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# Contiguous relations of hypergeometric series ☆

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**Abstract**

The 15 Gauss contiguous relations for ${}_2F_1$ hypergeometric series imply that any three ${}_2F_1$ series whose corresponding parameters differ by integers are linearly related (over the field of rational functions in the parameters). We prove several properties of coefficients of these general contiguous relations, and use the results to propose effective ways to compute contiguous relations. We also discuss contiguous relations of generalized and basic hypergeometric functions, and several applications of them.


*Keywords:* Hypergeometric function; Contiguous relations

## 1. Contiguous relations of ${}_2F_1$ series

In this paper, let

$$\mathbf{F}\left(\begin{matrix} a,b \\ c \end{matrix}\right) := {}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix}\,\middle|\, z\right) = 1 + \frac{ab}{c\cdot 1}z + \frac{a(a+1)b(b+1)}{c(c+1)\cdot 1\cdot 2}z^2 + \cdots \tag{1}$$

denote the Gauss hypergeometric function with the argument $z$. Two hypergeometric functions with the same argument $z$ are *contiguous* if their parameters $a$, $b$ and $c$ differ by integers. For example,

$$\mathbf{F}\left(\begin{matrix} a,b \\ c \end{matrix}\right) \quad \text{and} \quad \mathbf{F}\left(\begin{matrix} a+10, b-7 \\ c+3 \end{matrix}\right)$$

are contiguous. As is known [1, 2.5], for any three contiguous ${}_2F_1$ functions there is a *contiguous* relation, which is a linear relation, with coefficients being rational functions in the parameters $a$, $b$,

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$c$ and the argument $z$. For example,

$$a(z-1)\mathbf{F}\begin{pmatrix} a+1,b \\ c \end{pmatrix} + (2a-c-az+bz)\mathbf{F}\begin{pmatrix} a,b \\ c \end{pmatrix}$$
$$+(c-a)\mathbf{F}\begin{pmatrix} a-1,b \\ c \end{pmatrix} = 0, \tag{2}$$

$$a\mathbf{F}\begin{pmatrix} a+1,b \\ c \end{pmatrix} - (c-1)\mathbf{F}\begin{pmatrix} a,b \\ c-1 \end{pmatrix} + (c-a-1)\mathbf{F}\begin{pmatrix} a,b \\ c \end{pmatrix} = 0, \tag{3}$$

$$a\mathbf{F}\begin{pmatrix} a+1,b \\ c \end{pmatrix} - b\mathbf{F}\begin{pmatrix} a,b+1 \\ c \end{pmatrix} + (b-a)\mathbf{F}\begin{pmatrix} a,b \\ c \end{pmatrix} = 0. \tag{4}$$

In these relations two hypergeometric series differ just in one parameter from the third hypergeometric series, and the difference is 1. The relations of this kind were found by Gauss, there are 15 of them. A contiguous relation between any three contiguous hypergeometric functions can be found by combining linearly a sequence of Gauss contiguous relations. In the next section, we discuss this and other ways to compute contiguous relations.

The following theorem summarizes some properties of coefficients of contiguous relations. These results are useful in computations and applications of contiguous relations. We assume that the parameters $a$, $b$, $c$ and $z$ are not related, and by $\mathscr{S}_a$, $\mathscr{S}_b$, $\mathscr{S}_c$, we denote the shift operators $a \mapsto a+1$, $b \mapsto b+1$ and $c \mapsto c+1$, respectively.

**Theorem 1.1.** *For any integers $k$, $l$, $m$ there are unique functions $\mathbf{P}(k,l,m)$ and $\mathbf{Q}(k,l,m)$, rational in the parameters $a$, $b$, $c$ and $z$, such that*

$$\mathbf{F}\begin{pmatrix} a+k,b+l \\ c+m \end{pmatrix} = \mathbf{P}(k,l,m)\mathbf{F}\begin{pmatrix} a,b \\ c \end{pmatrix} + \mathbf{Q}(k,l,m)\mathbf{F}\begin{pmatrix} a+1,b \\ c \end{pmatrix}. \tag{5}$$

*These functions satisfy the same contiguous relations as $\mathbf{F}\left(\begin{smallmatrix} a,b \\ c \end{smallmatrix}\right)$, that is, if*

$$\sum_{n=1}^{3} f_n \mathbf{F}\begin{pmatrix} a+\alpha_n, b+\beta_n \\ c+\gamma_n \end{pmatrix} = 0, \qquad f_1, f_2, f_3 \in \mathbf{C}(a,b,c,z) \tag{6}$$

*is a contiguous relation, then $\sum_{n=1}^{3} (\mathscr{S}_a^k \mathscr{S}_b^l \mathscr{S}_c^m f_n)\mathbf{P}(k+\alpha_n, l+\beta_n, m+\gamma_n) = 0$, and similarly for $\mathbf{Q}(k,l,m)$.*

