The stability of a cubic type functional equation with the fixed point alternative

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Abstract

In this note we investigate the generalized Hyers–Ulam–Rassias stability for the new cubic type functional equation

\[ f(x + y + 2z) + f(x + y - 2z) + f(2x) + f(2y) = 2[f(x + y) + 2f(x + z) + 2f(y + z) + 2f(y - z) - f(x - z)] \]

by using the fixed point alternative. The first systematic study of fixed point theorems in nonlinear analysis is due to G. Isac and Th.M. Rassias [Internat. J. Math. Math. Sci. 19 (1996) 219–228].

Keywords: Stability; Cubic function; Fixed point alternative

1. Introduction

In 1940, S.M. Ulam [23] proposed the following question concerning the stability of group homomorphisms: Under what condition does there is an additive mapping near an approximately additive mapping between a group and a metric group?

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In the next year, D.H. Hyers [7] answers the problem of Ulam under the assumption that the groups are Banach spaces. A generalized version of the theorem of Hyers for approximately linear mappings was given by Th.M. Rassias [17]. Since then, the stability problems of various functional equation have been extensively investigated by a number of authors (for instance, [1–4,6,8,9,12,18–22]).

By regarding a large influence of S.M. Ulam, D.H. Hyers and Th.M. Rassias on the investigation of stability problems of functional equations the stability phenomenon that was introduced and proved by Th.M. Rassias [17] in the year 1978 is called the Hyers–Ulam–Rassias stability.

Consider the functional equation
\[ f( x + y) + f( x - y) = 2f(x) + 2f(y). \]
The quadratic function \( f(x) = cx^2 \) is a solution of this functional equation, and so one usually is said the above functional equation to be quadratic [1,5,11,13].

The Hyers–Ulam stability problem of the quadratic functional equation was first proved by F. Skof [22] for functions between a normed space and a Banach space. Afterwards, her result was extended by P.W. Cholewa [4] and S. Czerwik [5].

The cubic function \( f(x) = cx^3 \) satisfies the functional equation
\[ f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x). \] (1.1)

Hence, throughout this note, we promise that Eq. (1.1) is called a cubic functional equation and every solution of Eq. (1.1) is said to be a cubic function. The stability result of Eq. (1.1) was obtained by K.-W. Jun and H.-M. Kim [10] (see also [16]).

Now we introduce the new cubic type functional equation, that is,
\[ f(x + y + 2z) + f(x + y - 2z) + f(2x) + f(2y) = 2f(x + y) + 2f(x + z) + 2f(x - z) + 2f(y + z) + 2f(y - z) \] (1.2) satisfying the following algebraic identity:
\[ (x + y + 2z)^3 + (x + y - 2z)^3 + (2x)^3 + (2y)^3 = 2[(x + y)^3 + 2(x + z)^3 + 2(y + z)^3 + 2(x - z)^3 + 2(y - z)^3]. \]

The main goal of this note is to offer the generalized Hyers–Ulam–Rassias stability result for the functional equation (1.2) by using the fixed point alternative [14] as in [15].

2. Stability of Eq. (1.2)

For completeness, we will first present solutions of the functional equation (1.2).

Lemma 2.1. Let \( X \) and \( Y \) be real vector spaces. A function \( f : X \to Y \) satisfies the functional equation (1.2) if and only if \( f \) is cubic.

Proof. \((\Rightarrow)\) Substituting \( x = y = z = 0 \) in (1.2) yields \( f(0) = 0 \). Putting \( x = -z \) and \( y = z \) in (1.2) and then replacing \( z \) by \( z/2 \) in the result, we get
\[ f(z) + f(-z) = 0 \]

which implies that \( f \) is odd.

Letting \( y = 0 \) in (1.2) and employing the fact that \( f \) is odd, we obtain that
\[ f(x + 2z) + f(x - 2z) + 6f(x) = 4f(x + z) + 4f(x - z). \]  \hspace{1cm} (2.1)

Setting \( y = 0 = z \) in (1.2) gives the identity \( f(2x) = 8f(x) \), and so we replace \( x \) by \( 2x \) in (2.1) to get
\[ f(2x + z) + f(2x - z) = 2f(x + z) + 2f(x - z) + 12f(x) \]  \hspace{1cm} (2.2)

for all \( x, z \in X \) which implies that \( f \) is cubic.

