)],
\]

\[
A^{(7)} := [a_i + 9t_i, (m_i + 1), a_i + 9t_i + (t_i - 1)(m_i + 1)],
\]

\[
A^{(8)} := [a_i + 7t_i + (t_i - 1)m_i, (1), a_i + 9t_i + (t_i - 1)m_i],
\]

\[
A^{(9)} := [a_i + 4t_i + t_m_i, (m_i), a_i + 4t_i + (t_i + l_i)m_i],
\]

\[
A^{(1)} := \bigcup_{j=1}^9 A^{(j)}.
\]

We have for $m_i = 10t_i$, $t_i \geq 2$, $l_i \geq t_i$ (see Hämmerer and Hofmeister [3])

\[
A^{(1)} + A^{(1)} \supseteq [2a_i(1), 2a_i + 14t_i - 1 + (t_i + l_i)m_i].
\]

We must have

\[
a_{i+1} \leq n (2, A^{(0)} \cup \cdots \cup A^{(1)}) + 1
\]

and that the maximal distance of two consecutive elements of $A^{(0)} \cup \cdots \cup A^{(1)}$ is not greater than $m_i$. So we choose

\[
a_0 := 0, \quad a_{i+1} := a_i + 14t_i + (t_i + l_i)m_i,
\]

\[
t_{i+1} := \lfloor \alpha t_i \rfloor, \quad \alpha \geq 5, \quad \alpha \in \mathbb{R}
\]

(then $A^{(i+1, 1)} + (A^{(0)} \cup \cdots \cup A^{(1)}) \supseteq [a_{i+1}, 2a_{i+1} - 1]$),

\[
t_{i+1} := \lfloor \beta t_i \rfloor, \quad \beta \geq 1, \beta \in \mathbb{R}.
\]

Then

\[
C := \bigcup_{i=0}^\infty A^{(i)}
\]

is a basis.
Let \( n_i := a_{i+1} - 1 = 10i^2(1 + \beta)/(a^2 + 1)(a^2 - 1)(1 + o(1)) \). Then
\[
C(n_i) = t_0 (10 + \beta) \frac{x^{i+1}}{x-1} (1 + o(1))
\]
and
\[
\lim_{n \to \infty} \frac{C(n)}{\sqrt{n}} = \lim_{n \to \infty} \frac{C(n_i)}{\sqrt{n_i}} = \frac{x + 1}{x - 1} \frac{10 + \beta}{\sqrt{10} (1 + \beta)}.
\]
Looking at
\[
m_i := a_{i+1} + 7i + 1
\]
we find
\[
\lim_{n \to \infty} \frac{C(n)}{\sqrt{n}} = \lim_{n \to \infty} \frac{C(m_i)}{\sqrt{m_i}} = \frac{x + 1}{x - 1} \frac{6 + 4x + \beta}{\sqrt{10} (1 + \beta)}.
\]
Now, for given \( \epsilon > 0 \) we choose
\[
x = \frac{(1 + \epsilon)^2 + 1}{(1 + \epsilon)^2 - 1}, \quad \beta = 8;
\]
then
\[
\lim_{n \to \infty} \frac{C_e(n)}{\sqrt{n}} = 3 \frac{7}{5} \sqrt{\frac{5}{3}} (1 + \epsilon),
\]
\[
\lim_{n \to \infty} \frac{C_e(n)}{\sqrt{n}} = 3 \frac{7}{5} \sqrt{\frac{5}{3}} (1 + \epsilon) \left( \frac{7 + 2 \frac{(1 + \epsilon)^2 + 1}{(1 + \epsilon)^2 - 1}}{1 + \epsilon} \right) < \infty.
\]
Thus, we get

**Theorem 2.** (a) There is a basis \( B \) such that
\[
\lim_{n \to \infty} \frac{B(n)}{\sqrt{n}} = \sqrt{\frac{7}{2}} \quad \text{and} \quad \lim_{n \to \infty} \frac{B(n)}{\sqrt{n}} = \infty.
\]
(b) For each \( \epsilon > 0 \) there is a basis \( C_e \) such that
\[
\lim_{n \to \infty} \frac{C_e(n)}{\sqrt{n}} = 3 \frac{7}{5} \sqrt{\frac{5}{3}} (1 + \epsilon) \quad \text{and} \quad \lim_{n \to \infty} \frac{C_e(n)}{\sqrt{n}} < \infty.
3. AN UNIFORMLY THIN BASIS

