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Near-minimax complex approximation by four kinds of Chebyshev polynomial expansion

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Abstract

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Functions of the form $w(z)F(z)$ with F analytic and $w(z)=1$, $(z^2-1)^{1/2}$, $(z+1)^{1/2}$ or $(z-1)^{1/2}$ are approximated in the ellipse ξ_r: $|z+(z^2-1)^{1/2}|=r$ by $w(z)p_n(z)$, where p_n is a polynomial of degree n. Here p_n is obtained by the expansion of F in Chebyshev polynomials of the first, second, third or fourth kinds, corresponding to the above four respective weight functions. Bounds are established and computed for the norms on ξ_r of the corresponding projections, thus confirming that all resulting approximations are near-minimax within relative distances asymptotically proportional to $4\pi^{-2}\ln n$, and extending a known result (Geddes, 1978) for $w(z)=1$ and first kind Chebyshev polynomials.

Keywords: Chebyshev polynomial; near-minimax; complex approximation; Lebesgue constant.

1. Introduction

1.1. The Chebyshev polynomials

The Chebyshev polynomials of the first and second kinds are well known. In the case of a real variable x on $[-1, 1]$, they are defined by

$$T_n(x) = \cos n\theta, \tag{1.1}$$

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \tag{1.2}$$

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where the subscript n denotes the polynomial degree, and where $x = \cos\theta$. However, real Chebyshev polynomials of the third and fourth kinds may also be defined, and the relevant formulae are

$$V_n(x) = \frac{\cos(n+\frac{1}{2})\theta}{\cos(\frac{1}{2}\theta)}, \tag{1.3}$$

$$W_n(x) = \frac{\sin(n+\frac{1}{2})\theta}{\sin(\frac{1}{2}\theta)}. \tag{1.4}$$

These latter polynomials have appeared in various guises in the literature. They have been called "airfoil polynomials", since they are appropriate for approximating the single square root singularities that occur at the sharp end of an airfoil, and a discussion and set of references may be found in [3]. They are also sometimes referred to as "half angle shifted" Chebyshev polynomials. However their apt designation as third and fourth kind Chebyshev polynomials is apparently due to Gautschi [4], in consultation with colleagues in the field of orthogonal polynomials. In fact the four polynomials (1.1)–(1.4) form a natural quartet, since they are Jacobi polynomials for the four parameter pairs $\alpha = \pm\frac{1}{2}$, $\beta = \pm\frac{1}{2}$, being orthogonal on $[-1, 1]$ with respect to

$$(1-x^2)^{-1/2}, \quad (1-x^2)^{1/2}, \quad (1-x)^{1/2}(1+x)^{-1/2} \quad \text{and} \quad (1+x)^{1/2}(1-x)^{-1/2}.$$

The four kinds of Chebyshev polynomial are all readily generated from their common recurrence relation

$$p_n(x) = 2xp_{n-1}(x) - p_{n-2}(x), \qquad p_0(x) = 1, \tag{1.5}$$

but with differing choices of starting values

$$p_1(x) = x,\ 2x,\ 2x-1 \text{ and } 2x+1, \text{ respectively.} \tag{1.6}$$

In two recent papers of the present authors, some new results and properties were obtained on the real interval $[-1, 1]$. We considered classes $C_{\pm1}[-1, 1]$, $C_{-1}[-1, 1]$ and $C_{+1}[-1, 1]$ of functions continuous on $[-1, 1]$ but vanishing at ±1, -1 or $+1$, respectively [9]. We showed that expansions in $\{T_k(x)\}$, $\{(1-x^2)^{1/2}U_k(x)\}$, $\{(1+x)^{1/2}V_k(x)\}$ and $\{(1-x)^{1/2}W_k(x)\}$ yielded near-minimax approximations in $C[-1, 1]$, $C_{\pm1}[-1, 1]$, $C_{-1}[-1, 1]$ and $C_{+1}[-1, 1]$, respectively, and obtained explicit formulae for the norms of the associated projections. We also obtained corresponding results for interpolation at Chebyshev polynomial zeros. The first author [7] discussed the application of the four kinds of Chebyshev polynomials to certain problems involving indefinite integration or integral transforms.

