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A Class of N Nonlinear Hyperbolic Conservation Laws*

LARS HOLDEN

Norwegian Computing Center,
P.B. 114 Blindern, 0314 Oslo 3, Norway

AND

RAPHAEL HØEGH-KROHN†

Matematisk Institutt, Universitetet i Oslo,
Blindern, 0316 Oslo 3, Norway

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The Riemann problem for a class of nonlinear systems of first order hyperbolic conservation laws is studied. The class consists of systems where the derivative of the flux function is a lower triangular matrix. There are no assumptions on genuine nonlinearity and strict hyperbolicity. Existence and uniqueness are proved except in a set with measure zero in the phase space and a set with measure zero in the flux function space where there is a one-parameter family of solutions. Travelling waves are used as an entropy condition and examples show that the Lax or Liu entropy conditions are not sufficient. An example shows that the solution does not necessarily depend continuously on the data. The model may be used to describe three-phase and tracer flow and flow in a neighborhood of a heterogeneity in porous media.

1. INTRODUCTION

In this paper we study the Riemann problem for the system of differential equations

$$\frac{\partial}{\partial t} u_i + \frac{\partial}{\partial t} f_i(u_1, ..., u_i)_x = 0, \qquad i = 1, 2, ..., n, \tag{1.1}$$

where f_i is continuous and the partial derivatives of f_i are defined almost everywhere. Assume furthermore that $\partial f_i/\partial u_i$ is piecewise monotone in u_i

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† Deceased on January 24, 1988.

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with a finite number of intervals where it is monotone. In order to always have a solution it is also necessary to impose some restrictions on the behavior of f_i when $|u_i|$ is large.

The Riemann problem is a particular Cauchy problem where the initial condition is

$$u_i(x,0)=\begin{cases} u_{i,-} & \text{for} \quad x<0, \\ u_{i,+} & \text{for} \quad x>0, \end{cases} \quad i=1,2,...,n. \tag{1.2}$$

In problem (1.1) the matrix $\{\partial f_i/\partial u_j\}_{i,j}$ is lower triangular. The eigenvalues of this matrix are the diagonal elements. The problem is therefore always hyperbolic, and we will call it a triangular hyperbolic system. There are no assumptions on possible degeneracy of the eigenvalues. Therefore the class to be studied is not necessarily strictly hyperbolic. Genuine nonlinearity for this class of problem is equivalent to

$$\frac{\partial^2 f_i}{\partial u_i^2}\neq 0, \qquad i=1,...,n.$$

We will allow for loss of genuine nonlinearity in this paper.

For $n=1$, i.e., the scalar problem, existence and uniqueness were proved by Oleinik [22, 23]. For systems, existence and uniqueness were proved by Lax [19] and Glimm [5] assuming u_- and u_+ close, strict hyperbolicity, and genuine nonlinearity. Liu [20] extended this result for 2×2 systems with monotonicity assumptions instead of genuine nonlinearity and without assuming u_- and u_+ close. In order to attain existence and uniqueness Liu introduced an extended entropy condition. Several authors have extended the analysis to 2×2 systems which fail to be strictly hyperbolic. For a survey of this work, including the work of M. Gomes, H. Holden, E. Isaacson, B. Keyfitz, H. Kranzer, D. Marchesin, B. Plohr, D. Schaeffer, M. Shearer, and B. Temple, see [6].

The Riemann problem is a particular mathematical problem where it is possible to find an explicit solution. In addition it is used as a building block in the Cauchy problem with general initial data. In fact, the Riemann problem is used for both existence and uniqueness theorems and as a numerical method. It is used in both ways in the celebrated paper by Glimm [5]. See [10, 11] for a particular method to solve the Cauchy problem by solving Riemann problems in the scalar case. Godunov [8] uses the Riemann problem in a numerical method.

There are no smooth solutions of (1.1) with general initial data except for small t, no matter how smooth the flux function is. Therefore we are interested in weak solutions. There are several weak solutions of the problem. In this paper we use an entropy criterion with travelling waves in order to find the relevant solution. See Chapter 24 in [25] and Conley and

Smoller [1]. A shock with speed s and with values u_- and u_+ to the left and to the right, respectively, is deemed admissible iff there exists an integral curve

$$u'(\xi) = f(u(\xi)) - su(\xi) - (f(u_\pm) - su_\pm), \tag{1.3}$$

and

$$u(\xi) \to u_\pm \qquad \text{when} \quad \xi \to \pm\infty.$$

We call this integral curve an entropy curve in order to separate it from other integral curves.

The origin for the travelling wave entropy criterion is that the solution is the limiting solution when a second order term vanishes. Assuming a solution of the form $u(x, t) = v((x - st)/\varepsilon)$ of the regularized equation

$$u_t + f(u)_x = \varepsilon u_{xx},$$

the limiting solution when ε vanishes satisfies the entropy criterion above. We have used the identity matrix as the viscosity matrix, which is usual. See Conley and Smoller [1].

A solution satisfying the above entropy condition will also satisfy the Rankine–Hugoniot condition

$$s(u_+ - u_-) = f(u_+) - f(u_-). \tag{1.4}$$

The Hugoniot locus is defined as

$$H(u_-) = \{u; s(u_+ - u_-) = f(u_+) - f(u_-) \text{ for } s \in R\}.$$

There are other possible entropy criteria. Lax used the inequalities

$$\lambda_{k-1}(u_-) < s < \lambda_k(u_-) \qquad \text{and} \qquad \lambda_k(u_+) < s < \lambda_{k+1}(u_+),$$

where λ_k are the ordered eigenvalues. Keyfitz and Kranzer [14] introduced a generalized Lax entropy condition by allowing nonstrict inequalities. This entropy condition is used in many other papers on nonstrictly hyperbolic conservation laws; see, e.g. Keyfitz and Kranzer [14, 15] and Isaacson and Temple [13]. We will show in an example in the last section that the generalized Lax entropy condition in some cases gives a solution which differs from the solution one obtains with the travelling wave entropy condition. In this paper we have used the travelling wave entropy condition since it is the limiting solution when a second order term vanishes. See Conley and Smoller [1] and Keyfitz and Kranzer [14]. Liu [20] introduced another entropy criterion when he studied systems which are not genuinely nonlinear. This entropy condition is only well-defined for the connected Hugoniot locus. In triangular systems there are cases where all weak solutions have shocks to disconnected branches of the Hugoniot locus.

There are two main reasons to study a triangular hyperbolic system.

First, by restricting ourselves to the analysis of triangular systems, we are able to solve the Riemann problem for a large class of $n \times n$ systems with no assumptions on genuine nonlinearity and strict hyperbolicity and for a very general flux function. Some new phenomena are found. We will in particular mention that we find a one-parameter family of solution for $n = 3$. When the initial values approach the values where there is a one-parameter family of solutions, we get the two end points in the one-parameter family. Therefore the solution does not depend continuously on the initial values. Disconnected branches of the Hugoniot locus (see [6, 14]) are accepted as a solution. The generalized Lax entropy condition gives another solution other than the travelling wave entropy condition in some cases. In Section 3 we give examples of these phenomena. Both Shearer [24] and Gomes [9] find two solutions both satisfying the travelling wave entropy condition. But in their cases the two solutions are qualitatively different. In this paper there are two qualitatively equal solutions and a one-parameter family of admissible solutions connecting the two end point solutions.

Second, it is possible to approximate the solution of some physical problems by the solution of (1.1). We will mention three examples from incompressible flow in porous media.

In three-phase flow in porous media with oil, water, and gas, it is reasonable to approximate the fractional flow function of gas with a function which only depends on the gas saturation. Therefore the first equation in (1.1) models the flow of gas and f_1 is only a function of the gas saturation. f_2, the fractional flow of water, is a function of both water and gas saturations. These assumptions result in a triangular system. This model is worked out in detail in separate papers; see Gimse [3, 4] and Holden [12].