*Besides, the following expressions hold:*

$$\begin{aligned} &\mathbf{P}(0,0,0) = 1, \quad \mathbf{Q}(0,0,0) = 0, \\ &\mathbf{P}(1,0,0) = 0, \quad \mathbf{Q}(1,0,0) = 1. \end{aligned} \tag{7}$$

$$\mathbf{P}(k,l,m) = \frac{c-a-1}{(a+1)(1-z)}\,\mathscr{S}_a\mathbf{Q}(k-1,l,m). \tag{8}$$

$$\mathbf{Q}(k+k',l+l',m+m') = (\mathscr{S}_a^k\mathscr{S}_b^l\mathscr{S}_c^m\mathbf{Q}(k',l',m'))\mathbf{Q}(k+1,l,m)$$
$$+\left(\mathscr{S}_a^{k+1}\mathscr{S}_b^l\mathscr{S}_c^m\frac{c-a}{a(1-z)}\mathbf{Q}(k'-1,l',m')\right)\mathbf{Q}(k,l,m). \tag{9}$$

$$\begin{vmatrix} \mathbf{P}(k,l,m) & \mathbf{P}(k',l',m') \\ \mathbf{Q}(k,l,m) & \mathbf{Q}(k',l',m') \end{vmatrix} = \frac{(c)_m(c)_m z^{-m}(z-1)^{m-k-l}}{(a+1)_k(b)_l(c-a)_{m-k}(c-b)_{m-l}}$$
$$\times\mathscr{S}_a^k\mathscr{S}_b^l\mathscr{S}_c^m\mathbf{Q}(k'-k,l'-l,m'-m). \tag{10}$$

$$\mathbf{Q}(-k,-l,-m) = \frac{(-1)^{m+1}(-a)_k(1-b)_l z^m(1-z)^{k+l-m}}{(1-c)_m(1-c)_m(c-a)_{k-m}(c-b)_{l-m}}$$
$$\times\mathscr{S}_a^{-k}\mathscr{S}_b^{-l}\mathscr{S}_c^{-m}\mathbf{Q}(k,l,m). \tag{11}$$

Here $(a)_k = \Gamma(a+k)/\Gamma(a)$ *is the Pochhammer symbol*; *for positive* $k$ *it is equal to* $a(a+1)\cdots(a+n-1)$.

This theorem is proved in Section 3. In the following section, we discuss computational aspects and applications of contiguous relations, including contiguous relations of generalized and basic hypergeometric series.

## 2. Computational aspects and applications

*Computational aspects*. To compute a contiguous relation (6) between three ${}_2F_1$ series, one can take one of the series as $\mathbf{F}\binom{a,b}{c}$. If necessary, one can apply a suitable shift operator to the contiguous relation afterwards. Then the other two hypergeometric series are

$$\mathbf{F}\begin{pmatrix} a+k',\ b+l' \\ c+m' \end{pmatrix} \quad\text{and}\quad \mathbf{F}\begin{pmatrix} a+k'',\ b+l'' \\ c+m'' \end{pmatrix}$$

for some integers $k',l',m'$ and $k'',l'',m''$. According to Theorem 1.1, these two functions can be expressed linearly in

$$\mathbf{F}\begin{pmatrix} a,b \\ c \end{pmatrix} \quad\text{and}\quad \mathbf{F}\begin{pmatrix} a+1,b \\ c \end{pmatrix}$$

as in (5). Elimination of $\mathbf{F}\binom{a+1,b}{c}$ gives

$$\mathbf{Q}(k'',l'',m'')\mathbf{F}\left(\begin{matrix} a+k',b+l' \\ c+m' \end{matrix}\right) - \mathbf{Q}(k',l',m')\mathbf{F}\left(\begin{matrix} a+k'',b+l'' \\ c+m'' \end{matrix}\right)$$
$$= (\mathbf{P}(k',l',m')\mathbf{Q}(k'',l'',m'') - \mathbf{P}(k'',l'',m'')\mathbf{Q}(k',l',m'))\mathbf{F}\left(\begin{matrix} a,b \\ c \end{matrix}\right). \tag{12}$$

Due to formula (8) it is enough to be able to compute $\mathbf{Q}(k,l,m)$ for any integers $k$, $l$, $m$. For this one can take a finite sequence of integer triples $(k_i,l_i,m_i)$ which starts with $(0,0,0)$, $(1,0,0)$, ends with $(k,l,m)$, and any two neighboring triples differ just in one component precisely by 1. Then all $\mathbf{Q}(k_i,l_i,m_i)$ including $\mathbf{Q}(k,l,m)$ can be computed consequently using the simplest Gauss contiguous relations. A possible choice of the sequence is to take all triples with intermediate integer values in the first component between $(0,0,0)$ and $(k,0,0)$, then all "intermediate triples" between $(k,0,0)$ and $(k,l,0)$, and finally all triples between $(k,l,0)$ and $(k,l,m)$.