The main purpose of the present paper is to seek complex near-minimax series expansion results analogous to the real results of [9], by extending the definitions of Chebyshev polynomials appropriately to a complex variable z. However, several key differences arise in the complex case. Firstly, an elliptical domain is involved. Secondly, results depend on the size of the domain. Thirdly, analytic rather than continuous function spaces are the more natural setting. Fourthly, weighted rather than constrained function spaces are adopted. Finally, it is not apparently possible to provide *attainable* bounds for the norms of series projections.

The new results establish that weighted Chebyshev series expansions provide excellent (near-minimax) approximations to weighted analytic functions around square root singularities, and these results have practical relevancy, for example, to applications in airfoil modelling [3] and to crack and punch problems in fracture mechanics [6].

1.2. Generalised expansions

In an earlier discussion, the first author [7] considered the L_∞ approximation of $f(z)$ continuous on the elliptical contour

$$\xi_r\colon\ |z+(z^2-1)^{1/2}|=r,\quad r>1, \tag{1.7}$$

based on a generalised Chebyshev series projection

$$(J_{nn}f)(z)=\sum_{k=0}^{n}{}' c_kT_k(z)+(z^2-1)^{1/2}\sum_{k=1}^{n}d_kU_{k-1}(z), \tag{1.8}$$

where

$$c_k=\frac{1}{\pi \mathrm{i}}\int_{\xi_r}f(z)T_k(z)(z^2-1)^{1/2}\,\mathrm{d}z, \tag{1.9}$$

$$d_k=\frac{1}{\pi \mathrm{i}}\int_{\xi_r}f(z)U_{k-1}(z)\,\mathrm{d}z, \tag{1.10}$$

and where T_k and U_k are Chebyshev polynomials of the first and second kinds given by

$$T_n(z)=\tfrac{1}{2}(w^n+w^{-n}),\quad \text{for } z=\tfrac{1}{2}(w+w^{-1}),$$

and

$$U_n(z)=\frac{w^{n+1}-w^{-n-1}}{w-w^{-1}}=\tfrac{1}{2}(w^{n+1}-w^{-n-1})(z^2-1)^{-1/2}.$$

(The prime above the sum in (1.8) indicates that the first term is halved.) It is clear that these definitions of $T_n(z)$ and $U_n(z)$ are natural complex extensions of (1.1) and (1.2). Mason showed that

$$\|J_{nn}\|_\infty=\lambda_n, \tag{1.11}$$

where

$$\lambda_n=\frac{1}{\pi}\int_0^{2\pi}\left|\sum_{k=0}^{n}\cos k\phi\right|\mathrm{d}\phi=\frac{1}{\pi}\int_0^{2\pi}\left|\frac{\sin(n+\frac{1}{2})\phi}{\sin(\frac{1}{2}\phi)}\right|\mathrm{d}\phi \tag{1.12}$$

is the classical Lebesgue constant asymptotic to $4\pi^{-2}\ln n$ (see, for example, [2]). Since

$$\|f-J_{nn}f\|_\infty\leqslant(1+\|J_{nn}\|_\infty)\|f-f^{\mathrm{B}}\|_\infty, \tag{1.13}$$

where f^{B} is the minimax approximation of the form of the right-hand side of (1.8), it follows that $J_{nn}f$ is near-minimax within a relative distance λ_n. It further follows from [1] that J_{nn} is a minimal projection.

In the present paper we extend this idea, by looking first at classes of functions $f(z)$ and $(z^2-1)^{1/2}f(z)$ with f analytic, for which the expansion (1.8) reduces either to a first kind (T_k) expansion or a weighted second kind (U_k) expansion. The projection norm is bounded by λ_n in the first case and by a smaller value in the second case, found by exploiting an odd Fourier series kernel in place of the traditional even Dirichlet kernel. However, neither of the resulting projections are minimal.

