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Dodgson condensation: The historical and mathematical development of an experimental method ☆

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Abstract

Dodgson's condensation method has become a powerful tool in the automation of determinant evaluations. In this expository paper I describe its 19th century roots and the major steps on the path that began in the 20th century when the iteration of an identity derived by Dodgson first was studied, including its role in the discovery of the alternating sign matrix conjecture, the evaluation of an important 19th century determinant in partition theory as well as a combinatorial proof of it. I then discuss additional developments that have led the way to its use in modern experimental mathematics.

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1. Introduction

Charles L. Dodgson (Lewis Carroll, 1832–1898) made important mathematical discoveries many of which were properly recognized for the first time only in the second half of the last century. Of these it is his condensation method that arguably has had the greatest influence on subsequent mathematical discoveries.

☆ This paper is an expanded version of a talk given at Concordia University on 29 July 2007 commemorating the 175th birthday of Charles L. Dodgson.

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Originating independently from Desanot's 1819 "law of extensible minors" which Desanot proved for $n \times n$ matrices when $n \leqslant 6$, and proved in the general case by Jacobi in 1833, Dodgson's elegant condensation technique was announced in 1866 in his paper delivered to a meeting of the Royal Society by his friend and tutor, Bartholomew Price, professor of natural philosophy at Oxford [12,22,16,13]. The following year it appeared in three appendices to Dodgson's book, *An Elementary Theory of Determinants* [14]. In 1883, Muir published a proof of his law of extensible minors which is a generalized form of Desanot's law [21].

In this paper I will refer to the work of Desanot and of Muir as the Desanot/Muir law of extensible minors (DSM). Dodgson's method is both a special case of DSM and of Jacobi's 1833 adjoint matrix theorem, sometimes referred to as the Desanot/Jacobi adjoint matrix theorem (DJ) [23]. But Dodgson acknowledged neither Desanot or Jacobi by name in the published talk printed in the proceedings of the Royal Society nor in his book. However, he had written earlier to the Oxford mathematician and author of an 1851 book on determinants, Spottiswoode [26], and in a letter dated 2 April 1866 Spottiswoode replied

> "The Theorem upon which it [condensation] is founded is, as you are doubtless aware, known, but the application of it is, as far as I am aware, quite original." [2]

The theorem is DJ and Dodgson probably read it in Hoüel's 1861 translation of Baltzer's 1857 book, *Theorie und Anwendung der Determinanten* [7]. Dodgson used Jacobi's theorem to establish the validity of his own method. He wrote

> "I now proceed to give a proof of the validity of this process [condensation], deduced from a well-known theorem in determinants… The theorem referred to is the following: 'if the determinant of a block $= R$, the determinant of any minor of the mth degree of the adjugate block is the product of R^{m-1} and the coefficient which, in R, multiplies the determinant of the corresponding minor.' " [2]

For a given m by m minor, the *corresponding* m by m minor in the adjugate block is the one in which each entry has the same position as in the given minor. The *adjugate* block is a new block each of whose entries is the determinant of the complemental minor of the corresponding entry in the given block. For a given m by m minor, the entries of the *complemental* m by m minor are those that are not in any row or column of the given minor [1].

2. Dodgson condensation

Dodgson described his method this way: for any n by n block, i.e. matrix,

> "Compute the determinant of every 2 by 2 minor consisting of four adjacent terms. These values become the terms of a second block having $n - 1$ rows and $n - 1$ columns. [In modern usage, a *minor* is a determinant of a square sub-matrix formed from consecutive rows and columns.]
> Condense the second block in the same way, dividing each term, when found, by the corresponding term in the interior of the first block (the block that remains when the first and last rows and columns are erased).
> Repeat this process until the block is condensed to a single term, which will be the required value. Note that in condensing any block r in the series, the terms found must be divided by the corresponding terms in the interior of the $r - 1$th block." [2]

MATRIX 1

-2	-1	-1	-4
-1	-2	-1	-6
-1	-1	2	4
2	1	-3	-8

-2	-1		-1	-1		-1	-4
-1	-2		-2	-1		-1	-6
-1	-2		-2	-1		-1	-6
-1	-1		-1	2		2	4
-1	-1		-1	2		2	4
2	1		1	-3		-3	-8