Formula (9) allows to compute $\mathbf{Q}(k,l,m)$ by "divide and conquer" techniques, that is, by reducing the shift $(k,l,m)$ recursively by (more or less) a half at each step, compare with [3, 4.6.3]. A straightforward algorithm of this type is the following: compute $\mathbf{Q}(k,0,0)$ and $\mathbf{Q}(k+1,0,0)$ using (9) with shifts in the first parameter only (so there would be $O(\log k)$ intermediate $\mathbf{Q}$-terms, perhaps linearly growing in size); then use (4) to compute $\mathbf{Q}(k,1,0)$; and then alternate similar application of (9) and Gauss contiguity relations to compute $\mathbf{Q}(k,l,0)$, $\mathbf{Q}(k,l+1,0)$, $\mathbf{Q}(k,l,1)$ and $\mathbf{Q}(k,l,m)$. Notice that most formulas of Theorem 1.1 cannot be used when $a$, $b$ or $c$ are specialized, since the corresponding shift operators would not be defined then.

Returning to the computation of a contiguous relation (6) note that one can take any other of the given three functions in (12) as $\mathbf{F}\binom{a,b}{c}$. Proceeding in the same way one would obtain a contiguous relation involving, say, $\mathbf{Q}(-k',-l',-m')$ and $\mathbf{Q}(k''-k',l''-l',m''-m')$. The two contiguous relations must be the same up to a rational factor and corresponding shifts in the parameters. Formulas (10) and (11) are the explicit relations (between the coefficients) implied by this fact. Notice that combination of (12) and (10) gives an expression of a contiguous relation with coefficients linear in $\mathbf{Q}$'s.

Contiguous relations (5) and (12) can be also computed by combining Gauss contiguous relations themselves in similar ways: after choosing a sequence of contiguous hypergeometric series "connecting"

$$\mathbf{F}\left(\begin{matrix} a+k,b+l \\ c+m \end{matrix}\right) \quad \text{and} \quad \mathbf{F}\left(\begin{matrix} a,b \\ c \end{matrix}\right)$$

so that neighboring series differ just in one parameter by 1, or using a similar formula to (9). But this is a double work compared with computing the $\mathbf{Q}$'s.

Paule has shown that contiguous relations can be computed by a generalized version of Zeilberger's algorithm, see [4]. However, for large shifts $k$, $l$, $m$ this method is not efficient, because the growing degree (in the discrete parameters) of coefficients of contiguous relations imply larger linear problems. On the other hand, recurrence relations for hypergeometric functions can be computed as special cases of contiguous relations by the described methods, alternatively to Zeilberger's algorithm. Similar remarks hold also for computation of contiguous relations of generalized and basic hypergeometric series. On the website [8] there is a link to Maple package for computing contiguous relations for ${}_2F_1$ series.

*Generalization*. Generalized hypergeometric series is defined as

$$ {}_rF_s\left(\left.\begin{matrix} a_1\dots,a_r \\ b_1,\dots,b_s \end{matrix}\right| z\right) := \sum_{k=0}^{\infty} \frac{(a_1)_k\dots(a_r)_k}{(b_1)_k\dots(b_s)_k k!} z^k. \tag{13} $$

Two ${}_pF_q$ series are *contiguous* if their respective upper and lower parameters differ by integers. Like for ${}_2F_1$ series, there are linear relations between contiguous generalized hypergeometric series. In general they relate $1+\max(p,q+1)$ hypergeometric series, see [6, Section 48] and [1, 3.7]. In particular, relations (3) and (4) also hold for hypergeometric series with more upper and lower parameters. These relations allow to transform any difference operator to a difference operator with different shifts in one upper parameter only (by lowering all other upper parameters and raising the lower ones). It follows that general contiguous relations for ${}_pF_q$ series are generated by the relations of types (3–4), and a recurrence relation with the shifts in one upper parameter.

Like for ${}_2F_1$ series, contiguous relations for a class of ${}_pF_q$ functions can be computed by linearly combining a sequence of simplest contiguous relations. One can derive a formula analogous to (5) with at most $\max(p,q+1)$ fixed hypergeometric functions on the right-hand side. The coefficients to the fixed hypergeometric functions would satisfy the contiguity relations of ${}_pF_q$ functions, with corresponding initial conditions like (7). Those coefficients usually are not all related by a formula like (8), unless the hypergeometric functions under consideration satisfy three-term contiguous relations (say, ${}_3F_2(1)$ functions).