In two-phase flow the fractional flow function may change between different rock types. This is modelled in a triangular hyperbolic system by letting the first independent variable only depend on the rock type. The solution of the first equation is therefore only a shock with speed zero at the border between the different rock types. See Gimse [3, 4] and Glimm and Sharp [7].

Finally, in an oil reservoir water is often injected with tracers in order to keep track of the injected water. This is usually modelled by the system

$$s_t + f(s)_x = 0$$

$$(sc_i + a_i(c_i))_t + (c_i f(s))_x = 0, \qquad i = 1, ..., n,$$

where s is water saturation and c_i is the concentration of tracer i, $i = 1, ..., n$.

Notice that the tracers do not influence the water/oil flow. This model is studied in Johansen and Winther [17]. This model is a simplification of the polymer model described in Johansen and Winther [16].

The solution of the Riemann problem $u(s) = u(x/t)$ is made up of three types of elementary waves (solutions), namely

(i) constant states,

(ii) shock waves satisfying the entropy condition above, and

(iii) rarefaction waves, i.e., continuous solutions satisfying the ordinary differential equation

$$-su_s + f(u)_s = 0.$$

In order to always get a solution we have to accept adjacent shocks with the same speed. This is also necessary in the scalar equation and is admissible according to the generalized Lax entropy condition.

In the following section we prove existence of a solution of (1.1) and (1.2) for all initial values and uniqueness almost everywhere. Some characteristics of the solution are discussed in Section 3.

2. Existence and Uniqueness

We will first state the main theorem in the paper.

THEOREM 2.1. *Assume f_i is continuous and that all partial derivatives of f_i are defined almost everywhere. Assume furthermore that $(\partial f_i/\partial u_i)(u_1, ..., u_i)$ as a function of u_i with $u_1, ..., u_{i-1}$ fixed satisfies the following for $i = 1, ..., n$:*

(i) piecewise monotone and monotone in a finite number of intervals and

(ii) increases to ∞ or decreases to $-\infty$ for $u_i < u_{i,\min}$ and for $u_i > u_{i,\max}$ for some constants $u_{i,\min}$ and $u_{i,\max}$.

Then there exists a solution of the Riemann problem (1.1) *and* (1.2). *The solution is unique except for functions f in a set with measure zero in the supremum norm and for a given f initial values u_- and u_+ in a set with measure zero in the (u_-, u_+) plane. There is always uniqueness if $n < 3$.*

The proof of this theorem is given at the end of this section. In fact we will prove a sharper theorem. In Theorem 2.3 there are weaker assumptions on f_i for u_i large/small. The assumptions on f_i depend on the solution of the first $i - 1$ equations. In some applications this stronger theorem is needed since the flux functions f_i have a horizontal asymptote when $u_i \to \infty$ and when $u_i \to -\infty$. In these applications we utilize that all waves have positive speed.

We allow discontinuities in the derivatives of f since this is necessary in some applications. This does not weaken the results and does not give any complications in the proofs.

The system

$$u_t + (u^2)_x = 0$$
$$v_t + (uv)_x = 0$$

has been studied by Korchinski [18]. This system does not satisfy the assumption on f since f_2 is linear in $v = u_2$. In this system there is not always a weak solution in the usual sense since there are shocks where (1.4) is not satisfied. The solution of this problem, which is described in [18], may contain a delta function. This system does not satisfy the assumptions on the flux function in Theorem 2.1. We do not accept a delta function in the solution in Theorem 2.1.

The system (1.1) with initial value (1.2) is solved inductively. The first equation is a scalar equation and existence and uniqueness theorems are well known. This is stated as a separate theorem.

THEOREM 2.2. *The scalar Riemann problem*

$$u_t + f(u)_x = 0,$$

where f is continuous, f' is defined except at a finite number of points, and f' is piecewise monotone with a finite number of intervals where the function is monotone, and with the initial value

$$u(x, 0) = \begin{cases} u_+ & \text{for} \quad x > 0 \\ u_- & \text{for} \quad x < 0, \end{cases}$$

has a unique solution which may be described uniquely by a function $u(s)$, where $s = x/t$. $u(s)$ is piecewise continuous and there are a $s_{\min}$ and a $s_{\max}$ such that $u(s)$ is constant for $s < s_{\min}$ and $s > s_{\max}$. At a discontinuity of $u(s)$ there exists a unique entropy curve $w(\xi)$ such that

$$w'(\xi) = f(u(\xi)) - su(\xi) - (f(u(s_\pm)) - su(s_\pm)),$$

and

$$w(\xi) \to u(s_\pm) \qquad \text{when} \quad \xi \to \pm\infty.$$

At a discontinuity of $u(s)$ the values on the left and right hand side of the discontinuity are denoted u_- and u_+, respectively.

The entropy curve is unique up to change in parameter ξ.

Proof of this theorem with weaker assumption on f is given in [2, 11]. See also Theorem 3.1.

Assume that the problem (1.1) is solved for the first $i<n$ equations. We will then solve the problem for the first $i+1$ equations. The $(i+1)$st equation may be written as

$$v_t + g(u, v)_x = 0 \tag{2.1}$$

and

$$v(x, 0) = \begin{cases} v_- & \text{for} \quad x < 0 \\ v_+ & \text{for} \quad x > 0. \end{cases} \tag{2.2}$$

$u(s)$, $s = x/t$, is a known piecewise continuous function $u: R \to R^i$, which is constant for $s < s_{\min}$ and $s > s_{\max}$ for some $s_{\min}$ and $s_{\max}$. Where $u(s)$ is discontinuous, there exists an entropy curve

$$w'(\xi) = f(u(\xi)) - su(\xi) - (f(u(s_\pm)) - su(s_\pm)) \tag{2.3}$$

and

$$w(\xi) \to u(s_\pm) \qquad \text{when} \quad \xi \to \pm\infty.$$

Similarly the solution v will be described by a function $v(s)$ and for each discontinuity in $v(s)$ there is an entropy curve $y(\xi)$.

The induction step in the proof of Theorem 2.1 is stated as a separate theorem.

THEOREM 2.3. *Assume that*:

(i) *$u(s): R \to R^i$ is piecewise continuous and constant for $s < s_{\min}$ and $s > s_{\max}$ and where $u(s)$ is discontinuous there exists an entropy curve $w(\xi)$,*

(ii) *g is continuous,*

(iii) *the partial derivatives of g are defined almost everywhere,*

(iv) *g_v is piecewise monotone and monotone in a finite number of intervals, and*

(v) *there exist a $v_{\min}$ and a $v_{\max}$ such that*

$$\textit{either } g_v(u, v) < s_{\min} \textit{ or } g_v(u, v) > s_{\max} \textit{ for all } u, \textit{ when } v < v_{\min}$$

and

$$\textit{either } g_v(u, v) < s_{\min} \textit{ or } g_v(u, v) > s_{\max} \textit{ for all } u, \textit{ when } v > v_{\max}.$$

Then there exists a unique solution of the Riemann problem

$$v_t + g(u, v)_x = 0$$

$$v(x, 0) = \begin{cases} v_L & \textit{for} \quad x < 0 \\ v_R & \textit{for} \quad x > 0, \end{cases}$$

for arbitrary $v_L, v_R \in R$.

There also exists an entropy curve for each shock in v. This entropy curve is unique except for $g(u, v)$ *in a set with measure zero in the supremum norm and for a set with measure zero in the* (v_L, v_R) *plane which depends on the function* $g(u, v)$.

If the entropy curve $w(\xi)$ *is not unique, the solution* $v(x, t)$ *is not always unique.*

The theorem is proved at the end of the section. Define the set

$$C(v_L, s_0) = \{v_R \in R;\ \text{There is an admissible solution } v(s) \text{ for } x/t \leqslant s_0 \text{ such that } v(s) = v_L \text{ for } s < M \text{ for some constant } M \text{ and } v(s_0) = v_R\}.$$

We will prove that $C(v_L, s)$ has the following properties:

(P)

(i) $C(v_L, s) = \bigcup_{j=1}^{m} [a_j, b_j]$, where $a_1 \leqslant b_1 < a_2 \leqslant \cdots < b_m$ and $a_j, b_j \in R \cup \{-\infty, \infty\}$ for $j = 1, ..., m$,

(ii) If $v \in [a_j, b_j]$ for $1 \leqslant j \leqslant m$ then $g_v(u(s), v) \leqslant s$,

(iii) $g(u(s), a_j) = g(u(s), b_{j-1}) + s(a_j - b_{j-1})$ for $j = 2, ..., m$.