MATRIX 2

3	-1	2
-1	-5	8
1	1	-4

Interior of matrix 1 is

-2	-1
-1	2

Interior of matrix 2 is -5

3	-1	2
-1	-5	8
1	1	-4

MATRIX 3

3	-1		-1	2
-1	-5		-5	8
-1	-5		-5	8
1	1		1	-4

-16	2
4	12

Dividing the entries of the single entry form of matrix 3 by the entries in the interior of matrix 1 produces the matrix

8	-2
-4	6

Dividing each entry of this final matrix by the interior of matrix 2 (equivalently, dividing its determinant 40 by the interior of matrix 2) gives -8, the determinant of the original 4 by 4 matrix.

Fig. 1. Dodgson's condensation method.

Because the set of k by k connected minors can be arranged naturally as an $n-k+1$ by $n-k+1$ array, the set of new minors computed at each stage can be viewed as a matrix one dimension lower than the one formed in the previous stage. So as the algorithm proceeds, the original matrix "condenses" until the required determinant is the single entry in a 1 by 1 matrix. Fig. 1 illustrates his method when $n=4$.

Dodgson understood that his method had a fatal defect, i.e. the determinant of any interior matrix cannot be 0. Although some remedies like row/column exchanges (permutations) can be effective in eliminating the defect, they may not always work. But the main advantage is now obvious: unlike the standard Laplacian expansion, nowhere does a determinant of order greater than two have to be computed. The algorithm implies that the determinant of a square matrix is a rational function of all its connected minors of any two consecutive sizes. Commenting on the "fatal defect", Bressoud and Propp wrote,

> "Although the use of division may seem like a liability, it actually provides a useful form of error checking for hand calculations with integer matrices: when the algorithm is performed properly…all the entries of all the intervening matrices are integers, so that when a division fails to come out evenly, one can be sure that a mistake has been made somewhere. The method is also useful for computer calculations, especially since it can be executed in parallel by many processors." [9]

$$\mathbf{A_{n-2}(2,2)\ A_n(1,1) \;=\; A_{n-1}(1,1)\ A_{n-1}(2,2) \;-\; A_{n-1}(1,2)\ A_{n-1}(2,1) \qquad DDI}$$

For an n by n matrix A, let $A_r(i,j)$ denote the r by r minor consisting of r contiguous rows and columns of A, beginning with row i, column j. Note that $A_n(1,1)$ = det A, $A_{n-2}(2,2)$ is the central minor; $A_{n-1}(1,1)$, $A_{n-1}(2,2)$, $A_{n-1}(1,2)$, $A_{n-1}(2,1)$ are the northwest, southeast, northeast, and southwest minors, respectively.

$\mathbf{A_n(1,1)}$	$\mathbf{A_{n-2}(2,2)}$	$\mathbf{A_{n-1}(1,1)}$	$\mathbf{A_{n-1}(2,2)}$	$\mathbf{A_{n-1}(1,2)}$	$\mathbf{A_{n-1}(2,1)}$
-2 -1 -1 -4	-2 -1	-2 -1 -1	-2 -1 -6	-1 -1 -4	-1 -2 -1
-1 -2 -1 -6	-1 2	-1 -2 -1	-1 2 4	-2 -1 -6	-1 -1 2
-1 -1 2 4		-1 -1 2	1 -3 -8	-1 2 4	2 1 -3
2 1 -3 -8					

Iterating DDI:	$\mathbf{A_{n-2}(2,2)}$	$\mathbf{A_{n-1}(1,1)}$	$\mathbf{A_{n-1}(2,2)}$	$\mathbf{A_{n-1}(1,2)}$	$\mathbf{A_{n-1}(2,1)}$
-2 -1 -1	-2	-2 -1	-2 -1	-1 -1	-1 -2
-1 -2 -1		-1 -2	-1 -2	-2 -1	-1 -1
-1 -1 2					
-2 -1 -6	2	-2 -1	2 4	-1 -6	-1 2
-1 2 4		-1 2	-3 -8	2 4	1 -3
1 -3 -8					
-1 -1 -4	-1	-1 -1	-1 -6	-1 -4	-2 -1
-2 -1 -6		-2 -1	2 4	-1 -6	-1 2
-1 2 4					
-1 -2 -1	-1	-1 -2	-1 2	-2 -1	-1 -1
-1 -1 2		-1 -1	1 -3	-1 2	2 1
2 1 -3					