\( (\Leftarrow) \) Suppose that \( f \) is cubic. Putting \( x = 0 = y \) in (1.1), we get \( f(0) = 0 \). Setting \( x = 0 \) in (1.1) yields \( f(y) = -f(y) \) and by letting \( y = 0 \) in (1.1) and \( y = x \) in (1.1), we obtain that \( f(2x) = 8f(x) \) and \( f(3x) = 27f(x) \), respectively.

We substitute \( y := x + y \) in (1.1) and then \( y := x - y \) in (1.1) to obtain that
\[ f(3x + y) + f(x - y) = 2f(2x + y) - 2f(y) + 12f(x) \]  \hspace{1cm} (2.3)

and
\[ f(3x - y) + f(x + y) = 2f(2x - y) + 2f(y) + 12f(x). \]  \hspace{1cm} (2.4)

Adding (2.3) to (2.4) and then using (1.1), we see that
\[ f(3x + y) + f(3x - y) = 3f(x + y) + 3f(x - y) + 48f(x) \]  \hspace{1cm} (2.5)

for all \( x, y \in X \).

Replacing \( x \) and \( y \) by \( x + y \) and \( x - y \) in (2.5), respectively, we have
\[ f(4x + 2y) + f(2x + 4y) = 3f(2x) + 3f(2y) + 48f(x + y), \]
which, in view of the identity \( f(2x) = 8f(x) \), reduces to
\[ f(2x + y) + f(x + 2y) = 3f(x) + 3f(y) + 6f(x + y). \]  \hspace{1cm} (2.6)

Putting \( x := x + 3y \) and \( y := x - 3y \) in (2.6) and then using the identities \( f(2x) = 8f(x) \), \( f(3x) = 27f(x) \), we have
\[ 9f(x + y) + 9f(x - y) = f(x + 3y) + f(x - 3y) + 16f(x). \]  \hspace{1cm} (2.7)

Let us interchange \( x \) with \( y \) in (2.7) to get the identity
\[ 9f(x + y) - 9f(x - y) = f(3x + y) - f(3x - y) + 16f(y). \]  \hspace{1cm} (2.8)

Then, by adding (2.7) to (2.8), we lead to
\[ 18f(x + y) = f(x + 3y) + f(x - 3y) + f(3x + y) - f(3x - y) + 16f(x) + 16f(y). \]  \hspace{1cm} (2.9)

On the other hand, if we interchange \( x \) with \( y \) in (2.5), we get
\[ f(x + 3y) - f(x - 3y) = 3f(x + y) - 3f(x - y) + 48f(y). \]  \hspace{1cm} (2.10)

Hence, according to (2.5) and (2.10), we obtain that
\[ 6f(x + y) = f(3x + y) + f(3x - y) + f(x + 3y) - f(x - 3y) - 48f(x) - 48f(y). \]  
(2.11)

Now, by adding (2.9) and (2.11), we arrive at
\[ f(x + 3y) + f(3x + y) = 12f(x + y) + 16f(x) + 16f(y). \]  
(2.12)

Using (2.5), we have
\[ 16f(3x + z) + 16f(3x - z) + 16f(3y + z) + 16f(3y - z) = 48f(x + z) + 48f(x - z) + 768f(x) + 48f(y + z) + 48f(y - z) + 768f(y). \]  
(2.13)

Also, putting \( x := 3x + z \) and \( y := 3y + z \) in (2.12) and using (2.5), we deduce that
\[ 16f(3x + z) + 16f(3y + z) + 16f(3x - z) + 16f(3y - z) = f(3x + 9y + 4z) + f(9x + 3y + 4z) - 12f(3x + 3y + 2z) + f(3x + 9y - 4z) + f(9x + 3y - 4z) - 12f(3x + 3y - 2z) = 3f(x + 3y + 4z) + 3f(x + 3y - 4z) + 48f(x + 3y) + 3f(3x + y + 4z) + 3f(3x + y - 4z) + 48f(3x + y) - 36f(x + y + 2z) - 36f(x + y - 2z) - 576f(x) = 3f(3x + y + 4z) + 3f(3x + y - 4z) + 48f(3x + y)
\]
\[ + 3f(x + 3y + 4z) + 3f(x + 3y - 4z) + 48f(x + 3y)
\]
\[ = 48f(x + z) + 48f(x - z) + 768f(x) + 48f(y + z) + 48f(y - z) + 768f(y) + 36f(x + y + 2z) + 36f(x + y - 2z) + 576f(x + y). \]  
(2.14)