We also consider Chebyshev series projections for analytic $f(z)$ weighted by $(z+1)^{1/2}$ and $(z-1)^{1/2}$, respectively, based on Chebyshev polynomials of the third and fourth kinds:

$$V_n(z)=\frac{w^{n+1/2}+w^{-(n+1/2)}}{w^{1/2}+w^{-1/2}},\qquad W_n(z)=\frac{w^{n+1/2}-w^{-(n+1/2)}}{w^{1/2}-w^{-1/2}},\tag{1.14}$$

and we obtain bounds for these projections similar to λ_n in form. Clearly the complex definitions (1.14) are natural extensions of (1.3) and (1.4) above.

2. Series of first kind polynomials

Suppose that f belongs to $A(\xi_r)$, the class of functions analytic within the elliptical boundary ξ_r given by (1.7), and continuous on ξ_r. Then from (1.10), $d_k=0$ (since the integrand is analytic) and the expansion (1.8) involves only Chebyshev polynomials $T_k(z)$ of the first kind. Denoting the partial sum projection in this case by $(P_n^{(1)}f)(z)$, we readily deduce that

$$(P_n^{(1)}f)(z)=\sum_{k=0}^{n}c_kT_k(z),\tag{2.1}$$

and from (1.9) it follows that

$$(P_n^{(1)}f)(z)=\frac{1}{\pi}\int_{\xi_r}\sum_{k=0}^{n}\frac{T_k(z)T_k(t)}{(t^2-1)^{1/2}}f(t)\,\mathrm{d}t.$$

Now

$$2T_k(z)T_k(t)=2\cos ku\cos kv=\cos k\phi+\cos k\psi,\tag{2.2}$$

where $z=\cos u$, $t=\cos v$, $\phi=v-u$, $\psi=v+u$. Hence, on using a classical Dirichlet kernel approach,

$$(P_n^{(1)}f)(z)=\frac{1}{2\pi\mathrm{i}}\int_{\xi_r}f(t)\left(\sum_{k=0}^{n}{}'\cos k\phi+\sum_{k=0}^{n}{}'\cos k\psi\right)\frac{\mathrm{d}t}{(t^2-1)^{1/2}}.\tag{2.3}$$

By replacing v by $-v$ it is clear that the integrals for both sums in (2.3) coincide. Hence the substitution $t=\cos v$ in (2.3) gives

$$(P_n^{(1)}f)(z)=\frac{1}{\pi}\int_0^{2\pi}f(\cos v)\sum_{k=0}^{n}\cos k\phi\,\mathrm{d}\phi.\tag{2.4}$$

Now ϕ is readily seen to be real and to move on $[0, 2\pi]$, and hence from the definition (1.12) of the classical Lebesgue constant λ_n,

$$\|P_n^{(1)}f\|\leqslant\lambda_n\|f\|,\tag{2.5}$$

and hence

$$\|P_n^{(1)}\|\leqslant\lambda_n^{(1)}=\lambda_n.\tag{2.6}$$

Here and throughout the remainder of this paper we adopt the L_∞-norm, so that $\|\cdot\|$ denotes $\|\cdot\|_\infty$.

Thus $P_n^{(1)}f$ is near-minimax within a relative distance $\lambda_n^{(1)} = \lambda_n$. This result was first obtained in [5].

Note that we have not established equality in (2.6). We cannot be sure for example that it is possible to find an f in $A(\xi_r)$ which coincides on ξ_r with the function

$$f(t) = \operatorname{sgn}\left\{\sum_{k=0}^{n} \cos k\phi\right\}, \quad \text{for } z = z_0. \tag{2.7}$$

If there were such an analytic function, then its values within ξ_r would be completely determined (by Cauchy's integral formula) from the values (2.7) on ξ_r. It is not reasonable to assume that the infinite series expansion so defined should converge throughout the interior.