In the proof we use a fixed v_L value. Therefore we use the notation $C(s)$ instead of $C(v_L, s)$. The set $C(s)$ is most easily visualized by defining the function $h_s(v)$:

$$h_s(v) = \begin{cases} g(u(s), v) & \text{if} \quad v \in C(s) \\ g(u(s), a_1) + s(v - a_1) & \text{if} \quad v < a_1 \\ g(u(s), a_j) + s(v - a_j) & \text{if} \quad b_{j-1} < v < a_j,\ j = 1, ..., m \\ g(u(s), b_m) + s(v - b_m) & \text{if} \quad b_m < v. \end{cases}$$

From the properties (P) it is easily seen that h_s is a continuous function and $h_s' \leqslant s$ where defined.

See Fig. 2.1 for a typical $C(s)$, $h_s(v)$, and $g(u(s), v)$.

We may then start with the proofs. In the following lemmas we assume that g and u satisfy the assumptions in Theorem 2.3.

LEMMA 2.4. *If* $u(s) = u_-$ *for* $s \leqslant s_-$, *then* $C(s_-)$ *satisfies* (P).

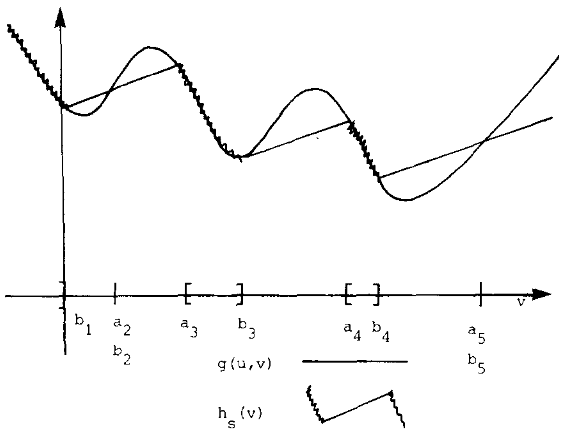

FIG. 2.1. A typical $g(u, v)$, $C(s) = \bigcup_{j=1}^{5} [a_j, b_j]$, and $h_s(v)$, $s > 0$.

Proof. When $u(s)$ is constant, the system (2.1) and (2.2) is equivalent to the scalar problem. The solution is then well known. If v_L is smaller than v_R, the solution is described by the convex envelope from v_L to v_R, and if v_L is larger than v_R, the solution is described by the concave envelope from v_L to v_R. It is easily seen that $C(s_-)$ satisfies (P). ∎

LEMMA 2.5. *Assume $C(s_0)$ satisfies the properties* (P) *and that $u(s)$ is continuous for $s \in [s_0, s_1]$. Then $C(s_1)$ satisfies the properties* (P).

Proof. When $u(s)$ is continuous, we will prove that the solution of (2.1)–(2.3) is a combination of constant states, smooth rarefaction waves, and shocks in the v variable.

Let $v_0 \in C(s_0)$ be arbitrary. Assume first that $g_v(u(s_0), v_0) \neq s_0$. Equation (2.1) may be rewritten as

$$-sv_s + g_u(u, v)\, u_s + g_v(u, v)\, v_s = 0.$$

Therefore there is a constant state or rarefaction wave starting in v_0 defined by

$$v(s_0) = v_0 \tag{2.4}$$

$$v_s(s) = \frac{g_u u_s}{s - g_v}. \tag{2.5}$$

Obviously there is only a constant state if $v_s = 0$. These curves are well-defined as long as $g_v(u(s), v(s)) \neq s$.

Assume that $g_v(u(s), v(s)) = s$ either for $s = s_0$ or for $s > s_0$. In the degenerate case where also $g_u u_s = 0$ the solution is described as in the scalar case. If $g_u u_s \neq 0$, there is a shock. Since $u(s)$ is continuous, this shock is exactly like a shock in the scalar equation, i.e., we may connect a value v_+ to the right to a v_- value with speed

$$s = \frac{g(u(s), v_-) - g(u(s), v_+)}{v_- - v_+}.$$

If $g(u(s), v) > v_- + s(v - v_-)$ for v between v_- and v_+, then $v_+ > v_-$, and if $g(u(s), v) < v_- + s(v - v_-)$ for v between v_- and v_+, then $v_- > v_+$.

Thus for every point v_0 where $h_{s_0}(v_0) = g(u(s_0), v_0)$, there starts a rarefaction or a combined rarefaction and shock curve in the v variable. When s increases from s_0 to s_1, these curves define the set $C(s)$. It is easily seen that $C(s_1)$ satisfies the properties (P). ∎

Thus we are left with the most difficult case where there is a shock in u. Assume $u(s)$ is discontinuous at s_0 and a single admissible shock connects the left and right values u_- and u_+, respectively, i.e., there exists a piecewise monotone entropy curve $w(\xi)$ such that

$$w(\xi) \to u_\pm \qquad \text{when} \quad \xi \to \pm\infty.$$

In order to simply the notation let us use $C(s_-)$, $C(s_+)$, $h_-(v)$, and $h_+(v)$ instead of $C(s_0-)$, $C(s_0+)$, $h_{s_0-}(v)$, and $h_{s_0+}(v)$, respectively.

Assume that $C(s_-)$ satisfies (P). We will first prove that from almost everywhere at $g(u_-, v)$ there starts an integral curve and from almost everywhere at $g(u_+, v)$ there ends an integral curve. Notice that these integral curves are defined such that if a curve starts from a point at $g(u_-, v)$ and ends at a point at $g(u_+, v)$ then the integral curve is an entropy curve (Fig. 2.2).

LEMMA 2.6. *Consider the integral curves*

$$v'_{b,c}(\xi) = g(w(\xi), v_{b,c}(\xi)) - s_0 v_{b,c}(\xi) - c, \qquad \xi \in R, \qquad v_{b,c}(0) = b,$$

where b and c are constants. Then

(i) *If $b_1 < b_2$, then $v_{b_1,c}(\xi) < v_{b_2,c}(\xi)$ for all ξ.*

(ii) *For all values of b, $v_{b,c}(\xi)$ converges to a v_- where $g(u_-, v_-) = c + s_0 v_-$ or diverges to ∞ or $-\infty$ when ξ decreases to $-\infty$.*

(iii) *For every value of v_- where $g(u, v) - s_0 v$ increases in v in a neighbourhood of v_-, there exist d and e such that for $d < b < e$, $v_{b,c}(\xi)$ converges to v_- when ξ decreases to $-\infty$.*

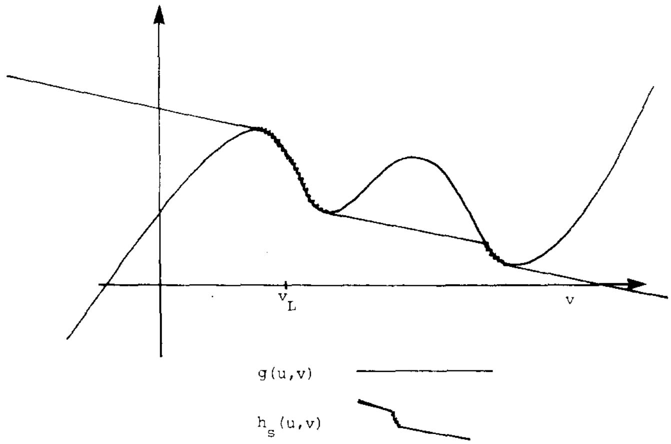

FIG. 2.2. $g(u, v)$ and $h_s(v)$ for $u(s)$ constant, $s<0$.