Our final result is

**Theorem 3.** There is a basis $D$ such that

$$\lim_{n \to \infty} \frac{D(n)}{\sqrt{n}} = 10 \cdot \frac{7}{6} \cdot \frac{\sqrt{2}}{\sqrt{3}}.$$ 

In order to prove this we define for real $\alpha, \beta, \alpha \leq \beta - 1, m \in \mathbb{N}$,

$$P := \left[ \alpha, (m), \beta \right] := \left\{ \left\lfloor \alpha \right\rfloor + tm \mid 0 \leq t \leq \left\lfloor \frac{[\beta] - \left\lfloor \alpha \right\rfloor}{m} \right\rfloor \right\}$$

and

$$l = l(\alpha, m, \beta) := 2P + \left\lfloor \frac{[\beta] - \left\lfloor \alpha \right\rfloor}{m} \right\rfloor + 1.$$ 

For $[\alpha, (1), \beta]$ we also write $[a, \beta]$.

We need some basic knowledge about the Frobenius problem: Let $m_1, m_2 \in \mathbb{N}$ be coprime. Then $g(m_1, m_2)$ denotes the greatest integer with no representation

$$n = x_1 m_1 + x_2 m_2, \quad x_i \in \mathbb{N}_0.$$ (1)

Then $G(m_1, m_2) := g(m_1, m_2) + 1$ is minimal such that each $n \geq G(m_1, m_2)$ has a representation (1). If $m_1, m_2$ is defined by

$$m_1 \cdot \tilde{m}_1 \equiv 1 \pmod{m_2}, \quad 0 < \tilde{m}_1 \leq m_2$$
$$m_2 \cdot \tilde{m}_2 \equiv 1 \pmod{m_1}, \quad 0 < \tilde{m}_2 \leq m_1$$

then (as essentially known to Sylvester [11])

$$g(m_1, m_2) = -m_1 + (m_1 - 1) m_2 = (m_2 - 1) m_1 - m_2, \quad (2)$$
$$G(m_1, m_2) = (m_1 - 1)(m_2 - 1) = (\tilde{m}_1 - 1) m_1 + (\tilde{m}_2 - 1) m_2, \quad (3)$$
$$h(m_1, m_2) := m_1 m_2 \quad \text{is the greatest number} \ h \ \text{with no representation} \quad (4)$$

$$h = y_1 m_1 + y_2 m_2, \quad y_j \in \mathbb{N},$$
$$l = \tilde{m}_1 m_1 + \tilde{m}_2 m_2 - m_1 m_2. \quad (5)$$
Lemma 2. Let \( P_1 = [x_1, (m_1), \beta_1], P_2 = [x_2, (m_2), \beta_2], \) \( m_1 \leq \# P_2 =: l_2, \)
m_2 \leq \# P_1 =: l_1, \( g \in d (m_1, m_2) = 1. \)

Then
\[ P_1 + P_2 \geq [x_1 + x_2 + G(m_1, m_2) + 1, \beta_1 + \beta_2 - G(m_1, m_2) - m_1 - m_2]. \]

Proof. Let \( \tilde{n} := n - [x_1] - [x_2] \) and observe that if
\[ \tilde{n} = x_1 m_1 + x_2 m_2, \quad 0 \leq x_1 \leq l_1 - 1, \quad 0 \leq x_2 \leq l_2 - 1 \]
then \( n \in P_1 + P_2. \)

Now,
\[ \tilde{n}_0 := l_1 m_1 + l_2 m_2 - m_1 m_2 \]
has no representation (6) (otherwise we would have \( m_1 m_2 = (l_1 - x_1) m_1 + (l_2 - x_2) m_2 \) which is a contradiction to (4)).