3. Series of second kind polynomials

Suppose now that f belongs to the space

$$\left\{f(z) = (z^2-1)^{1/2}F(z) \colon F \text{ in } A(\xi_r)\right\},$$

and again consider a norm restricted to the elliptical contour ξ_r. Then from (1.9), $c_k = 0$ (since the integrand is analytic) and the expansion (1.8) involves only Chebyshev polynomials $U_{k-1}(z)$ of the second kind. Denoting the partial sum projection in this case by $(P_n^{(2)}f)(z)$, it follows that

$$(P_n^{(2)}f)(z) = (z^2-1)^{1/2}\sum_{k=1}^{n} d_k U_{k-1}(z),$$

and from (1.10) that

$$(P_n^{(2)}f)(z) = -\frac{1}{\pi \mathrm{i}}\int_{\xi_r}\sum_{k=1}^{n}(z^2-1)^{1/2}U_{k-1}(z)U_{k-1}(t)f(t)\,\mathrm{d}t. \tag{3.1}$$

Now,

$$2(z^2-1)^{1/2}U_{k-1}(z)(t^2-1)^{1/2}U_{k-1}(t) = 2\sin ku \sin kv = \cos k\phi - \cos k\psi,$$

where $z = \cos u$, $t = \cos v$, $\phi = u - v$, $\psi = u + v$, and hence

$$(P_n^{(2)}f)(z) = \frac{-1}{4\pi \mathrm{i}}\int_{\xi_r} f(t)K(u, v)\frac{\mathrm{d}t}{(t^2-1)^{1/2}},$$

where

$$K(u, v) = 2\sum_{k=0}^{n}{}' \cos k(u-v) - 2\sum_{k=0}^{n}{}' \cos k(u+v). \tag{3.2}$$

Here we have replaced each sum from 1 to n by a sum from 0 to n, which is justified since the constant terms cancel. Changing the variable to v and then ϕ, and observing that ϕ takes real values, we obtain

$$(P_n^{(2)}f)(z) = -\frac{1}{4\pi}\int_{u}^{u+2\pi} f(\cos v)K(u, v)\,\mathrm{d}v = \frac{1}{4\pi}\int_{0}^{2\pi} f(\cos v)K_1(\phi, \psi)\,\mathrm{d}\phi,$$

where

$$K_1(\phi, \psi) = \frac{\sin(n+\frac{1}{2})\phi}{\sin(\frac{1}{2}\phi)} - \frac{\sin(n+\frac{1}{2})\psi}{\sin(\frac{1}{2}\psi)} = K(u, v)$$
$$= \frac{\sin(n+\frac{1}{2})(u-v)}{\sin(\frac{1}{2}(u-v))} - \frac{\sin(n+\frac{1}{2})(u+v)}{\sin(\frac{1}{2}(u+v))}.$$

We have simplified the two sums in (3.2) by geometric series. Hence

$$\| P_n^{(2)} f \| \leqslant \| f \| \lambda_n^{(2)}, \tag{3.3}$$

where

$$\lambda_n^{(2)} = \sup_{z \in \xi_r} \frac{1}{4\pi} \int_0^{2\pi} | K_1(\phi, \psi) | \, \mathrm{d}\psi, \tag{3.4}$$

and $\psi = 2u - \phi = 2 \cos^{-1} z - \phi$.

This formula can be further simplified. Writing

$$z = \tfrac{1}{2}(r + r^{-1}) \cos \theta + \tfrac{1}{2}\mathrm{i}(r - r^{-1}) \sin \theta, \quad \text{on } \xi_r \text{ for } 0 \leqslant \theta \leqslant 2\pi,$$

we deduce that $u = \theta + \mathrm{i} \ln r$, and hence that

$$\lambda_n^{(2)}(r) = \sup_{0 \leqslant \theta \leqslant 2\pi} \int_0^{2\pi} | K_1(\phi, \psi) | \, \mathrm{d}\phi, \tag{3.5}$$

where $\psi = 2\theta + 2\mathrm{i} \ln r - \phi$. Clearly $\lambda_n^{(2)}$ depends on r, and so a different bound on $\| P_n^{(2)} \|$ is obtained for every ellipse. From (3.3) it follows that

$$\| P_n^{(2)} \| \leqslant \lambda_n^{(2)}, \tag{3.6}$$

and hence that $P_n^{(2)} f$ is near-minimax within a relative distance $\lambda_n^{(2)}$ given by (3.4).