Similarly, there are contiguous relations for basic hypergeometric (or $q$-hypergeometric) series; see [1, 10.9] for the definition of ${}_r\phi_s$ series. Two such series are contiguous if their corresponding upper and lower parameters differ by a power of the base $q$. Moreover, multiplicative $q$-shifts in the argument of these functions can also be allowed, since the $q$-shift in the argument can be expressed in $q$-shifts of the parameters, say

$$ a\,{}_2\phi_1\left(\left.\begin{matrix} a,b \\ c \end{matrix}\right| q;qz\right) = (a-1)\,{}_2\phi_1\left(\left.\begin{matrix} aq,b \\ b \end{matrix}\right| q;z\right) + {}_2\phi_1\left(\left.\begin{matrix} a,b \\ c \end{matrix}\right| q;z\right). $$

For any three contiguous ${}_2\phi_1$ series, where also $q$-shifts in the argument $z$ are allowed, there is a contiguous relation. Allowing $q$-shifts in the argument $z$ is natural, because many transformations of basic hypergeometric series mix the parameters $a,b,c$ and the argument $z$, see [1 (10.10.1)]. A $q$-differential equation for ${}_r\phi_s$ series can be interpreted as a contiguous relation in this more general sense, since it can be seen as a $q$-difference equation.

*Evaluation*. Contiguous relations can be used to evaluate a hypergeometric function which is contiguous to a hypergeometric series which can be satisfactorily evaluated. For example, Kummer's identity [1, Cor. 3.1.2]

$$ {}_2F_1\left(\left.\begin{matrix} a,b \\ 1+a-b \end{matrix}\right| -1\right) = \frac{\Gamma(1+a-b)\Gamma(1+a/2)}{\Gamma(1+a)\Gamma(1+a/2-b)} \tag{14} $$

can be generalized to

$$ {}_2F_1\left(\left.\begin{matrix} a+n, b \\ a-b \end{matrix}\,\right|\, -1\right) = P(n)\,\frac{\Gamma(a-b)\Gamma((a+1)/2)}{\Gamma(a)\Gamma((a+1)/2-b)} + Q(n)\,\frac{\Gamma(a-b)\Gamma(a/2)}{\Gamma(a)\Gamma(a/2-b)}. \tag{15} $$

Here $n$ is an integer, the factors $P(n)$ and $Q(n)$ can be expressed for $n \geqslant 0$ as

$$ P(n) = \frac{1}{2^{n+1}}\,{}_3F_2\left(\left.\begin{matrix} -n/2, -(n+1)/2, a/2-b \\ 1/2, a/2 \end{matrix}\,\right|\, 1\right), \tag{16} $$

$$ Q(n) = \frac{n+1}{2^{n+1}}\,{}_3F_2\left(\left.\begin{matrix} -(n-1)/2, -n/2, (a+1)/2-b \\ 3/2, (a+1)/2 \end{matrix}\,\right|\, 1\right) \tag{17} $$

and similarly for $n < 0$, see [7]. In fact, formula (15) is a contiguous relation between

$$ {}_2F_1\left(\left.\begin{matrix} a+n, b \\ a-b \end{matrix}\,\right|\, -1\right), \quad {}_2F_1\left(\left.\begin{matrix} a-1, b \\ a-b \end{matrix}\,\right|\, -1\right) \quad \text{and} \quad {}_2F_1\left(\left.\begin{matrix} a, b \\ 1+a-b \end{matrix}\,\right|\, -1\right), $$

where the last two terms are evaluated in terms of $\Gamma$-function using Kummer's identity (14), and the coefficients are expressed as terminating ${}_3F_2(1)$ series.

Formulas (35) and (36) in [7] present similar evaluations of

$$ {}_2F_1\left(\left.\begin{matrix} -a, 1/2 \\ 2a+3/2+n \end{matrix}\,\right|\, \frac{1}{4}\right) \quad \text{and} \quad {}_3F_2\left(\left.\begin{matrix} a+n, b, c \\ a-b, a-c \end{matrix}\,\right|\, 1\right) $$

when $n$ is an integer. These series are contiguous to the ${}_2F_1(1/4)$ and ${}_3F_2(1)$ series evaluable by Gosper's or Dixon's (respectively) identities. Both new formulas are also three-term contiguous relations, where two hypergeometric terms are evaluated and the coefficients are written as terminating hypergeometric series. In both cases all series contiguous to Gosper's ${}_2F_1(1/4)$ or well-posed ${}_3F_2(1)$ series can be evaluated by contiguous relations, but general expressions for the coefficients in the final three-term expression like (15) are not known.

Formulas like (15) may specialize to one-term evaluations. For example, $P(-5)=0$ if $2a^2-4ab+b^2-12a+17b+12=0$, and under this condition formula (15) may be brought to

$$ {}_2F_1\left(\left.\begin{matrix} a-5, b \\ a-b \end{matrix}\,\right|\, -1\right) = \frac{a-b-1}{a-2b}\,\frac{\Gamma(a-b-2)\Gamma(a/2-1)}{\Gamma(a-3)\Gamma(a/2-b)}. \tag{18} $$

By parameterizing the curve given by the relation between $a$ and $b$ one gets an exotic formula, see [7, (33)]. Apparently, this kind of formula can be obtained only by using contiguous relations and a known evaluation.