On account of (2.12) and (2.5), the left-hand side of (2.13) can be written in the form
\[ 16f(3x + z) + 16f(3y - z) + 16f(3x - z) + 16f(3y + z) = f(3x + 9y - 2z) + f(9x + 3y + 2z) - 12f(3x + 3y) + f(9x + 3y - 2z) + f(3x + 9y + 2z) - 12f(3x + 3y) = 3f(x + 3y + 2z) + 3f(x + 3y - 2z) + 48f(x + 3y) + 3f(3x + y + 2z) + 3f(3x + y - 2z) + 48f(3x + y)
\]
\[ + 3f(3x + y - 2z) + 48f(3x + y) - 648f(x + y) = 768f(x) + 768f(y) + 48f(x + z) + 48f(x - z) + 48f(y + z) + 48f(y - z) + 36f(x + y + 2z) + 36f(x + y - 2z) - 72f(x + y). \]  
(2.15)

Replacing \( z \) by \( 2z \) in (2.15) and then applying (2.14), we obtain
\[ 16f(3x + 2z) + 16f(3y - 2z) + 16f(3x - 2z) + 16f(3y + 2z) = 3f(x + 3y + 4z) + 3f(x + 3y - 4z) + 48f(x + 3y)
\]
\[ + 3f(x + 3y + 4z) + 3f(x + 3y - 4z) + 48f(x + 3y) - 648f(x + y) = 768f(x) + 768f(y) + 48f(x + z) + 48f(x - z) + 48f(y + z) + 48f(y - z) + 36f(x + y + 2z) + 36f(x + y - 2z) - 72f(x + y). \]  
(2.16)

Again, making use of (2.12) and (2.5), we get
\[
16f(3x + 2z) + 16f(3x - 2z) + 16f(3y + 2z) + 16f(3y - 2z) \\
= f(12x + 4z) + f(12x - 4z) - 12f(6x) + f(12y + 4z) \\
+ f(12y - 4z) - 12f(6y) \\
= 64f(3x + z) + 64f(3x - z) - 2592f(x) + 64f(3y + z) \\
+ 64f(3y - z) - 2592f(y) \\
= 64\left[3f(x + z) + 3f(x - z) + 48f(x) + 3f(y + z) + 3f(y - z) + 48f(y)\right] \\
- 2592f(x) - 2592f(y) \\
= 192f(x + z) + 192f(x - z) + 480f(x) + 192f(y + z) \\
+ 192f(y - z) + 480f(y).
\]

Finally, if we compare (2.16) with (2.17), then we conclude that
\[
f(x + y + 2z) + f(x + y - 2z) + 8f(x) + 8f(y) \\
= 2f(x + y) + 4f(x + z) + 4f(x - z) + 4f(y + z) + 4f(y - z)
\]
which, by considering \(f(2x) = 8f(x)\), gives
\[
f(x + y + 2z) + f(x + y - 2z) + f(2x) + f(2y) \\
= 2\left[f(x + y) + 2f(x + z) + 2f(x - z) + 2f(y + z) + 2f(y - z)\right]
\]
for all \(x, y, z \in X\). This completes the proof of the lemma. \(\square\)

For explicitly later use, we state the following theorem:

**Theorem 2.2** [14]. (The alternative of fixed point.) Suppose that we are given a complete generalized metric space \((\Omega, d)\) and a strictly contractive mapping \(T : \Omega \to \Omega\) with Lipschitz constant \(L\). Then, for each given \(x \in \Omega\), either
\[
d(T^nx, T^{n+1}x) = \infty \quad \text{for all } n \geq 0,
\]
or there exists a natural number \(n_0\) such that
\[
\begin{align*}
&\bullet \ d(T^nx, T^{n+1}x) < \infty \text{ for all } n \geq n_0; \\
&\bullet \ The \ sequence \ (T^nx) \text{ is convergent to a fixed point } y^* \text{ of } T; \\
&\bullet \ y^* \text{ is the unique fixed point of } T \text{ in the set } \Delta = \{y \in \Omega : d(T^mx, y) < \infty\}; \\
&\bullet \ d(y, y^*) \leq \frac{1}{1-L}d(y, Ty) \text{ for all } y \in \Delta.
\end{align*}
\]

Utilizing the above-mentioned fixed point alternative, we now obtain our main result, i.e., the generalized Hyers–Ulam–Rassias stability of the functional equation (1.2).