An alternative bound for $\| P_n^{(2)} \|$ can be obtained which is independent of r, by rewriting (3.1) above in the form

$$(P_n^{(2)} f)(z) = \frac{1}{\pi \mathrm{i}} \int_{\xi_r} \sum_{k=1}^{n} \left\{ T_k(z) T_k(t) - (z^2-1)^{1/2} (t^2-1)^{1/2} U_{k-1}(z) U_{k-1}(t) \right\}$$
$$\times \frac{f(t) \, \mathrm{d}t}{(t^2-1)^{1/2}},$$

since the first term in the integral vanishes (by the analyticity of F). Hence

$$(P_n^{(2)} f)(z) = \frac{1}{\pi \mathrm{i}} \int_{\xi_r} f(t) \sum_{k=0}^{n}{}' \cos k\phi \frac{\mathrm{d}t}{(t^2-1)^{1/2}}.$$

(The $k = 0$ term may be included since the integral vanishes in this case.) Thus

$$(P_n^{(2)} f)(z) = \frac{1}{\pi} \int_0^{\pi} f(\cos v) \left(\frac{\sin(n+\frac{1}{2})\phi}{\sin(\frac{1}{2}\phi)} \right) \mathrm{d}\phi,$$

where $\phi = v - u$, $z = \cos u$ and $t = \cos v$. Thus,

$$\| P_n^{(2)} \| \leqslant \lambda_n^{(1)}, \tag{3.7}$$

and we have the pair of bounds $\lambda_n^{(2)}(r)$ and $\lambda_n^{(1)}$ for $\| P_n^{(2)} \|$.

The bound $\lambda_n^{(2)}(r)$ is significantly smaller than $\lambda_n^{(1)}$ as $r \to 1$, when the ellipse collapses to the interval $[-1, 1]$. However, the values of $\lambda_n^{(2)}(r)$ increase rapidly with r, and hence the bound $\lambda_n^{(1)}$ is generally superior.

4. Series of third and fourth kind polynomials

Any function $g(w)$ continuous on C_r: $|w| = r$ has a Fourier series expansion on C_r whose partial sums may be written in the form

$$g_n = \sum_{k=0}^{n} \left[c_k^* \tfrac{1}{2}(w^{k+1} + w^{-k}) + d_k^* \tfrac{1}{2}(w^{k+1} - w^{-k}) \right]. \tag{4.1}$$

Now we know that

$$\left[\tfrac{1}{2}(w^{1/2} + w^{-1/2}) \right]^2 = \tfrac{1}{4}(w + w^{-1}) + \tfrac{1}{2} = \tfrac{1}{2}(z + 1),$$

$$\left[\tfrac{1}{2}(w^{1/2} - w^{-1/2}) \right]^2 = \tfrac{1}{4}(w + w^{-1}) - \tfrac{1}{2} = \tfrac{1}{2}(z - 1).$$

Hence the continuous function $w^{-1/2} g(w) = f(z)$ may be approximated by the partial sum

$$\begin{aligned} w^{-1/2} g_n(w) &= \sum_{k=0}^{n} \left[c_k^* \tfrac{1}{2}(w^{k+1/2} + w^{-(k+1/2)}) + d_k^* \tfrac{1}{2}(w^{k+1/2} - w^{-(k+1/2)}) \right] = f_n(z) \\ &= \left[\tfrac{1}{2}(z+1) \right]^{1/2} \sum_{k=0}^{n} c_k^* V_k(z) + \left[\tfrac{1}{2}(z-1) \right]^{1/2} \sum_{k=0}^{n} d_k^* W_k(z), \end{aligned} \tag{4.2}$$

where $z = \frac{1}{2}(w + w^{-1})$ and where the third and fourth kind polynomials $V_n(z)$ and $W_n(z)$ are defined by (1.14) above. Thus

$$f_n(z) = (J_{nn}^* f)(z), \tag{4.3}$$

where J_{nn}^* is a generalised "balanced" Chebyshev series projection in which (in contrast to the projection J_{nn} given by (1.8)) the singularities of $(z + 1)^{1/2}$ and $(z - 1)^{1/2}$ are balanced between the two parts of the approximation.