*Transformations.* General transformations of hypergeometric series can be derived from the symmetries of their contiguous relations. For example, all terms in the relations between 24 Kummer's ${}_2F_1$ functions (see [2, 2.9]) satisfy not only the same hypergeometric differential equation, but also the same contiguous relations with the same shifts in the parameters $a$, $b$ and $c$. To show this statement, one can check that two functions

$$u_1 = {}_2F_1\left( \begin{matrix} a,b \\ c \end{matrix} \middle| z \right), \quad u_2 = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\, {}_2F_1\left( \begin{matrix} a,b \\ a+b+1-c \end{matrix} \middle| 1-z \right), \tag{19}$$

satisfy basic contiguous relations (2–4). Then the functions

$$u_3 = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)}(-z)^{-a}{}_2F_1\left( \begin{matrix} a,a+1-c \\ a+1-b \end{matrix} \middle| \frac{1}{z} \right), \tag{20}$$

$$u_4 = \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}(-z)^{-b}{}_2F_1\left( \begin{matrix} b+1-c,b \\ b+1-a \end{matrix} \middle| \frac{1}{z} \right), \tag{21}$$

$$u_5 = \frac{\Gamma(c)\Gamma(1-a)\Gamma(1-b)}{\Gamma(2-c)\Gamma(c-b)\Gamma(c-a)}\, z^{1-c}{}_2F_1\left( \begin{matrix} a+1-c,b+1-c \\ 2-c \end{matrix} \middle| z \right), \tag{22}$$

$$u_6 = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}{}_2F_1\left( \begin{matrix} c-a,c-b \\ c+1-a-b \end{matrix} \middle| 1-z \right), \tag{23}$$

satisfy the same contiguous relations as well, since the coefficients in the relations

$$u_6 = u_1 - u_2, \qquad \frac{\mathbf{e}^{\mathrm{i}\pi a}\sin(\pi b)}{\sin(\pi(b-a))}u_3 = u_1 - \frac{\mathbf{e}^{\mathrm{i}\pi(c-a)}\sin(\pi b)}{\sin(\pi(a+b-c))}u_6, \tag{24}$$

$$u_4 = u_1 - u_3, \quad u_5 = u_1 - \frac{\sin(\pi a)\sin(\pi b)}{\sin(\pi c)\sin(\pi(a+b-c))}u_6, \tag{25}$$

are constants with respect to integral shifts in the parameters $a$, $b$, $c$. Other 18 hypergeometric functions are alternative representations of $u_1, \ldots, u_6$, see [2, 2.9]. All relations between 24 Kummer's functions are generated by (24–25), so the statement follows. Mind that different expressions of $u_i$'s as hypergeometric functions may have different arguments.

In principle, relations (24–25) can be found by showing that the three terms satisfy the same second-order recurrence relation (with respect to integral shifts in some parameter), and comparing their asymptotics as the corresponding parameter approaches $\infty$ and $-\infty$. However, this approach may be cumbersome. Quadratic or higher order algebraic transformations can also be found in this way.

An interesting question is whether the symmetries of contiguous relations can classify all identities between hypergeometric series. For ${}_2F_1$ series this boils down to the symmetries of $\mathbf{Q}(k,l,m)$. In the remainder of this section, we demonstrate a way to investigate the symmetries of recurrence relations (a special case of contiguous relations) of hypergeometric series.

*Symmetries of recurrence relations.* To obtain recurrence relations for some hypergeometric series one can introduce the discrete parameter $n$ by replacing one or several continuous parameters by $a \mapsto a+2n$ (and similarly) in different ways. Then a recurrence is a contiguous relation between sufficiently many hypergeometric functions with successive $n$. We are interested in situations when two hypergeometric series (with a common discrete parameter $n$) satisfy the same recurrence relation, perhaps after multiplying the functions by a hypergeometric term (or equivalently, a solution of a first-order recurrence relation with coefficients rational in $n$, see [5, 5.1]). Therefore we call two hypergeometric functions with discrete parameter $n$ *equivalent* if they differ by such a factor.

More specifically, suppose that a hypergeometric series satisfies a second order recurrence relation (for $n \geqslant 0$)

$$A(n)S(n+1) + B(n)S(n) + C(n)S(n-1) = 0 \tag{26}$$

with $A(n)$, $B(n)$ and $C(n)$ being rational functions in $n$. For convenience, we assume that $S(n)$ does not satisfy a first-order recurrence relation, and that $B(n) \neq 0$ for all $n \geqslant 0$. Then $S(n)$ is equivalent to

$$Z(n) = (-1)^n \frac{A(0)\ldots A(n-1)}{B(0)\ldots B(n-1)} S(n).$$

The sequence $Z(n)$ satisfies the recurrence

$$Z(n+1) - Z(n) + \frac{C(n)A(n-1)}{B(n)B(n-1)} Z(n-1) = 0. \tag{27}$$

In fact, $Z(n)$ is the unique sequence equivalent to $S(n)$ and satisfying a recurrence relation of form $Z(n+1) - Z(n) + H(n)Z(n-1) = 0$. (Note that the second order recurrence for $Z(n)$ is unique up to a factor, and that we want to keep the coefficients of two terms.) It follows that an equivalence class of hypergeometric functions is determined by the rational function

$$\frac{B(n)B(n-1)}{C(n)A(n-1)}. \tag{28}$$

Hypergeometric series with a discrete parameter $n$ can be classified by their equivalence class function. One can compute this function for various types of hypergeometric functions and different ways of introducing the discrete parameter $n$. Then one can look for the cases when different equivalence class functions are equal (perhaps under some relations between their continuous parameters).