(iv) *For every value of v_- where $g(u, v)-s_0 v$ decreases in v in a neighbourhood of v_-, there exists a unique b such that $v_{b,c}(\xi)$ converges to v_- when ξ decreases to $-\infty$.*

When ξ increases to ∞, (ii), (iii), *and* (iv) *are still valid but there we have a unique value of b when $g(u, v)-s_0 v$ increases and convergence for b in an interval when $g(u, v)-s_0 v$ decreases.*

Proof. It is trivial to prove (i) and (ii). We will only prove the lemma for $s_0=0$ and when ξ decreases to $-\infty$.

Let $g(u_-, v_-)=c$ and $g(u_-, v)$ be monotone decreasing/increasing in a neighbourhood of v_-. Then according to the assumptions on $g(u, v)$ and $w(\xi)$, there exists a ξ_0 such that for $\xi<\xi_0$, $g(w(\xi), v)$ is monotone decreasing/increasing for $v \in (d, e)$ where $v_-<e$. Therefore there exists a unique function $a(\xi)$ such that $g(w(\xi), a(\xi))=c$ for $\xi<\xi_0$ and $a(\xi) \rightarrow v$, when ξ decreases to $-\infty$.

Assume $g(u, v)$ is increasing in a neighbourhood of v_-. Then the point $a(\xi)$ is attractive; $v(\xi)$ is moving towards $a(\xi)$ when $\xi<\xi_0$ if $v(\xi)$ is close to v. Therefore $v_{b,c}(\xi)$ converges to v_- when $d<v_{b,c}(\xi_0)<e$. $d<v_{b,c}(\xi_0)<e$ corresponds to $d'<v_{b,c}(0)<e'$ for some constants d' and e'. This proves (iii).

Assume $g(u, v)$ is decreasing in a neighborhood of v_-. Then the point $a(\xi)$ is repulsive; $v(\xi)$ is always moving away from $a(\xi)$. According to (ii), either $v_{b,c}(\xi)$ converges to v_- or diverges to ∞ or $-\infty$. We will prove that $v_{b,c}(\xi)$ always converges to v_- or diverges to the left/right of v for b in an open interval. Let us first prove that there is an open interval to the left.

For general initial value problems we have

$$\lim_{b \to b_0} v_{b,c}(\xi) = v_{b_0,c}(\xi) \qquad \text{for all } \xi.$$

Assume $v_{b,c}(\xi)$ converges to the left of v_-. Then $v_b(\xi) < d - \delta$ for $\xi < \xi_1$ for some ξ_1. Then also $v_{b+\varepsilon,c}(\xi) < d - \delta/2$ for $\varepsilon > 0$ small and therefore also $v_{b+\varepsilon,c}(\xi)$ converges to the3 left of v_-. Then the interval of convergence to the left of v_- is open. Convergence to the right of v_- is proved similarly. Therefore there exists at least one point between these two open intervals where $v_{b,c}(\xi)$ converges to v_-.

Assume $v_{b_1,c}(\xi)$ and $v_{b_2,c}(\xi)$ both converge to v_-. It is easily seen that $|v_{b_1,c}(\xi) - v_{b_1,c}(\xi)|$ increases when ξ decreases, so there must be a unique value of b such that $v_{b,c}(\xi)$ converges to v_- when ξ decreases to $-\infty$. ∎

Figure 2.3a shows a typical function $g(u_-, v)$ and a c value. Figure 2.3b shows where $v_{b,c}(\xi)$ converges when ξ decreases to $-\infty$ with the $g(u_-, v)$ function in Fig. 2.3a.

LEMMA 2.7. *Let $v_i(\xi)$, $i = 1, 2$, be two integral curves satisfying*

$v_i'(\xi) = g(w(\xi), v_i(\xi)) - s_0 v_i(\xi) - c_i$,

$v_i(\xi)$ *converges to* $v_{i,-}$ *when* ξ *decreases to* $-\infty$,

$h_-(v_{i,-}) = g(u_-, v_{i,-})$ *for* $i = 1, 2$, *and*

$v_{1,-} < v_{2,-}$.

Then $v_1(\xi) < v_2(\xi)$ *for all* ξ.

Proof. For simplicity we assume $s_0 = 0$.
$v_i'(\xi)$ vanishes when ξ decreases to $-\infty$. Therefore

$$c_i = g(u_-, v_{i,-}) = h_-(v_{i,-}).$$

According to the assumptions on $h_-(v)$ and since $v_{1,-} < v_{2,-}$, we have $c_1 \geqslant c_2$.

Assume the lemma is not correct. Let ξ_0 be the smallest value of ξ such that $v_1(\xi_0) = v_2(\xi_0)$. Then $v_1(\xi_0 - \varepsilon) < v_2(\xi_0 - \varepsilon)$ for $\xi > 0$ small. But

$$v_1'(\xi_0) = g(w(\xi), v_1(\xi_0)) - c_1 < g(w(\xi), v_2(\xi_0)) - c_2 = v_2'(\xi)$$

and therefore $v_1(\xi_0 - \varepsilon) > v_2(\xi_0 - \varepsilon)$. This is a contradiction and therefore $v_1(\xi) < v_2(\xi)$ for all ξ. ∎

Now we can consider the case where $u(s)$ is discontinuous, i.e., there is a shock in one of the equations higher up in the system of equations.

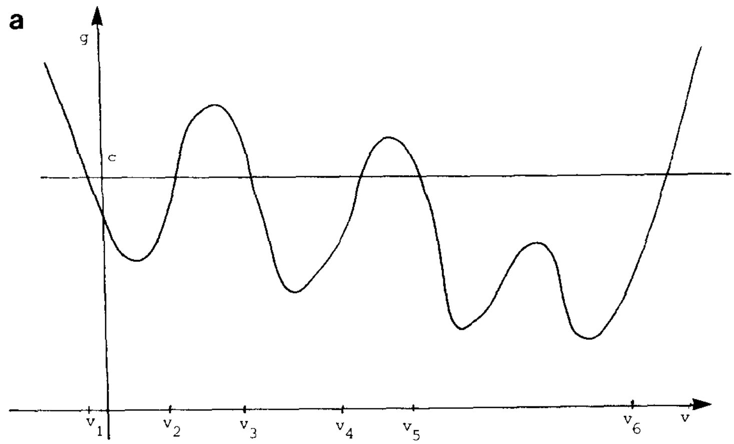

FIG. 2.3a. $g(u_-, v)$ and the constant c.

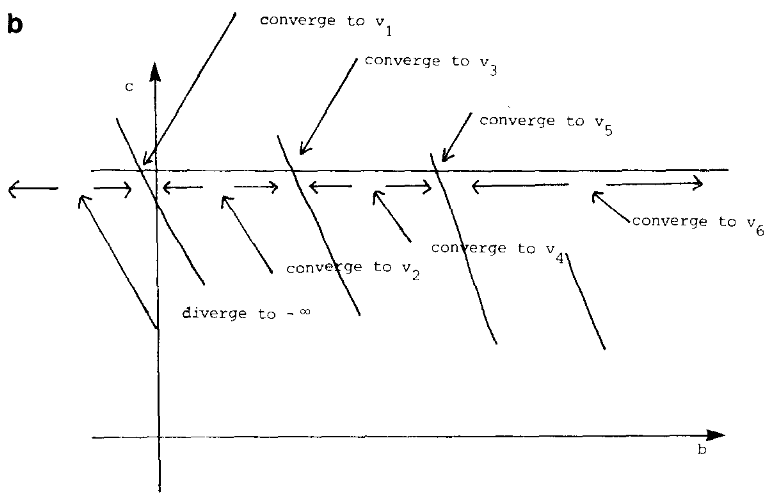

FIG. 2.3b. Convergence when $\xi \to -\infty$.

LEMMA 2.8. *Assume that $u(s)$ is discontinuous for $s = s_0$ and that $C(s_-)$ satisfies* (P). *Then $C(s_+)$ satisfies* (P).