From now on, let \(X\) be a real vector space and \(Y\) be a real Banach space. Given a mapping \(f : X \to Y\), we set
\[
Df(x, y, z) := f(x + y + 2z) + f(x + y - 2z) + f(2x) + f(2y) \\
- 2\left[f(x + y) + 2f(x + z) + 2f(x - z) + 2f(y + z) + 2f(y - z)\right]
\]
for all \(x, y, z \in X\).
Let \( \phi : X \times X \times X \to [0, \infty) \) be a function such that
\[
\lim_{n \to \infty} \frac{\phi(\lambda_i^n x, \lambda_i^n y, \lambda_i^n z)}{\lambda_i^{3n}} = 0
\]
(2.18)
for all \( x, y, z \in X \), where \( \lambda_i = 2 \) if \( i = 0 \) and \( \lambda_i = 1/2 \) if \( i = 1 \).

**Theorem 2.3.** Suppose that a function \( f : X \to Y \) satisfies the functional inequality
\[
\| Df(x, y, z) \| \leq \phi(x, y, z)
\]
(2.19)
for all \( x, y, z \in X \) and \( f(0) = 0 \). If there exists \( L = L(i) < 1 \) such that the function \( x \mapsto \psi(x) = \phi(0, x, 0) \) has the property
\[
\psi(x) \leq L \cdot \lambda_i^3 \cdot \psi \left( \frac{x}{\lambda_i} \right)
\]
(2.20)
for all \( x \in X \), then there exists a unique cubic function \( C : X \to Y \) such that the inequality
\[
\| f(x) - C(x) \| \leq \frac{L^{1-i}}{1-L} \psi(x)
\]
(2.21)
holds for all \( x \in X \).

**Proof.** Consider the set
\[
\Omega := \{ g : g : X \to Y, g(0) = 0 \}
\]
and introduce the generalized metric on \( \Omega \),
\[
d(g, h) = d_\phi(g, h) = \inf \left\{ K \in (0, \infty) : \| g(x) - h(x) \| \leq K \psi(x), \ x \in X \right\}.
\]
It is easy to see that \( (\Omega, d) \) is complete.

Now we define a function \( T : \Omega \to \Omega \) by
\[
Tg(x) = \frac{1}{\lambda_i^3} g(\lambda_i x)
\]
for all \( x \in X \). Note that for all \( g, h \in \Omega \),
\[
d(g, h) < K \quad \Rightarrow \quad \| g(x) - h(x) \| \leq K \psi(x), \ x \in X,
\]
\[
\Rightarrow \quad \left\| \frac{1}{\lambda_i^3} g(\lambda_i x) - \frac{1}{\lambda_i^3} h(\lambda_i x) \right\| \leq \frac{1}{\lambda_i^3} K \psi(\lambda_i x), \ x \in X,
\]
\[
\Rightarrow \quad \left\| \frac{1}{\lambda_i^3} g(\lambda_i x) - \frac{1}{\lambda_i^3} h(\lambda_i x) \right\| \leq LK \psi(x), \ x \in X,
\]
\[
\Rightarrow \quad d(Tg, Th) \leq LK.
\]
Hence we see that
\[
d(Tg, Th) \leq Ld(g, h)
\]
for all \( g, h \in \Omega \), that is, \( T \) is a strictly self-mapping of \( \Omega \) with the Lipschitz constant \( L \).
If we put \( x = 0 = z \) in (2.19) and use (2.20) with the case \( i = 0 \), then we see that
\[
\| f(2y) - 8f(y) \| \leq \varphi(0, y, 0)
\]
(2.22)
which is reduced to
\[
\| f(y) - \frac{1}{32} f(2y) \| \leq \frac{1}{23} \psi(2y) \leq L \psi(y)
\]
for all \( y \in X \), that is, \( d(f, Tf) \leq L = L^1 < \infty \).

If we substitute \( y := y/2 \) in (2.22) and use (2.20) with the case \( i = 1 \), then we see that
\[
\| f(y) - 2^3 f\left(\frac{y}{2}\right) \| \leq \psi(y)
\]
for all \( y \in X \), that is, \( d(f, Tf) \leq L^0 = L^0 < \infty \).