The coefficients c_k^* and d_k^* are given, from (4.1), by the Laurent coefficient formulae

$$\tfrac{1}{2}(c_k^* + d_k^*) = \frac{1}{2\pi \mathrm{i}} \int_{C_r} g(w) w^{-(k+2)} \, \mathrm{d}w$$

and

$$\tfrac{1}{2}(c_k^* - d_k^*) = \frac{1}{2\pi \mathrm{i}} \int_{C_r} g(w) w^{(k-1)} \, \mathrm{d}w.$$

Hence,

$$c_k^* = \frac{1}{2\pi i}\int_{C_r} g(w)(w^{k+1/2} + w^{-(k+1/2)})w^{-3/2}\, dw$$
$$= \frac{1}{2\pi i}\int_{\xi_r} f(z)\left[\tfrac{1}{2}(z+1)\right]^{1/2} V_k(z)\left[\tfrac{1}{4}(z^2-1)\right]^{-1/2} dz.$$

Thus,

$$c_k^* = \frac{1}{2\pi i}\int_{\xi_r} f(z)\left[\tfrac{1}{2}(z-1)\right]^{-1/2} V_k(z)\, dz. \tag{4.4}$$

Similarly,

$$d_k^* = \frac{1}{2\pi i}\int_{\xi_r} f(z)\left[\tfrac{1}{2}(z+1)\right]^{-1/2} W_k(z)\, dz. \tag{4.5}$$

Clearly (4.4) and (4.5) are respectively formulae for the coefficients of $V_k(z)$ and $W_k(z)$ in the series expansion of which (4.2) is a partial sum. This partial sum reduces to one involving $V_k(z)$ only or $W_k(z)$ only according as $f(z)$ has the form $\frac{1}{2}(z+1)^{1/2}F(z)$ or $\frac{1}{2}(z-1)^{1/2}F(z)$, where $F(z)$ is analytic within ξ_r and continuous on ξ_r.

4.1. Partial sum of third kind series

Consider then a function

$$f(z) = \left[\tfrac{1}{2}(z+1)\right]^{1/2} F(z), \quad F \text{ in } A(\xi_r).$$

From (4.5), $d_k^* = 0$, and hence the approximation (4.2) reduces to

$$f_n(z) = (P_n^{(3)}f)(z) = \left[\tfrac{1}{2}(z+1)\right]^{1/2} \sum_{k=0}^{n} c_k^* V_k(z), \tag{4.6}$$

a weighted {by $\frac{1}{2}(z+1)^{1/2}$} partial sum of the expansion of $F(z)$ in Chebyshev polynomials of the third kind. Hence, from (4.4),

$$(P_n^{(3)}f)(z) = \left[\tfrac{1}{2}(z+1)\right]^{1/2} \frac{1}{2\pi i}\int_{\xi_r} \sum_{k=0}^{n} \frac{V_k(z)V_k(t)}{\left[\tfrac{1}{2}(t-1)\right]^{1/2}} f(t)\, dt. \tag{4.7}$$

Now,

$$2\left[\tfrac{1}{2}(z+1)\right]^{1/2}\left[\tfrac{1}{2}(t+1)\right]^{1/2} V_k(z)V_k(t) = 2\cos(k+\tfrac{1}{2})u \cos(k+\tfrac{1}{2})v$$
$$= \cos(k+\tfrac{1}{2})\phi + \cos(k+\tfrac{1}{2})\psi,$$

where $z = \cos u$, $t = \cos v$, $\phi = v - u$, $\psi = v + u$, and hence by comparison with the method used in (2.3) and (2.4) above,

$$(P_n^{(3)}f)(z) = \frac{1}{2\pi i}\int_{\xi_r} (t^2-1)^{-1/2} \sum_{k=0}^{n} \left[\cos(k+\tfrac{1}{2})\phi + \cos(k+\tfrac{1}{2})\psi\right] f(t)\, dt$$
$$= \frac{1}{\pi}\int_0^{2\pi} f(\cos v) \sum_{k=0}^{n} \cos(k+\tfrac{1}{2})\phi\, d\phi, \tag{4.8}$$

where ϕ is real and moves on $[0, 2\pi]$.