Proof. For simplicity we assume $s_0 = 0$.
In Lemma 2.7 we defined the integral curves

$$v_{b,c}(0) = b \qquad \text{and} \qquad v'_{b,c}(\xi) = g(w(\xi), v_{b,c}(\xi)) - c.$$

$v_{b,c}$ converges to v_- or diverges to ∞ or $-\infty$ when ξ decreases to $-\infty$. In Lemma 2.6 we proved that if $g(u, v)$ is monotone decreasing in v in a

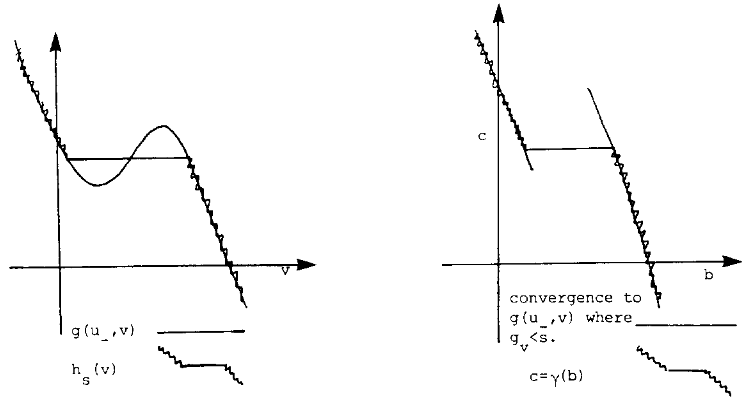

FIG. 2.4. $h_s(v)$ and $\gamma(b)$. $s=0$ in figure.

neighbourhood of v, then there is a unique value of (b, c) such that $v_{b,c}(\xi)$ converges to v_- when ξ decreases to $-\infty$. Lemma 2.7 shows that integral curves which converge to points on $h_-(v)$ do not cross each other. Then we may define the function $\gamma(b)=c$ if $v_{b,c}(\xi)$ converges to v_- and $h_-(v_-)=c$. It is easy to see that $\gamma(b)$ is well-defined and continuous. See Fig. 2.4. $\gamma(b)$ is a monotone decreasing function of b.

In a similar way the integral curve $v_{b,c}(\xi)$ converges to some v_+ or diverges to ∞ or $-\infty$ when ξ increases to ∞. Let us study the values of (b, c) where the curve $v_{b,c}(\xi)$ converges to v_+ when ξ increases to ∞. For convergence to v_+ the situation is changed; there is a single value of (b, c) for which $v_{b,c}(t)$ converges to a point where $g(u_+, v)$ is monotone increasing in a neighborhood of v_+ and an interval with values of b for which $v_{b,c}(\xi)$ converges to a v_+ where $g(u_+, v)$ is monotone decreasing in a neighbourhood of v_+. See Fig. 2.5, where the different values of (b, c),

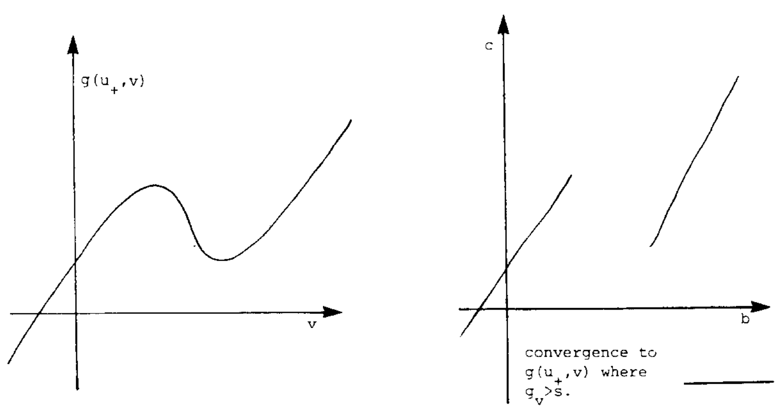

FIG. 2.5. Convergence to $g(u_+, v)$.

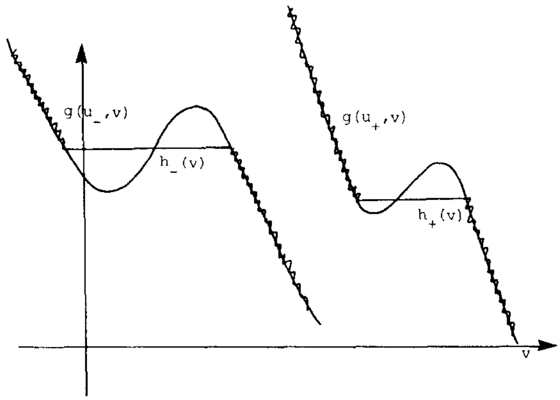

FIG. 2.6. $h_-(v)$ and $h_+(v)$.

where $v_{b,c}(\xi)$ converges to v_+ and $g(u_+, v)$ is monotone increasing in a neighbourhood of v_+, are shown.

Since $\gamma(b)=c$ is continuous, it intersects the curves where the corresponding integral curve $v_{b,c}(\xi)$ converges to v_+ and $g(u_+, v)$ is monotone increasing in a neighbourhood of v_+.

If $\gamma(b)=c$ and $v_{b,c}(\xi)$ converges to v when ξ increases to ∞, then $v \in C(s_+)$ and $h_+(v)=c$. It is trivial to see that $C(s_+)$ satisfies (P). Figure 2.6 shows typical $h_-(v)$ and $h_+(v)$. ▮

Proof of Theorem 2.3. $u(s)$ is piecewise continuous and constant for s small and s large. It follows from Lemma 2.4, Lemma 2.5, and Lemma 2.8 that $C(s)$ satisfies (P) for $s \in R$. By the assumptions on $u(s)$, $g(u, v)$ and the properties (P) of $C(s)$ there is a s_+ such that for $s > s_+$, $C(s) = R$ (i.e., $h_s(v) \equiv g(u, v)$). Thus it is possible to connect the fixed v_L to all values of v_R. Since v_L is arbitrary, there is a solution for all initial values. From the construction in Lemmas 2.4–2.8 it is obvious that the solution $v(s)$ is unique.

It is left to prove that the entropy curves are unique except for $g(u, v)$ in a set with measure zero and for a set with measure zero in the (v_L, v_R) plane which depends on the function $g(u(s), v)$.

Lemma 2.6 states that the entropy curve $y(\xi)$ from v_- to v_+, when $g(u_-, v) - sv$ is monotone decreasing in a neighbourhood of v_-, is unique. Luckily there are only a finite number of values of $v_- \in C(s_-)$ where $g(u_-, v) - sv$ is not monotone decreasing. This follows from (P). Therefore the entropy curves are unique except for a finite number of values of v_-.

Each value of v_R defines a $v(s)$ function by the construction in Lemma 2.4, Lemma 2.5, and Lemma 2.8. Let us stress the dependence of v_R by using the notation $v(s, v_R)$. For a finite number of v_+ values $v(s, v_R) = v_+$ for v_R in an interval . See Fig. 2.7 for an example. In Fig. 2.7a,

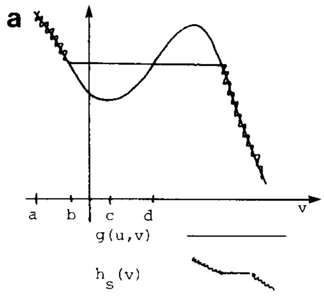

FIGURE 2.7a

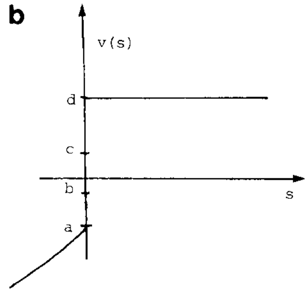

FIG. 2.7b. $v(s)$ for $v_R = d$.

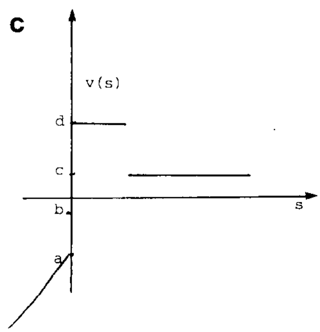

FIG. 2.7c. $v(s)$ for $v_R = c$.