Now, from the fixed point alternative in both cases, it follows that there exists a fixed point \( C \) of \( T \) in \( \Omega \) such that
\[
C(x) = \lim_{n \to \infty} f(\lambda_n x)\lambda_n^3
\]
(2.23)
for all \( x \in X \) since \( \lim_{n \to \infty} d(T^n f, C) = 0 \).

To show that the function \( C : X \to Y \) is cubic, let us replace \( x, y \) and \( z \) by \( \lambda_n x, \lambda_n^i y \) and \( \lambda_n^i z \) in (2.19), respectively, and divide by \( \lambda_n^3 \). Then it follows from (2.18) and (2.23) that
\[
\| Df(x, y, z) \| = \lim_{n \to \infty} \| Df(\lambda_n^i x, \lambda_n^i y, \lambda_n^i z) \| \leq \lim_{n \to \infty} \frac{\varphi(\lambda_n^i x, \lambda_n^i y, \lambda_n^i z)}{\lambda_n^3} = 0
\]
for all \( x, y, z \in X \), that is, \( C \) satisfies the functional equation (1.2). Therefore Lemma 2.1 guarantees that \( C \) is cubic.

According to the fixed point alternative, since \( C \) is the unique fixed point of \( T \) in the set \( \Delta = \{ g \in \Omega : d(f, g) < \infty \} \), \( C \) is the unique function such that
\[
\| f(x) - C(x) \| \leq K \psi(x)
\]
for all \( x \in X \) and some \( K > 0 \). Again using the fixed point alternative, we have
\[
d(f, C) \leq \frac{1}{1 - L} d(f, Tf),
\]
and so we obtain the inequality
\[
d(f, C) \leq \frac{L^{1-i}}{1 - L}
\]
which yields the inequality (2.21). This completes the proof of the theorem.

From Theorem 2.3, we obtain the following corollary concerning the Hyers–Ulam–Rassias stability [17] of the functional equation (1.2).

**Corollary 2.4.** Let \( X \) and \( Y \) be a normed space and a Banach space, respectively. Let \( p \geq 0 \) be given with \( p \neq 3 \). Assume that \( \delta \geq 0 \) and \( \varepsilon \geq 0 \) are fixed. Suppose that a function \( f : X \to Y \) satisfies the functional inequality
\[
\| Df(x, y, z) \| \leq \delta + \varepsilon (\| x \|^p + \| y \|^p + \| z \|^p)
\]
(2.24)
for all \( x, y, z \in X \). Furthermore, assume that \( f(0) = 0 \) and \( \delta = 0 \) in (2.24) for the case \( p > 3 \). Then there exists a unique cubic function \( C : X \to Y \) such that the inequality
\[
\| f(x) - C(x) \| \leq \frac{\delta}{2^{3-p} - 1} + \frac{\varepsilon}{8 - 2^p} \| x \|^p
\]
holds for all \( x \in X \), where \( p < 3 \) if \( i = 0 \) and \( p > 3 \) if \( i = 1 \), that is, the relation (2.18) is true.

Since the inequality
\[
\frac{1}{\lambda_i} \psi(\lambda_i x) = \frac{\delta}{\lambda_i^{p-3}} + \frac{\lambda_i^{p-3}}{2^p} \varepsilon \| x \|^p \leq \lambda_i^{p-3} \psi(x)
\]
holds for all \( x \in X \), where \( p < 3 \) if \( i = 0 \) and \( p > 3 \) if \( i = 1 \), we see that the inequality (2.20) holds with either \( L = 2^{p-3} \) or \( L = 1/2^{p-3} \). Now the inequality (2.21) yields the inequalities (2.25) and (2.26) which complete the proof of the corollary.

The following corollary is the Hyers–Ulam stability [7] of the functional equation (1.2).

**Corollary 2.5.** Let \( X \) and \( Y \) be a normed space and a Banach space, respectively. Assume that \( \theta \geq 0 \) is fixed. Suppose that a function \( f : X \to Y \) satisfies the functional inequality
\[
\| Df(x, y, z) \| \leq \theta
\]
for all \( x, y, z \in X \). Then there exists a unique cubic function \( C : X \to Y \) such that the inequality
\[
\| f(x) - C(x) \| \leq \frac{1}{21} \theta
\]
holds for all \( x \in X \).

**Proof.** In Corollary 2.4, putting \( \delta := 0 \), \( p := 0 \) and \( \varepsilon := \theta/3 \), we arrive at the conclusion of the corollary.

**References**