For example, the equivalence class function for

$${}_0F_1\left(\left.\begin{array}{c} - \\ \hat{c}+n \end{array}\right| \hat{z}\right)$$

is the polynomial $-(n+\hat{c}-1)(n+\hat{c}-2)/\hat{z}$. This is also the equivalence class function for

$${}_0F_1\left(\left.\begin{array}{c} - \\ 2-\hat{c}-n \end{array}\right| \hat{z}\right).$$

The equivalence class function for

$${}_1F_1\left(\left.\begin{array}{c} a+n \\ c+2n \end{array}\right| z\right)$$

is

$$-\frac{(2n+c-1)(2n+c-3)(4n^2+4cn-4n+c^2+2az-cz-2c)(4n^2+4cn-12n+\cdots)}{z^2(n+a-1)(n+c-a-1)(2n+c)(2n+c-4)}.$$

When $c=2a$, this rational function is also a polynomial of degree two, namely $-4(2n+2a-1)(2n+2a-3)/z^2$. By comparing these two polynomials and corresponding transformations of the recurrences to normalized form (27), we conclude that there must be a linear relation between three functions

$$ {}_1F_1\left(\left.\begin{matrix} a+n \\ 2a+2n \end{matrix}\right| 4z\right), \quad {}_0F_1\left(\left.\begin{matrix} - \\ a+1/2+n \end{matrix}\right| z^2\right) \quad \text{and}$$

$$\frac{\Gamma(a+1/2+n)}{\Gamma(3/2-a-n)} z^{1-2a-2n} {}_0F_1\left(\left.\begin{matrix} - \\ 3/2-a-n \end{matrix}\right| z^2\right),$$

where coefficients are constants with respect to the shift $n \to n+1$. Note that the first two functions are bounded as $n \to \infty$, whereas the last one is unbounded (for fixed generic $a$ and $z$). Hence the coefficient to the last function is zero. By comparing the limits of the first two functions one concludes that

$$ {}_1F_1\left(\left.\begin{matrix} a+n \\ 2a+2n \end{matrix}\right| 4z\right) = \mathbf{e}^{2z} {}_0F_1\left(\left.\begin{matrix} - \\ a+n+1/2 \end{matrix}\right| z^2\right),$$

see also [1, (4.1.12)].

The author computed a number of equivalence class functions for ${}_rF_1$ (with $r=0,1,2$) and ${}_3F_2(1)$ functions. They do not imply new identities, except plenty of exotic two-term identities such as (18), and some interesting consequences of known identities. Straightforward computations are too cumbersome when the degree of numerators or denominators of equivalence class functions exceeds 10. However, this method generalizes easily to $q$-hypergeometric functions. A more intelligent consideration of the symmetries of recurrence (and contiguous) relations could be helpful in finding relations between hypergeometric functions when one cannot use symmetries of differential equations for them, say when the variable $z$ is specialized, or the parameters and $z$ are related.

## 3. Proof of Theorem 1

To show the existence of (5), observe that contiguous relations (2–4) express

$$\mathbf{F}\begin{pmatrix} a-1,b \\ c \end{pmatrix}, \quad \mathbf{F}\begin{pmatrix} a,b \\ c-1 \end{pmatrix}, \quad \mathbf{F}\begin{pmatrix} a,b+1 \\ c \end{pmatrix}$$

in terms of

$$\mathbf{F}\begin{pmatrix} a,b \\ c \end{pmatrix} \quad \text{and} \quad \mathbf{F}\begin{pmatrix} a+1,b \\ c \end{pmatrix}.$$

There are similar expressions for

$$\mathbf{F}\begin{pmatrix} a,b \\ c+1 \end{pmatrix} \quad \text{and} \quad \mathbf{F}\begin{pmatrix} a,b-1 \\ c \end{pmatrix}.$$

Using shifted versions of these relations one can express

$$\mathbf{F}\begin{pmatrix} a+k,b+l \\ c+m \end{pmatrix}$$

linearly in hypergeometric functions without shifts in parameters $b$ and $c$, and then leave only terms

$$\mathbf{F}\begin{pmatrix} a,b \\ c \end{pmatrix} \quad \text{and} \quad \mathbf{F}\begin{pmatrix} a+1,b \\ c \end{pmatrix}.$$

If expression (5) is not unique, then

$$\mathbf{F}\begin{pmatrix} a+1,b \\ c \end{pmatrix} \Big/ \mathbf{F}\begin{pmatrix} a,b \\ c \end{pmatrix}$$

is a rational function in the parameters, just as

$$\mathbf{F}\begin{pmatrix} a+1,b+1 \\ c \end{pmatrix} \Big/ \mathbf{F}\begin{pmatrix} a+1,b \\ c \end{pmatrix}$$

(by the symmetry of the upper parameters). Then

$${}_2F_1\left(\left.\begin{matrix} a+1,b+1 \\ 1+a-b \end{matrix}\right| -1\right) \Big/ {}_2F_1\left(\left.\begin{matrix} a,b \\ 1+a-b \end{matrix}\right| -1\right)$$

is a rational function in $a$ and $b$. But this function has unbounded set of poles according to Kummer's identity (14). Hence a contradiction.