$g(u(0), \cdot)$ and $h_0(\cdot)$ are shown. $u(s)$ is constant for $s \geqslant 0$. Then $v(0, v_R) = d$ for $b < v_R < d$. Figure 2.7b shows $v(s, d)$ and Fig. 2.7c shows $v(s, c)$. In both figures there is a jump from a to d with speed 0. If the connection from v_L to $v_- = c$ is not unique, then there is not uniqueness for v_R in an interval. It is therefore essential that none of the finite number of values of v_-, which do not have a unique connection to v_L, is connected to the finite number of values of v_+, where $v(s, v_R) = v_+$ for v_R in an interval. It is easily seen that this only happens for $g(u, v)$ in a set with measure zero. ∎

We now prove Theorem 2.1.

Proof of Theorem 2.1. The theorem is proved by induction. For $n = 1$ the theorem is the well-known result stated in Theorem 2.2. Theorem 2.3 is used for the induction step.

For $n = 2$ there may be several entropy curves, but the solution is obviously still unique. For $n > 2$ the nonuniqueness in entropy curves may lead to several solutions. If (u_-, u_+) is in a set with measure zero, then also the composed (u_-, v_-, u_+, v_+) is in a set with measure zero. Similarly if f is in a set with measure zero, then also the composed function (f, f_{i+1}) is in a set with measure zero. Therefore the solution is unique except for the flux function in a set with measure zero and for the initial value in a set with measure zero. ∎

3. Some Characteristics of the Solution

In this section we study some of the characteristics of the solution of triangular hyperbolic systems. First we show in an example that the generalized Lax entropy condition and the travelling wave entropy condition give different solutions. In the example it is easy to find a solution which satisfies the generalized Lax entropy condition and find out that the solution is unique. Furthermore we demonstrate that the solution does not depend continuously on the data. An example shows a flux function where there is a one-parameter family of solutions for the initial values in a set with positive measure. Finally, we give an example where all weak solutions have a shock to a disconnected branch of the Hugoniot locus. Thus no solution satisfies the Liu entropy condition.

For genuinely nonlinear and strictly hyperbolic systems the Lax inequalities are satisfied for local solutions (i.e., u_L and u_R close). In triangular hyperbolic systems the eigenvalues equal $\lambda^i = \partial f_i/\partial u_i$. Notice that the superscript does not indicate the order of the eigenvalue. Assume that there is a simple rarefaction solution in equations $1, ..., k-1$. Then there is a shock with speed s in equation k. This shock influences the solution in equations $k+1, ..., n$. Thus λ^i is larger or smaller than s on both sides of the shock for $i = 1, ..., k-1$. For $i = k$ the eigenvalues appear as in the scalar equation, i.e., $\lambda^k(u_+) \leqslant s \leqslant \lambda^k(u_-)$. According to the proof Lemma 2.8 it is easily seen that for $i > k$, $\lambda^i(u_-)$, $\lambda^i(u_+) < s$ or $\lambda^i(u_-)$, $\lambda^i(u_+) > s$ for u_- and u_+ close. This is according to the Lax entropy inequalities. However, if u_- and u_+ are not close, we may have $\lambda^i(u_-) \leqslant s \leqslant \lambda^i(u_+)$ of $\lambda^i(u_+) \leqslant s \leqslant \lambda^i(u_-)$. $\lambda^i(u_+) \leqslant s \leqslant \lambda^i(u_-)$ corresponds to the situation where the solution is not unique. In the second part of this section we show this in an example.

Let us first give an example where the generalized Lax entropy condition and the travelling wave entropy condition give different solutions. In the example $n = 2$. We consider one equation at a time.

$$f_1(u_1) = -u_1^2$$

and

$$u_1(x, 0) = \begin{cases} -1 & \text{for} \quad x < 0 \\ 1 & \text{for} \quad x > 0. \end{cases}$$

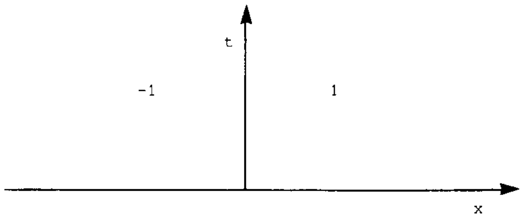

FIG. 3.1. $u_1(x, t)$.

See Fig. 3.1. $f_2(u_1, u_2)$ are defined by

$$f_2(u_1, u_2) = (u_1 + u_2 + 1)^2.$$

See Fig. 3.2. The initial condition is

$$u_2(x, 0) = -2.$$

Using the argument in the proof of Theorem 2.3 it is easy to find out that the solution which satisfies the travelling wave entropy condition is

$$u_2^{\mathrm{TW}}(x, t) = \begin{cases} -2 & \text{for } x/t < -4 \\ x/2t & \text{for } -4 < x/t < -2 \\ -1 & \text{for } -2 < x/t < 1 \\ -2 & \text{for } 1 < x/t \end{cases}$$

and that the solution satisfying the generalized Lax entropy condition reads

$$u_2^{\mathrm{GL}}(x, t) = \begin{cases} -2 & \text{for } x/t < -4 \\ x/2t & \text{for } -4 < x/t < 0 \\ -2 & \text{for } 0 < x/t. \end{cases}$$

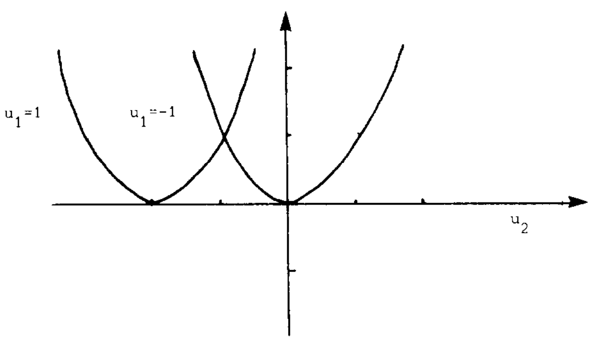

FIG. 3.2. $f_2(u_1, u_2)$.

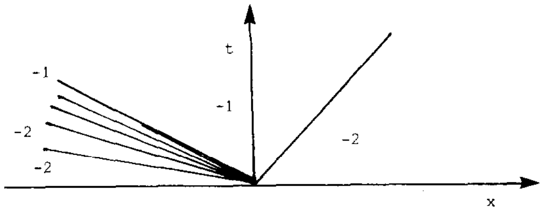

FIG. 3.3. $u_2(x, t)$, travelling wave solution.

See Figs. 3.3 and 3.4. It is easy to check that both these solutions are unique and do not satisfy the other entropy condtion. The solution of the Riemann problem of the scalar equation is monotone. This example shows that the solution of triangular hyperbolic systems is not monotone in general. $u_2(x, t) = -2$ is not a solution since the shock from $(-1, -2)$ to $(1, -2)$ with speed 0 does not satisfy the Rankine–Hugoniot condition.

The solution of the scalar equation depends continuously on the initial data; see Lucier [21] and Holden, Holden, and Høegh-Krohn [10, 11]. For the scalar equation the following theorem is valid.