Computing all the 2 by 2 minors, the determinant of the first 3 by 3 minor is -16/-2 = 8; that of the second 3 by 3 minor is 12/2 = 6; that of the third 3 by 3 minor is 2/-1 = -2; and that of the fourth 3 by 3 minor is 4/1 = -4. So the determinant of the original 4 by 4 matrix is 40/-5 = -8

Fig. 2. Dodgson's determinantal identity.

3. Alternating sign matrices

Dodgson's condensation method emerged from relative obscurity in 1986 – it previously had been used in some linear algebra texts published early in the 20th century – when Robbins and Rumsey studied the iteration of the identity that we call DDI, to which Dodgson condensation is implicitly related [25].

In Fig. 2 the same 4 by 4 matrix is "condensed" using the recurrence DDI above.
Note that A_n is itself a connected minor and that condensation is used repeatedly to compute all the connected minors of A_n. The determinant of a 0 by 0 matrix is assigned the value 1, so this equation reduces to the standard expression for computing a 2 by 2 determinant.

If we compare the computations in the two closely related methods on the same example we see that although DDI requires more calculations, they do not increase its computational complexity, and the recurrence DDI is a better algorithm than plain Dodgson condensation.

The study of DDI led Robbins and Rumsey to their discovery of the alternating sign matrix conjecture (ASM). An alternating sign matrix is a generalized permutation matrix, where every entry is 0, ± 1 in each row and column; the nonzero entries alternate starting and ending with 1, and

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

Fig. 3. The ASM conjecture.

they sum to 1. For the general 3 by 3 matrix there are seven ASMs, six of which are permutation matrices, and the seventh is the one below:

$$\begin{matrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{matrix}$$

The general 4 by 4 matrix has 42 ASMs of which 24 are permutation matrices. Trying to answer the natural question, how many ASMs are there? They found a sequence that they had never seen before,

$$A_n = 1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, \ldots$$

Dividing the ASMs into classes suitably, and noticing many patterns in the arrangement of the members of these classes, they were able to express the number of the set of n by n ASMs as the conjecture given in Fig. 3.

Note the correspondence of the terms in the sequence given by A_n with the conjectured terms, e.g. when $n = 4$, we have $(1/24)(1/5)(7)(720) = 42$, the fourth term in A_n.

In 1995 Zeilberger's proof of the ASM conjecture was accepted by the 88 referees and one computer of his paper which he had first submitted three years earlier [8,27].

4. Plane partitions

The number of ASMs is connected to a problem about plane partitions, i.e. counting the number of descending plane partitions [6]. A plane partition is a rectangular array of nonnegative integers with the property that all rows and columns are non-increasing. For example, the six plane partitions of 3 are

3	2 1	2	1 1 1	1 1	1
		1		1	1
					1

But if the integer whose partitions we want is very large, the number of its partitions is extremely large. To find them we need a generating function, a power series, whose coefficients give the number of plane partitions of the integer. A generating function also provides recursive formulas that suit algorithms implemented on a computer.

MacMahon discovered that function in 1897, and published it in 1912 [18,19]. Seventy-five years later Gessell and Viennot were able to simplify his proof and find an equivalent determinant for their generating function by considering plane partitions as configurations of nonintersecting paths in a two dimensional integral lattice [15]. Using a formula given by Krattenthaler in 1990 the determinant can be evaluated [17]. General determinant evaluation formulas are not new – they go back to the 18th century. A common derivation of the important ones from the 18th and 19th centuries and a nice discussion of them can be found in [11].