The uniqueness of (5) implies that $\mathbf{P}(k,l,m)$ and $\mathbf{Q}(k,l,m)$ satisfy the same contiguous relations of ${}_2F_1$ series, check the (triple) substitution of (5)–(6).

The "initial" conditions (7) are obvious.

To prove (8), apply (5) to both sides of

$$\mathbf{F}\begin{pmatrix} a+k,b \\ c \end{pmatrix} = \mathscr{S}_a\mathbf{F}\begin{pmatrix} a+k-1,b \\ c \end{pmatrix},$$

and use (2) to eliminate $\mathbf{F}\left(\begin{smallmatrix} a-1,b \\ c \end{smallmatrix}\right)$ on the right-hand side.

Formula (9) is obtained after several applications of (5) to

$$\mathbf{F}\begin{pmatrix} a+k+k',b+l+l' \\ c+m+m' \end{pmatrix}.$$

The first intermediate step is

$$\mathbf{F}\begin{pmatrix} a+k+k', b+l+l' \\ c+m+m' \end{pmatrix} = (\mathscr{S}_a^k \mathscr{S}_b^l \mathscr{S}_c^m \mathbf{P}(k',l',m'))\mathbf{F}\begin{pmatrix} a+k, b+l \\ c+m \end{pmatrix} + (\mathscr{S}_a^k \mathscr{S}_b^l \mathscr{S}_c^m \mathbf{Q}(k',l',m'))\mathbf{F}\begin{pmatrix} a+k+1, b+l \\ c+m \end{pmatrix}.$$

Now apply (5) to get terms with

$$\mathbf{F}\begin{pmatrix} a, b \\ c \end{pmatrix} \quad \text{and} \quad \mathbf{F}\begin{pmatrix} a+1, b \\ c \end{pmatrix}$$

only, compare the terms to $\mathbf{F}\left(\begin{smallmatrix} a+1,b \\ c \end{smallmatrix}\right)$, and eventually use (8) once to replace $\mathbf{P}(k',l',m')$.

For a proof of the last two formulas, let us introduce

$$W_{p,q,r}(k,l,m) := \begin{vmatrix} \mathbf{P}(k,l,m) & \mathbf{P}(k+p,l+q,m+r) \\ \mathbf{Q}(k,l,m) & \mathbf{Q}(k+p,l+q,m+r) \end{vmatrix}. \tag{29}$$

We assume that $p$, $q$, $r$ (just as $k$, $l$, $m$) are integers.

**Lemma 3.1.** *The following properties of the* $W$*-symbol hold.*

(i) $W_{0,0,0}(k,l,m) = 0$.
(ii) $W_{p,q,r}(0,0,0) = \mathbf{Q}(p,q,r)$.
(iii) *For fixed* $k,l,m$ *the determinants* $W_{p,q,r}$ *satisfy the contiguous relations of* ${}_2F_1$ *functions. More precisely, if* (6) *is a contiguous relation, then* $\sum_{n=1}^{3}(\mathscr{S}_a^{k+p}\mathscr{S}_b^{l+q}\mathscr{S}_c^{m+r} f_n) W_{p+\alpha_n,q+\beta_n,r+\gamma_n}(k,l,m) = 0$.
(iv) $W_{1,0,0}$ *satisfies first-order recurrence relations*

$$W_{1,0,0}(k+1,l,m) = \left(\mathscr{S}_a^{k+1}\mathscr{S}_c^m \frac{a-c}{a(1-z)}\right) W_{1,0,0}(k,l,m),$$

$$W_{1,0,0}(k,l+1,m) = \left(\mathscr{S}_b^{l}\mathscr{S}_c^m \frac{b-c+1}{b(1-z)}\right) W_{1,0,0}(k,l,m),$$

$$W_{1,0,0}(k,l,m+1) = \left(\mathscr{S}_a^{k}\mathscr{S}_b^{l}\mathscr{S}_c^m \frac{c^2(z-1)}{(c-a)(c-b)z}\right) W_{1,0,0}(k,l,m).$$