THEOREM 3.1. *If f and g are Lipschitz continuous functions, u_0 and $v_0 \in BV(R)$, and u and v are the solutions of*

$$u_t + f(u) = 0 \qquad \text{for} \quad x \in R \text{ and } t > 0$$

$$u(x, 0) = u_0(x) \qquad \text{for} \quad x \in R$$

and

$$v_t + g(v)_x = 0 \qquad \text{for} \quad x \in R \text{ and } t > 0$$

$$v(x, 0) = v_0(x) \qquad \text{for} \quad x \in R,$$

then for any $t > 0$

$$\|u(\cdot, t) - v(\cdot, t)\|_{L_1} \leqslant \|u_0(x) - v_0(x)\|_{L_1} + \|f - g\|_{\text{Lip}} \min(|u_0|_{BV(R)}, |v_0|_{BV(R)}),$$

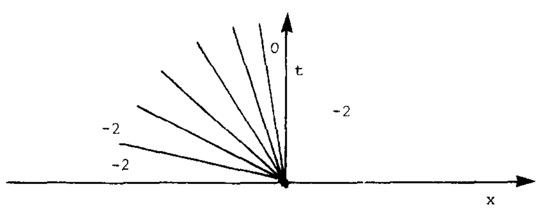

FIG. 3.4. $u(x, t)$, generalized Lax entropy condition solution.

where we have defined

$$\|g\|_{\text{Lip}} = \sup_{x \neq y} \left| \frac{g(x) - g(y)}{x - y} \right|.$$

In triangular hyperbolic systems the solution does not depend continuously on the data. This is connected with the nonuniqueness of the solution. In the following example we approach a point where the solution is not unique along different curves where the solution is unique. In the example some of the functions have discontinuous derivatives. The only reason for this is that it makes the example simpler. If the discontinuities are smoothed out, then the solutions still have the same characteristics.

We now consider an example with $n = 3$. $f_1(u_1)$ and $u(x, 0)$ are defined as in the previous example. The definition of f_2 is more complicated, namely

$$f_2(u_1, u_2) = \begin{cases} g_1(u_2) & \text{for} \quad u_1 < -1 \\ \frac{1}{2}(1 - u_1)\, g_1(u_2) + \frac{1}{2}(1 + u_1)\, g_2(u_2) & \text{for} \quad -1 < u_1 < 1 \\ g_2(u_2) & \text{for} \quad 1 < u_1, \end{cases}$$

where

$$g_1(u) = \begin{cases} |u| & \text{for} \quad u < 1 \\ 2 - u & \text{for} \quad u > 1 \end{cases}$$

and

$$g_2(u) = -2 - u.$$

See Fig. 3.5 for the definition of f_2. We use two different initial values in the Riemann problem. The initial values are

$$u_2^+(x, 0) = \begin{cases} -1 & \text{for} \quad x < 0 \\ -2 + \varepsilon & \text{for} \quad x > 0 \end{cases}$$

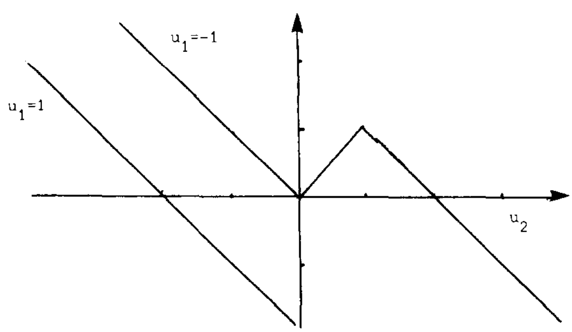

FIG. 3.5. $f_2(u_1, u_2)$.

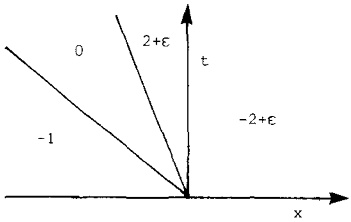

FIG. 3.6. $u_2^+(x, t)$.

respectively

$$u_2^-(x, 0) = \begin{cases} -1 & \text{for} \quad x < 0 \\ -2 - \varepsilon & \text{for} \quad x > 0, \end{cases}$$

for $\varepsilon > 0$. The exact solutions are

$$u_2^+(x, t) = \begin{cases} -1 & \text{for} \quad x/t < -1 \\ 0 & \text{for} \quad -1 < x/t < -\delta \\ 2 + \varepsilon & \text{for} \quad -\delta < x/t < 0 \\ -2 + \varepsilon & \text{for} \quad 0 < x/t \end{cases}$$

and

$$u_2^-(x, t) = \begin{cases} -1 & \text{for} \quad x/t < -1 \\ -\varepsilon & \text{for} \quad -1 < x/t < 0 \\ -2 - \varepsilon & \text{for} \quad 0 < x/t \end{cases}$$

for $\delta > 0$. δ depends on ε, and δ vanishes when ε vanishes. See Figs. 3.6 and 3.7. We see that when the right hand value approaches -2 then these two solutions become identical. But the entropy curves with speed 0 do not converge. This becomes more evident when we add the third equation

$$f_3(u_2, u_3) = \begin{cases} g_3(u_3) & \text{for} \quad u_2 < 0 \\ \frac{1}{2}(2 - u_2)\, g_3(u_3) + u_2\, g_4(u_3) & \text{for} \quad 0 < u_2 < 2 \\ g_4(u_3) & \text{for} \quad 2 < u_2, \end{cases}$$

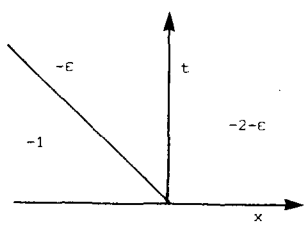

FIG. 3.7. $u_2^-(x, t)$.

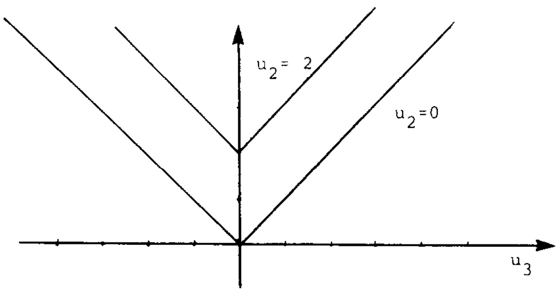

FIG. 3.8. $f_3(u_2, u_3)$.

where

$$g_3(u) = |u|$$

and

$$g_4(u) = |u| + 2.$$

See Fig. 3.8. The initial value is

$$u_3(x, 0) = \begin{cases} -1 & \text{for} \quad x < 0 \\ 1 & \text{for} \quad x > 0. \end{cases}$$

The solution depends on the initial value for u_2.

$$u_3^+(x, t) = \begin{cases} -1 & \text{for} \quad x/t < -1 \\ -2-\varepsilon & \text{for} \quad -1 < x/t < -\delta \\ 0 & \text{for} \quad -\delta < x/t < 0 \\ 2 & \text{for} \quad 0 < x/t < 1 \\ 1 & \text{for} \quad 1 < x/t \end{cases}$$

and

$$u_3^-(x, t) = \begin{cases} -1 & \text{for} \quad x/t < -1 \\ 0 & \text{for} \quad -1 < x/t < 1 \\ 1 & \text{for} \quad 1 < x/t. \end{cases}$$

See Fig. 3.9 and Fig. 3.10. When the right hand initial value for u_2 equals -2, there is a one-parameter family of entropy curves between the two entropy curves we find when the initial value approaches -2 from both sides. The corresponding solution for u_3 is changing from u_3^+ to u_3^-. The sector with value 0 is increasing and finally ends up as in u_3^-.

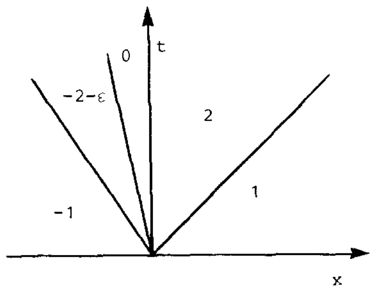

FIG. 3.9. $u_3^+(x, t)$.

