Discrete Mathematics 74 (1989) 331–332
North-Holland

NOTE

CONSTRUCTING 3-DESIGNS FROM SPREADS AND LINES

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Received 28 September 1987

Using the elements of a partial spread of PG(d, q) as points, a class of 3-designs is constructed.

1. Introduction

A t-(v, K, Λ) *design* is an incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ satisfying the following conditions:

(i) $\mathcal{P}$ has v elements, which we call *points*.

(ii) Any element of $\mathcal{B}$ (which is called a *block*) is incident with precisely k points, being $k \in K$.

(iii) The number of blocks through any t points is an element of Λ. If $|K| = |\Lambda| = 1$, say $K = \{k\}$ and $\Lambda = \{\lambda\}$, then $\mathcal{D}$ is also called a t-(v, k, λ) design.

A *partial t-spread* [5, 3] of a finite projective space $\boldsymbol{P} = \mathrm{PG}(d, q)$ is a set $\mathcal{S}$ of mutually skew t-dimensional subspaces. The partial t-spread $\mathcal{S}$ is said to be a *t-spread* if any point of $\boldsymbol{P}$ is on (exactly) one element of $\mathcal{S}$. It is well-known that PG(d, q) has a t-spread if and only if $t + 1$ divides $d + 1$ (cf. [4] or [2], II,7.4.2).

By Θ_s we denote the number of points in an s-dimensional subspace of a projective space of order q, this is $\Theta_s = q^s + q^{s-1} + \cdots + q + 1$.

Many connections between spreads and finite structures can be found in [1].

In this note we shall construct a class of 3-(v, K, Λ) designs from partial t-spreads in PG(d, q). In an important special case we will get (ordinary) 3-(v, k, λ) designs.

2. The construction

Let $\mathcal{S}$ be a partial t-spread in $\boldsymbol{P} = \mathrm{PG}(d, q)$. We define the design $\boldsymbol{D} = \boldsymbol{D}(\mathcal{S})$ as follows.

The *points* of $\boldsymbol{D}$ are the elements of $\mathcal{S}$.

The *blocks* of $\boldsymbol{D}$ are those lines of $\boldsymbol{P}$ which are not contained in any element of $\mathcal{S}$ but intersect at least three elements of $\mathcal{S}$.

0012-365X/89/$3.50

A point $\mathcal{U}$ (that is an element $\mathcal{U} \in \mathcal{S}$) is *incident* with a block ℓ if and only if $\mathcal{U} \cap \ell \neq \emptyset$.

The following theorem is easy to prove.

Theorem. *$\boldsymbol{D}$ is a 3-(v, K, Λ) design with*

$$v = |\mathcal{S}|, \qquad K \subseteq \{3, 4, \ldots, q+1\}, \qquad \Lambda \subseteq \{\Theta_{-1}, \Theta_0, \Theta_1, \ldots, \Theta_t\}.$$

Proof. We have only to show that any three points of $\boldsymbol{D}$ are incident with $\Theta_{-1}, \Theta_0, \Theta_1, \ldots, \Theta_t$ blocks. In order to prove this, let $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{U}_3$ be three distinct elements of $\mathcal{S}$. Let $\mathcal{U}_3$ intersect the subspace $\langle \mathcal{U}_1, \mathcal{U}_2 \rangle$ in a subspace $\mathcal{U}'$ of dimension u $(-1 \leqslant u \leqslant t)$. Since through any point of $\mathcal{U}'$ there is exactly one line which intersects $\mathcal{U}_1$ and $\mathcal{U}_2$, there are exactly Θ_u transversal lines of $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{U}_3$.

Hence the theorem is proved completely. □

Corollary 1. *If $\mathcal{S}$ is a (total) t-spread of $\boldsymbol{P} = \mathrm{PG}(d, q)$, then $\boldsymbol{D}(\mathcal{S})$ is a 3-$(|\mathcal{S}|, q+1, \Lambda)$ design with $\Lambda \subseteq \{\Theta_{-1}, \Theta_0, \ldots, \Theta_t\}$.*

Now we consider the smallest possible dimension.

Corollary 2. *If $\mathcal{S}$ is a partial t-spread of $\boldsymbol{P} = \mathrm{PG}(2t+1, q)$, then $\boldsymbol{D}(\mathcal{S})$ is a 3-$(|\mathcal{S}|, K, \Theta_t)$ design with $K \subseteq \{3, 4, \ldots, q+1\}$.*

The proof is immediate, since any two elements of $\mathcal{S}$ generate the whole space $\boldsymbol{P}$.

Corollary 3. *If $\mathcal{S}$ is a (total) t-spread of $\boldsymbol{P} = \mathrm{PG}(2t+1, q)$, then $\boldsymbol{D}(\mathcal{S})$ is a 3-$(q^{t+1}+1, q+1, \Theta_t)$ design.*

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