But
\[ \tilde{n}_0 - 1 = (l_1 - m_1) m_1 + (l_2 - m_2) m_2 \]
has such a representation.

Let \( \tilde{n}_0 - (k - 1) \) have a representation (6):
\[ \tilde{n}_0 - (k - 1) = x_1 m_1 + x_2 m_2, \quad 0 \leq x_1 \leq l_1 - 1, \quad 0 \leq x_2 \leq l_2 - 1. \]
We ask: when does \( \tilde{n}_0 - k \) have a representation (6)?

We have (because of (5))
\[ \tilde{n}_0 - k = (x_1 - \tilde{m}_1) m_1 + (x_2 - \tilde{m}_2) m_2 + m_1 m_2. \]

If \( x_1 - \tilde{m}_1 + m_2 \geq l_1 \) and \( x_2 - \tilde{m}_2 + m_1 \geq l_2 \) we get the contradiction
\[ \tilde{n}_0 - (k - 1) \geq (l_1 + \tilde{m}_1 - m_2) m_1 + (l_2 + \tilde{m}_2 - m_1) m_2 = \tilde{n}_0 + 1. \]

Thus \( x_1 - \tilde{m}_1 + m_2 \leq l_1 - 1 \) or \( x_2 - \tilde{m}_2 + m_1 \leq l_2 - 1. \)

But then \( \tilde{n}_0 - k \) has a representation
\[ \tilde{n}_0 - k = y_1 m_1 + y_2 m_2, \quad y_1 \leq l_1 - 1, \quad y_2 \leq l_2 - 1. \]

If \( x_1 \geq \tilde{m}_1 \) or \( x_2 \geq \tilde{m}_2 \), then with (7) we can in (8) choose \( y_1 \geq 0 \) and \( y_2 \geq 0. \)
The least \( k \), for which this choice is not possible is determined by
\[ x_1 = \tilde{m}_1 - 1 \quad \text{and} \quad x_2 = \tilde{m}_2 - 1. \]
For this $k$ we have (because of (2))

$$n - k = -m_1 - m_2 + m_1 m_2 = g(m_1, m_2).$$

Therefore, each $n$ with

$$G(m_1, m_2) \leq n \leq (l_1 - m_1) m_1 + (l_2 - m_2) m_2$$

has a representation (6).

Obviously

$$\lfloor x_1 \rfloor + \lfloor x_2 \rfloor + G(m_1, m_2) \leq \lfloor x_1 + x_2 \rfloor + G(m_1, m_2) + 1$$

and

$$\lfloor x_1 \rfloor + \lfloor x_2 \rfloor + (l_1 - m_1) m_1 + (l_2 - m_2) m_2$$

$$= \lfloor x_1 \rfloor + l_1 m_1 + \lfloor x_2 \rfloor + l_2 m_2 - (m_1 - 1) m_1 - (m_2 - 1) m_2 - m_1 - m_2$$

$$> \beta_1 + \beta_2 - G(m_1, m_2) - m_1 - m_2$$

and the lemma follows.

**Remark.** We want to apply this lemma to the set

$$D_{i_0} := \bigcup_{i = i_0}^{\infty} P_i,$$

where $P_i := \left\lfloor x^{\beta^2} q_i, (x + \beta^n) \beta^{2i} - q_i \right\rfloor$, $\alpha > 0$, $\beta > 1$ real, $n \geq 3$ integer (later we will choose $n = 3$, $\beta = \sqrt{2}$, $\alpha = \beta^n/(\beta^2 - 1)$), $q_i$ denotes the greatest prime number $\leq q_i - 1$, $i_0$ is chosen so large that for all $i \geq i_0$ all $q_i$ are distinct, in particular pairwise coprime.