By summing the series, we may deduce that

$$\sum_{k=0}^{n} \cos\left(k+\tfrac{1}{2}\right)\phi = \frac{\tfrac{1}{2}\sin(n+1)\phi}{\sin\left(\tfrac{1}{2}\phi\right)}.$$

Hence,

$$\left(P_n^{(3)}f\right)(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\cos v)\frac{\sin(n+1)\phi}{\sin\left(\tfrac{1}{2}\phi\right)}\,\mathrm{d}\phi, \tag{4.9}$$

and thus,

$$\| P_n^{(3)} \| \leqslant \frac{1}{\pi}\int_0^{\pi}\left|\frac{\sin(n+1)\phi}{\sin\left(\tfrac{1}{2}\phi\right)}\right|\mathrm{d}\phi = \lambda_n^{(3)}. \tag{4.10}$$

Thus $P_n^{(3)}f$, given by (4.7), is near-minimax within a relative distance $\lambda_n^{(3)}$ given by (4.10).

4.2. Partial sum of fourth kind series

For the function

$$f(z) = \left[\tfrac{1}{2}(z-1)\right]^{1/2} F(z), \quad F \text{ in } A(\xi_r),$$

the approximation (4.2) reduces ($c_k^* = 0$) to

$$f(z) = \left(P_n^{(4)}f\right)(z) = \left[\tfrac{1}{2}(z-1)\right]^{1/2}\sum_{k=0}^{n} d_k^* W_k(z),$$

and it follows in precisely the same way as in Section 4.1 (but with $z+1$ replaced by $z-1$) that

$$\| P_n^{(4)} \| \leqslant \lambda_n^{(4)} = \lambda_n^{(3)}.$$

5. Numerical values and behaviour of $\boldsymbol{\lambda_n^{(1)}}$, $\boldsymbol{\lambda_n^{(2)}}$, $\boldsymbol{\lambda_n^{(3)}}$, $\boldsymbol{\lambda_n^{(4)}}$

The Lebesgue constant $\lambda_n^{(1)}$ is already well known to behave asymptotically as $4\pi^{-2}\ln n$, and its numerical values may be calculated from (1.12). The constant $\lambda_n^{(3)} = \lambda_n^{(4)}$ is readily seen to behave similarly, since

$$\left|\frac{\sin(n+1)\phi}{\sin\left(\tfrac{1}{2}\phi\right)}\right| \leqslant \left|\frac{\sin\left(n+\tfrac{1}{2}\right)\phi}{\sin\left(\tfrac{1}{2}\phi\right)}\right| \left|\cos\left(\tfrac{1}{2}\phi\right)\right| + \left|\cos\left(n+\tfrac{1}{2}\right)\phi\right| \leqslant \left|\frac{\sin\left(n+\tfrac{1}{2}\right)\phi}{\sin\left(\tfrac{1}{2}\phi\right)}\right| + 1,$$

and hence from (4.10) and (1.12),

$$\lambda_n^{(3)} = \lambda_n^{(4)} \leqslant \lambda_n^{(1)} + 1. \tag{5.1}$$

Thus $\lambda_n^{(3)}$ and $\lambda_n^{(4)}$ also behave asymptotically as $4\pi^{-2}\ln n$. Numerical values of the Lebesgue constants are shown in Table 1. Numerical evidence shows that $\lambda_n^{(3)} = \lambda_n^{(4)}$ is asymptotically the same as $\lambda_n^{(1)}$.

Table 1

n	$\lambda_n^{(1)}$	$\lambda_n^{(2)}(1)$	$\lambda_n^{(3)} = \lambda_n^{(4)}$
3	1.778	1.565	1.832
4	1.880	1.658	1.923
6	2.039	1.797	2.059
10	2.223	1.980	2.242
20	2.494	2.241	2.504
50	2.860	2.596	2.864
100	3.139	2.870	3.141

The limiting case $r \to 1$ is considered for $\| P_n^{(2)} \|$, and the corresponding numerical bounds $\lambda_n^{(2)}(1)$ are given. These are significantly smaller than $\lambda_n^{(1)}$. However, the bound $\lambda_n^{(1)}$ is valid for $\| P_n^{(2)} \|$ and is superior to $\lambda_n^{(2)}(r)$ as r increases away from 1, since $\lambda_n^{(2)}(r)$ is $O(r^n)$ for large r.

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