**Proof.** The first two properties are straightforward.

If (6) is a contiguous relation, then

$$\sum_{n=1}^{3}(\mathscr{S}_a^{k+p}\mathscr{S}_b^{l+q}\mathscr{S}_c^{m+r}f_n)W_{p+\alpha_n,q+\beta_n,r+\gamma_n}(k,l,m)$$

$$=\begin{vmatrix}\mathbf{P}(k,l,m) & \sum_{n=1}^{3}(\mathscr{S}_a^{k+p}\mathscr{S}_b^{l+q}\mathscr{S}_c^{m+r}f_n)\mathbf{P}(k+p+\alpha_n,l+q+\beta_n,m+r+\gamma_n)\\ \mathbf{Q}(k,l,m) & \sum_{n=1}^{3}(\mathscr{S}_a^{k+p}\mathscr{S}_b^{l+q}\mathscr{S}_c^{m+r}f_n)\mathbf{Q}(k+p+\alpha_n,l+q+\beta_n,m+r+\gamma_n)\end{vmatrix}$$

$$=\begin{vmatrix}\mathbf{P}(k,l,m) & 0\\ \mathbf{Q}(k,l,m) & 0\end{vmatrix}=0.$$

The recurrence relation for $W_{1,0,0}$ with respect to $k$ follows from contiguous relation (2):

$$W_{1,0,0}(k+1,l,m)=\begin{vmatrix}\mathbf{P}(k+1,l,m) & \tilde{A}\mathbf{P}(k,l,m)+\tilde{B}\mathbf{P}(k+1,l,m)\\ \mathbf{Q}(k+1,l,m) & \tilde{A}\mathbf{Q}(k,l,m)+\tilde{B}\mathbf{Q}(k+1,l,m)\end{vmatrix}$$

$$=-\tilde{A}W_{1,0,0}(k,l,m),\quad \text{with } \tilde{A}=\mathscr{S}_a^{k+1}\mathscr{S}_c^{m}\frac{c-a}{a(1-z)}.$$

To prove the recurrence relation for $W_{1,0,0}$ with respect to $l$ we use contiguous relations (4) and [2, 2.8.(36)] to derive

$$\mathbf{P}(k,l+1,m)=A\mathbf{P}(k,l,m)+B\mathbf{P}(k+1,l,m),$$

$$\mathbf{P}(k+1,l+1,m)=C\mathbf{P}(k,l,m)+D\mathbf{P}(k+1,l,m),$$

with $A=\mathscr{S}_a^{k}\mathscr{S}_b^{l}\frac{b-a}{b}$, $B=\mathscr{S}_a^{k}\mathscr{S}_b^{l}\frac{a}{b}$, $C=\mathscr{S}_a^{k}\mathscr{S}_b^{l}\mathscr{S}_c^{m}\frac{c-a-1}{b(1-z)}$, $D=\mathscr{S}_a^{k}\mathscr{S}_b^{l}\mathscr{S}_c^{m}\frac{a+b-c+1}{b(1-z)}$. There are similar expressions for $\mathbf{Q}(k,l+1,m)$ and $\mathbf{Q}(k+1,l+1,m)$. Then

$$W_{1,0,0}(k,l+1,m)=\begin{vmatrix}A\mathbf{P}(k,l,m)+B\mathbf{P}(k+1,l,m) & C\mathbf{P}(k,l,m)+D\mathbf{P}(k+1,l,m)\\ A\mathbf{Q}(k,l,m)+B\mathbf{Q}(k+1,l,m) & C\mathbf{Q}(k,l,m)+D\mathbf{Q}(k+1,l,m)\end{vmatrix}$$

$$=(AD-BC)W_{1,0,0}(k,l,m),$$

which is the required relation. The recurrence with respect to $m$ can be proved similarly, using relations (34) and (38) in [2, 2.8]. This completes the proof of the lemma. □

The recurrence relations for $W_{1,0,0}(k,l,m)$ imply that

$$W_{1,0,0}(k,l,m)=\frac{(c)_m(c)_m z^{-m}(z-1)^{m-k-l}}{(a+1)_k(b)_l(c-a)_{m-k}(c-b)_{m-l}}. \tag{30}$$

The recurrence relations for $W_{p,q,r}$ and an initial condition (i) imply that

$$W_{p,q,r}(k,l,m)=W_{1,0,0}(k,l,m)\cdot\mathscr{S}_a^{k}\mathscr{S}_b^{l}\mathscr{S}_c^{m}\mathbf{Q}(p,q,r). \tag{31}$$

Formula (10) follows from these two equations.

To prove (11) note that the determinant in (10) can be expressed in two ways as the symbol $W$. This gives

$$W_{k-k',l-l',m-m'}(k',l',m') = -W_{k'-k,l'-l,m'-m}(k,l,m). \tag{32}$$

By setting $k' = l' = m' = 0$ and changing the sign of $k$, $l$ and $m$ one obtains

$$W_{-k,-l,-m}(0,0,0) = -W_{k,l,m}(-k,-l,-m). \tag{33}$$

Statement (ii) of Lemma 3.1, formulas (31) and (30), and transformation of Pochhammer symbols give the last formula of Theorem 1.1. □

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