Using the prime number theorem (compare, e.g., Scheid [8]) we get for all sufficiently large $i$

$$\left(1 - \frac{1}{\sqrt{i \log \beta}}\right) \beta^n \leq q_i \leq \beta^n - 1.$$  

For all sufficiently large $i$ we have

$$\#P_i = l_i = \left\lfloor \frac{(x + \beta^n) \beta^{2i} - \lfloor x \beta^{2i} \rfloor}{q_i} \right\rfloor + 1$$

$$\geq \frac{(x + \beta^n) \beta^{2i} - x \beta^{2i} - 2}{\beta^i - 1} - 1 > \beta^{i+n} - 1 \geq q_{i+n}.$$
For all these \( i \) we can apply the lemma to

\[
P_{i+n} + P_i, \quad P_{i+n} + P_{i+1}, \ldots, \quad P_{i+n} + P_{i+n-1}
\]

and then to

\[
P_{i+n+1} + P_{i+1}, \quad P_{i+n+1} + P_{i+2}, \ldots, \quad P_{i+n+1} + P_{i+n}
\]

and so on. All these sums produce intervals of consecutive integers (in the sense of the lemma) and of course we want no integer gaps between two consecutive intervals. By the lemma we have for \( j = i, i+1, \ldots, i+n \) that

\[
P_{i+n} + P_j \equiv [\alpha (\beta^{2i+2n} + \beta^{2j}) + \beta^{i+n+j}, (\alpha + \beta^n)(\beta^{2i+2n} + \beta^{2j}) - \beta^{i+n+j}],
\]

where we have used that

\[
q_{i+n} q_j + q_{i+n} + q_j + 1 = (q_{i+n} + 1)(q_j + 1) \leq \beta^{i+n+j}.
\]

We postulate for \( j = i, i+1, \ldots, i+n-2 \)

\[
(\alpha + \beta^n)(\beta^{2i+2n} + \beta^{2j}) - \beta^{i+n+j} \geq \alpha (\beta^{2i+2n} + \beta^{2j} + 2) + \beta^{i+n+j+1}
\]

or

\[
\alpha (\beta^2 - 1) \leq \beta^{2i+2n-2j} + \beta^n - \beta^{i+n-j} - \beta^{i+n-j+1} =: \sigma_j.
\]

Because of \( \beta > 1 \) we have

\[
\sigma_j < \beta \sigma_j < \sigma_{j-1}.
\]

Therefore, we have no gaps between the intervals generated by

\[
P_{i+n} + P_i, \ldots, P_{i+n} + P_{i+n-1}
\]

if only

\[
\alpha (\beta^2 - 1) < \beta^n + \beta^n - \beta^2 - \beta
\]

(9)

and no gap between the intervals from

\[
P_{i+n} + P_{i+n-1} \quad \text{and} \quad P_{i+n+1} + P_{i+1}
\]

if

\[
\alpha (\beta^{2n} - \beta^{2n-2} - \beta^{2n-4} + 1) < \beta^{3n} - \beta^{3n-4} - \beta^{2n-3} - \beta^n.
\]

(10)

Then

\[
D_0 + D_0 \sim \mathbb{N}_0
\]
and
\[ \vec{D} := D_0 \cup I \quad (I \text{ a suitable finite interval}) \]
is a basis.

Furthermore we wish that the difference between the greatest element of \( P_i \) and the least element of \( P_{i+1} \) is greater or equal to \( q_i \). This means
\[ (x + \beta^n) \beta^{2i} \leq z \beta^{2i+2} \]
or
\[ x (\beta^2 - 1) \geq \beta^n. \]

We choose \( x = \beta^n / (\beta^2 - 1) \).

Starting with
\[ u_i = \lfloor x \beta^{2i} \rfloor + t_i q_i + r_i, \quad 0 \leq t_i \leq l_i - 1, \quad 0 \leq r_i \leq q_i - 1 \]
we get \( \lim \vec{D}(m)/\sqrt{m} \) by considering \( f(t_i) := D(u_i)/\sqrt{u_i} \) and solving \( f'(t_i) = 0 \). We get
\[ t_i = \frac{\beta^{i+n}}{\beta + 1} \left( 1 + \frac{1}{\sqrt[i]{\beta}} \right) \]
giving
\[ \lim_{\sqrt{m}} \frac{\vec{D}(m)}{\sqrt{m}} = 2 \sqrt{x \beta}. \]