Polynomials play a central role in the study of partition identities. When one is counting, a generating function provides a way of keeping track of the number of items being counted. For

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q = \prod_{i=1}^{m} \frac{1-q^{n+i}}{1-q^i}$$

Fig. 4. Gaussian binomial (or q-binomial) coefficients for positive integers m, n.

example, in the finite binomial theorem $\binom{n}{k}$ counts the number of ways $n-k$ x's and k y's can be arranged:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

We can introduce a parameter q to do this counting by setting $yx = qxy$. By letting $xq = qx$ and $yq = qy$, we can put all the qs together. Then the *q-binomial coefficients*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q$$

can be defined by

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k} y^k. \tag{1}$$

We see that the q-binomial coefficients are polynomials in q with integer coefficients. Since

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \tag{2}$$

and

$$(x+y)^{n+1} = (x+y)^n (x+y), \tag{3}$$

we can rewrite (1) and substitute $q^k xy^k$ for $y^k x$ to obtain a *q-analog* of (2), the q-binomial coefficients,

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q q^k + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q.$$

These polynomials are also known as Gaussian binomial coefficients and are EXPRESSED more generally in Fig. 4. In chapter 10 of their book, Andrews and his colleagues extend this example in interesting ways [5].
Setting $q = 1$ yields the ordinary binomial coefficients.

5. Dodgson condensation and combinatorics

Zeilberger published a combinatorial proof of DDI. He called his theorem, "Dodgson's Determinant Evaluation Rule Proved by Two-Timing Men and Women" [29]. Several aspects of this proof are important. It is bijective and the accompanying Maple package contains the programs that implement the main mappings. In enumerative combinatorics bijective proofs give the *number* of configurations in integer partition identities. The proof type itself resembles closely the design principles used to construct efficient algorithms. Generally, the difference between an algebraic and a combinatorial approach to a proof is that the former begins by showing that both sides of the algebraic identity are invariant. The latter begins by translating the identity into an equation for two sums and then giving an explicit bijection to prove the equation. The bijection itself is valuable because its properties can point the way to new results.

Zeilberger's proof inspired Brualdi and his student, Adam Berliner, in 2006 to extend Zeilberger's proof to prove what they call the Dodgson/Muir Combinatorial Identity. Recall that this identity, proved by Muir in 1883 that he called the law of extensible minors in determinants, includes Dodgson condensation as a special case [21]. Their theorem in Fig. 5 asserts that a homogeneous determinantal identity for the minors of a matrix remains valid when all the index sets are enlarged by the same disjoint index set [10].

In 1996, Zeilberger algebraically proved that a q-analog of an important determinant evaluation concerning the enumeration of plane partitions first developed by MacMahon in section X, Chapter 1 of [20] is a direct result of DDI. There MacMahon obtains an expression in determinant form of the inner lattice function on which Zeilberger comments, "Of course, MacMahon for expository reasons, only presents the $n = 4$ case, but his method is obviously general." (Private communication).

Zeilberger gave as the title of his paper, "Reverend Charles to the aid of Major Percy and Fields-Medalist Enrico" [28].

The proof itself is very short and illustrates a standard method used in combinatorial proofs that essentially generalizes proof by induction. We check that both sides of the equation we want to prove satisfy the same recurrence and the same boundary conditions. Since each side is uniquely determined by the boundary values and the recurrence, the two sides are equal.

A 1997 paper with Amdeberhan (where Zeilberger used the alias Shalosh B. Ekhad) is in the same spirit and form as the previous paper. In it they prove the determinant identity in Fig. 7 conjectured by Greg Kuperberg and Jim Propp follows immediately from DDI [3].

$$\det A \det A[\{k+1,\dots,n\},\{k+1,\dots,n\}]^{k-1} = \sum_{\sigma\in S_k}(-1)^{\iota(\sigma)}\prod_{j=1}^{k} \det A[\{j, k+1,\dots,n\},\{\sigma(j), k+1,\dots,n\}]$$

where A is a matrix of order n, $\iota(\sigma)$ is the number of inversions of the permutation σ, S_k is the set of all permutations of {1, 2, ..., k}. Dodgson condensation is the special case when k = 2.