We will then give an example with a flux function which gives a one-parameter of solutions for the initial values in a set with positive measure. The flux function and the initial data are only a minor modification of the previous example.

$$f_1^*(u_1) = \begin{cases} -u_1^2 & \text{for} \quad x < 1 \\ 2u_1 - 3 & \text{for} \quad x > 1 \end{cases}$$

and

$$u_1(x, 0) = \begin{cases} -1 & \text{for} \quad x < 0 \\ 2 & \text{for} \quad x > 0. \end{cases}$$

The solution is easily found to be

$$u_1^*(x, t) = \begin{cases} -1 & \text{for} \quad x/t < 0 \\ 1 & \text{for} \quad 0 < x/t < 2 \\ 2 & \text{for} \quad 2 < x/t. \end{cases}$$

See Figs. 3.11 and 3.12. The definition of f_2^* is

$$f_2^*(u_1, u_2) = \begin{cases} f_2(u_1, u_2) & \text{for} \quad u_1 < 1 \\ (2 - u_1) f_2(u_1, u_2) + (u_1 - 1) g_5(u_2) & \text{for} \quad 1 < u_1 < 2 \\ g_5(u_2) & \text{for} \quad 2 < u_1, \end{cases}$$

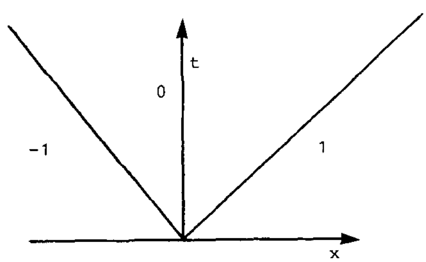

FIG. 3.10. $u_3^-(x, t)$.

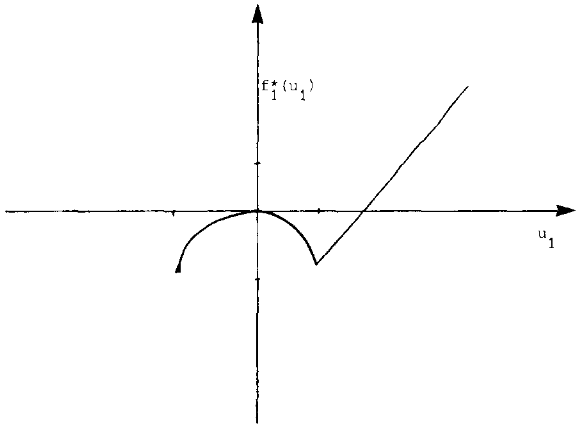

FIGURE 3.11.

where

$$g_5(u) = \begin{cases} 3|u+3|-3 & \text{for } u < -1 \\ 2-u & \text{for } u > -1. \end{cases}$$

See Fig. 3.13 for the definition of f_2^*. The initial value in this equation is

$$u_{2,a}^*(x,0) = \begin{cases} -1 & \text{for } x < 0 \\ a & \text{for } x > 0, \end{cases}$$

where $-3.2 < a < -0.8$. The solution is then

$$u_{2,a}^*(x,t) = \begin{cases} -1 & \text{for } x/t < -1 \\ 0 & \text{for } -1 < x/t < 0 \\ -2 & \text{for } 0 < x/t < s(a) \\ a & \text{for } s(a) < x/t, \end{cases}$$

where $s(a) = (f_2^*(2,-2) - f_2^*(2,a))/(-2-a)$. We see that $2 < s(a) < 3$. In the solution of $u_2^*(x,t)$ there is a one-parameter of entropy curves in the shock with speed 0 exactly as in the previous example. There is also a shock in $u_2^*(x,t)$ with speed 2, but $u_2^*(x,t)$ is equal to -2 on both sides. In the shock with speed 2 we have that $\partial f_2^*/\partial u_2$ is smaller than the speed of the shock before the shock and larger after the shock. Therefore the

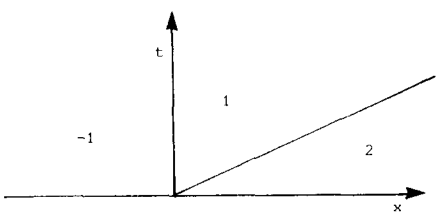

FIG. 3.12. $u_1^*(x,t)$.

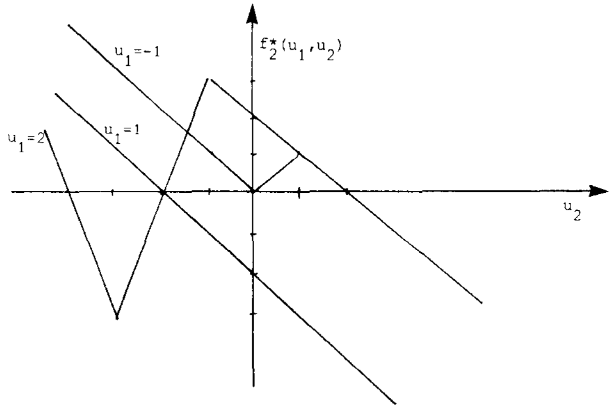

FIGURE 3.13

generalized Lax entropy condition is not satisfied. Furthermore, by modifying this example one can construct a system for which the generalized Lax entropy condition gives nonuniqueness.

In the third equation we chose $f_3^*(u_2, u_3)=f_3(u_2, u_3)$ and $u_3^*(x, 0)=u_3(x, 0)$. The solutions are $u_3^+(x, t)$, $u_3^-(x, t)$ both with $\varepsilon=0$, and a one-parameter family of solutions connecting these two end points. We may perturb the initial values and still get a one-parameter family of solutions. The flux function may not be perturbed since it is essential that $f_2^*(u_1, u_2)=0$ for $1 \leqslant u_1 \leqslant 2$ and $u_2=-2$. The example shows that there exist flux functions such that there is a one-parameter family of solutions for the initial values in a set with positive measure.

Finally, we give an example where all weak solutions have a shock to a disconnected branch of the Hugoniot locus, i.e., the graph

$$H(u_-)=\{u; s(u_+-u_-)=f(u_+)-f(u_-), s \in R\}$$

does not connect u_+ and u_-. Then the shock from u_- to u_+ is not admissible according to the Liu entropy condition; see [20]. Let $f_1(u_1)$ and $u_1(x, 0)$ be as in the first example. Then

$$u_1(x, t)=\begin{cases} -1 & \text{for} \quad x<0 \\ 1 & \text{for} \quad x>0. \end{cases}$$

Define $f_2^n(u_1, u_2)$ for $n>10$ as

$$f_2^n(u_1, u_2)=\begin{cases} u_1+|u_2| & \text{for} \quad |u_1| \geqslant 1/n \\ u_1+|u_2|+\sqrt{n}\,|u_1-1/n| & \text{for} \quad |u_1|<1/n. \end{cases}$$

See Fig. 3.14. Define

$$u_2(x, 0)=\begin{cases} -3 & \text{for} \quad x<0 \\ 1 & \text{for} \quad x>0. \end{cases}$$

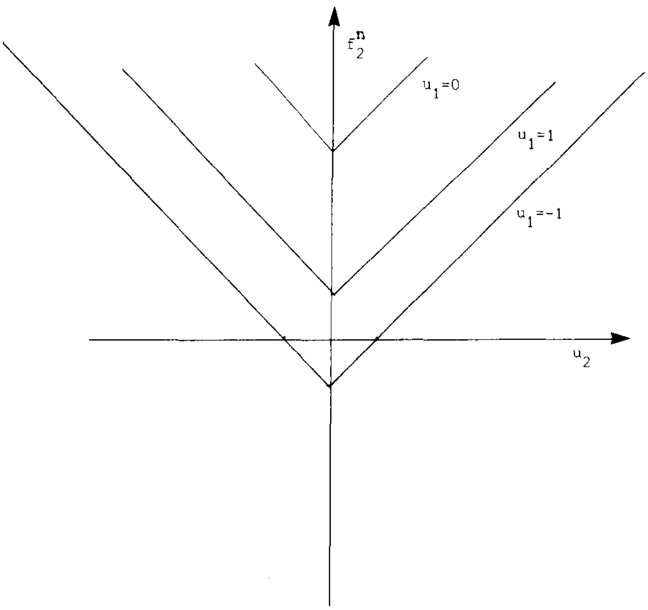

FIGURE 3.14

The solution is then for all n values,

$$u_2(x, t) = \begin{cases} -3 & \text{for } x < 0 \\ 1 & \text{for } x > 0. \end{cases}$$

It is easily seen that for n sufficiently large the shock with speed zero is to a disconnected branch of the Hugoniot locus since the Hugoniot locus cannot pass the line $u_1 = 0$.

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