On the other side with
\[ v_i = \lfloor (x + \beta^n) \beta^{2i} \rfloor - q_i \]
we find
\[ \lim_{\sqrt{m}} \frac{\vec{D}(m)}{\sqrt{m}} = (\beta + 1) \sqrt{x}. \]

The set \( \vec{D} \) is a basis, but \( \lim \vec{D}(m)/\sqrt{m} \) does not exist. In order to get a set \( D \supseteq \vec{D} \) with \( \lim \frac{D(m)}{\sqrt{m}} \) existing we denote by
\[ \vec{d}_0 < \vec{d}_1 < \vec{d}_2 < \cdots \]
the consecutive elements of $\mathcal{D}$ and find

$$
\lim_{j \to \infty} \frac{d_{j+1} - d_j}{\sqrt{d_j}} \asymp \frac{q_i}{\sqrt{(\alpha + \beta^n)^{1/2} q_i}} = \frac{1}{\beta \sqrt{\alpha}}.
$$

Now, a result of Cassels [2] (Hilfssatz 3) comes into the picture: Let $A: a_0 < a_1 < a_2 < \cdots$ be a sequence of integers with

$$
\lim a_{r+1} - a_r = \lambda, \tag{11}
$$

where $h > 1$ and $\lambda > 0.$ Then there is a sequence of integers $B: b_0 < b_1 < b_2 < \cdots$ with $A \subseteq B$ such that

$$
\lim_{r \to \infty} \frac{b_r}{r^h} = \left(\frac{\lambda}{h}\right)^h.
$$

If strict inequality holds in (11) one can even get

$$
b_r = \left(\frac{\lambda}{h}\right)^h r^h + O(r^{h-1}).
$$

In our application (with $h = 2$ and $\lambda = \frac{1}{d^2}$) of this result we replace $A$ by $\mathcal{D}$ and get a sequence of integers $D: d_0 < d_1 < d_2 < \cdots$ with $A \subseteq \mathcal{D}$ such that

$$
\lim d_r = \left(\frac{\lambda}{2}\right)^2 \text{ or } d_r = \left(\frac{\lambda}{2}\right)^2 r^2(1 + o(1)), \tag{12}
$$

Let $m \in \mathbb{N}_0$. Then there is a unique $k \in \mathbb{N}_0$ such that

$$
d_k \leq m < d_{k+1}.
$$

This gives $D(m) = k$ and together with (12)

$$
d_k = \left(\frac{\lambda}{2}\right)^2 k^2(1 + o(1)) \leq m \leq \left(\frac{\lambda}{2}\right)^2 (k + 1)^2(1 + o(1)) = \left(\frac{\lambda}{2}\right)^2 k^2(1 + o(1))
$$

and therefore

$$
m = \left(\frac{\lambda}{2}\right)^2 k^2(1 + o(1))
$$

or

$$
k = \frac{2}{\lambda} \sqrt{m(1 + o(1))}.\n$$
It follows
\[
\lim \frac{D(m)}{\sqrt{m}} = \frac{2}{\beta} \sqrt{\frac{m}{2}} = 2 \sqrt{\frac{p+2}{p^2 - 1}}
\]
which becomes minimal for \( \beta_{\text{min}} = \frac{\alpha + 2}{n} \).

For \( n = 2 \) the inequality (10) does not hold. For \( n = 3 \) we find
\[
\lim \frac{D(m)}{\sqrt{m}} = 10 \cdot \frac{1}{\sqrt{6}} \cdot \frac{\sqrt{5}}{\sqrt{3}} = 4.638 \ldots.
\]

Remark. Compared with
\[
\lim \frac{\tilde{D}(m)}{\sqrt{m}} = 4.082 \ldots,
\]
\[
\lim \frac{D(m)}{\sqrt{m}} = 4.115 \ldots,
\]
we see that we have to add more than \( \frac{\sqrt{m}}{n} \) elements to \( \tilde{D} \) up to \( m \) in order to get the existence of \( \lim \frac{D(m)}{\sqrt{m}} \).

REFERENCES