Fig. 5. Dodgson/Muir determinantal identity.

$$\det\left[\binom{a+i}{b+j}_{1\le i,j\le n}\right]_q = \frac{(a+n)!!(n-1)!!(a-b-1)!!b!!}{(a)!!(a-b+n-1)!!(b+n)!!}$$

where n!! = 1!2!...n!.

Fig. 6. q-Analog of MacMahon's determinant evaluation.

$$\det\left[\binom{i+j}{i}\binom{2n-i-j}{n-i}_{0\leqslant i,j\leqslant n}\right]_q=\frac{(2n+1)!^{n+1}}{(2n+1)!!},$$

where $n!!$ is defined as in Fig. 6

Fig. 7. Kuperberg/Propp determinantal identity.

6. Experimental mathematics and DDI

In 2001 Amdeberhan and Zeilberger used DDI to prove 15 explicit determinant evaluations that were conjectured and then proved using computer-assisted methods. The Maple package, *Lewis*, accompanying their paper automates a few of them. They write

> “We believe that in many cases, (*Lewis*) should…be useful, by extending the *ansatz* to a larger class, that for us humans looks messy, but that computers won’t mind.” [4]

1819	P. Desanot	Law of extensible minors
1833	C. Jacobi	Adjoint matrix theorem
1866	C.L. Dodgson	Condensation method
1883	T. Muir	Generalized law of extensible minors
1916	P.A. MacMahon	Determinant evaluation method
1986	D. Robbins and H. Rumsey	Alternating Sign Matrix Conjecture (ASM);Dodgson’s determinantal identity (DDI)
1996	D. Zeilberger	Proof of the ASM
1996	D. Zeilberger	Proof of a q-analog of MacMahon’s determinant method
1997	D. Zeilberger	Combinatorial proof of Dodgson’s determinantal identity
1997	T. Amdeberhan & S.B. Ekhad	Proof of a new determinantal identity using DDI
2001	T. Amdeberhan & D. Zeilberger	Ansatz-based automated proofs
2003	D. Zeilberger	New research methodology
2005	D. Robbins	DDI applied to floating point matrices
2006	A. Berliner & R. Brualdi	Combinatorial proof of the Dodgson/Muir determinantal identity

Fig. 8. Time line.

Automated theorem proving can be divided into two broad types of which the logic-based type is better known. The *ansatz*-based type considers objects that belong to a well-defined algebraic class that has canonical or normal forms. Here, for the first time, they speak of DDI in the context of computer assisted proofs of determinant identities as a *new paradigm*.

Finally, in a paper appearing in 2003, Zeilberger shows that if you are solving a problem that involves evaluating a determinant, if you can guess the required result correctly, then DDI permits an inductive proof of that determinant evaluation. And even the determinant evaluation itself can be proved using DDI. He also discusses where this paper, the three previous ones, and his bijective proof of DDI are leading. He writes

> "[W]e need methodologies for creating new algorithms that will enable computers to discover, and *prove*, new results without knowing, beforehand, whether it will succeed, but with a fair chance that it will." [30]

Zeilberger's goal is a research methodology to make the proof of determinant identities *completely* automatic using software that Maple or Mathematica provides.

In another extension of Dodgson condensation, Robbins, in perhaps his last paper written in 2003 and published in 2005, described a non-Archimedean approximate form of the method and stated this conjecture which he checked in billions of cases:

> "Suppose a determinant is computed with approximate n-digit floating point Dodgson condensation. If the condensation error for the computation is e, then, after conversion to fixed point, the result will be correct mod p^{n-e}." [24]

Recall that each application of DDI requires a division by a previously computed det C, the central minor. When computing a given minor if the maximum of the exponents of all the divisors of det C is e, then Robbins defines the *condensation error* of that minor to be e. The matrix entries are n-digit floating point integers, i.e. each is a pair (a, e), where a is invertible mod p^n and e is an integer $\geqslant 0$; a and e play the roles of the mantissa and exponent in ordinary floating point arithmetic.

7. Conclusion

Dodgson condensation has shown itself to be an extraordinarily fruitful concept and implemented as DDI, a widely applicable computer algorithm. The time line in Fig. 8 tracks its almost 200 year history.

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