The Construction of Translation Planes from Projective Spaces*

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Can every (nonDesarguesian) projective plane be imbedded (in some natural, geometric fashion) in a (Desarguesian) projective space ? The question is new but important, for, if the answer is yes, two entirely separate fields of research can be united. This paper provides a conceptually simple geometric construction which yields an affirmative answer for a broad class of planes. A plane π is given by the construction precisely when π is a translation plane with a coordinatizing right Veblen-Wedderburn system which is finite-dimensional over its left-operator skew-field. The condition is satisfied by all known translation planes, including all finite translation planes.

1. INTRODUCTION

One might say, with some justice, that projective geometry, in so far as present day research is concerned, has split into two quite separate fields. On the one hand, the researcher into the foundations of geometry tends to regard Desarguesian spaces as completely known. Since the only possible non Desarguesian spaces are planes, his attention is restricted to the theory of projective planes, especially the nonDesarguesian planes. On the other hand stand all those researchers—and especially, the algebraic geometers—who are unwilling to be bound to two-dimensional space and uninterested in permitting nonDesarguesian planes to assume an exceptional role in their theorems. For the latter group of researchers, there are no projective spaces except the Desarguesian spaces.

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In the present paper we present a construction which, we hope, may do just a little to span the chasm between the two fields of projective geometry. Specifically, we show how to construct a class of nonDesarguesian planes (which occur most naturally in affine form) in terms of the elements (certain points and certain projective subspaces) of (Desarguesian) projective spaces of even dimension. The construction is given in Section 4.

The researcher in the theory of planes will want to know what planes we have constructed. There are two answers: (1) Only translation planes. (2) All finite translation planes and, more specifically, precisely those translation planes coordinatized by a right Veblen-Wedderburn system which is finite-dimensional over its left-operator skew-field. See Sections 6, 7; in particular, Theorem 7.1.

Since this paper has two authors, the following remarks may be appropriate: The present construction hinges upon the concept of a *spread* (of an odd-dimensional projective space; see Section 3). This concept was introduced by Bruck, with the construction of planes in mind (and as a natural sequel to the concluding part of his 1963 lectures to the Saskatoon Seminar of the Canadian Mathematical Congress; see ref. 2) but with a different objective. The construction presented here is entirely due to Bose and evolved, essentially, from considering spreads of 3-dimensional projective space (which are maximal sets of skew lines) in terms of 5-dimensional projective space. And, finally, the analytical details of the paper, including the precise relation with translation planes, were supplied by Bruck.

The first example of a nonDesarguesian translation plane was given in 1907 by Veblen and Wedderburn [10]. The first examples of spreads (in the sense used in this paper) seem to have been given in 1945 and 1946 by C. Radha-krishna Rao [8, 9]. (Equivalent examples are given in Section 3. However, in ref. 9, Rao also gives a solution of the original Kirkman Schoolgirl Problem by partitioning the 35 lines of projective 3-space over $GF(2)$ into 7 disjoint spreads, each spread consisting of 5 skew lines containing, by threes, the 15 points of space.) The first published use of spreads for the construction of nonDesarguesian planes is that in the present paper.

2. PROJECTIVE SPACES IN TERMS OF VECTOR SPACES

Since most algebraists are more familiar with vector spaces than with projective spaces, we wish to recall a classical representation. This representa-tion is thoroughly studied, for example, in ref. 1.

Let F be a skew field (that is, an associative division ring which may or may not be commutative) and let V be a vector space with F as a ring of (say) left operators. The dimension of V over F may be finite or infinite but (to

avoid trivialities) should be at least 3. From V we define a projective space $\Sigma = \Sigma(V/F)$ in the following manner: A point (or 0-dimensional projective subspace) of Σ is a 1-dimensional vector subspace of V over F. More generally, for each nonnegative integer s, an s-dimensional projective subspace of Σ is an $(s + 1)$-dimensional vector subspace of V over F. And incidence in Σ is defined in terms of the containing relation in V. The axiom of Desargues is a theorem of Σ. The axiom of Pappus is valid in Σ precisely when F is a field.

Conversely, if $d \geqslant 2$ is a positive integer and if Σ is a d-dimensional projective space (and if Σ satisfies the axiom of Desargues in case $d = 2$) then there exists a skew field F, uniquely defined to within an isomorphism, and a $(d + 1)$-dimensional vector space V over F such that Σ is isomorphic to $\Sigma(V/F)$.

Since, by the Theorem of Wedderburn, the only finite skew fields are the Galois fields $GF(q)$, one for each prime power q, the only finite d-dimensional Desarguesian projective spaces are the projective spaces $PG(d, q)$, one for each $d \geqslant 2$ and each prime power q, where $PG(d, q)$ is defined as above, with V $(d + 1)$-dimensional over $GF(q)$.

3. Spreads

Let Σ be a (finite or infinite) projective space of odd dimension $2t - 1$. Let S be a collection of $(t - 1)$-dimensional projective subspaces of Σ. We call S a *spread* of Σ provided that each point of Σ is contained in one and only one member of Σ. Note that, if $t = 1$, Σ is a projective line and Σ has precisely one spread, namely the collection consisting of all the points of Σ.

In the special case that $\Sigma = PG(2t - 1, q)$, a simple calculation shows that a spread of Σ is merely a collection of $1 + q^t$ distinct $(t - 1)$-dimensional projective subspaces which are *skew* in the sense that no two have a common point.

The existence of a spread of $PG(2t - 1, q)$ may be shown quite simply in terms of the representation described in Section 2. Set $L = GF(q^{2t})$. Let $K = GF(q^t)$ be the unique subfield of L of indicated order and let $F = GF(q)$ be the unique subfield of L and K of indicated order. Then L is a 2-dimensional vector space over K, and K is a t-dimensional vector space over F, and L is a $(2t)$-dimensional vector space over F. Hence, in the sense of isomorphism, $PG(2t - 1, q) = \Sigma(L/F)$ and $PG(1, q^t) = \Sigma(L/K)$. The set, S, of all 1-dimensional vector subspaces of L over K is also a set of (some but not all) t-dimensional vector subspaces of L over F. And S is, simultaneously, a spread of $PG(1, q^t)$ and a spread of $PG(2t - 1, q)$.—As we shall see later, not all spreads of $PG(2t - 1, q)$ can be obtained in this manner if $t > 1$.

Before turning to our construction we should like to raise a point which may be of some interest. Again let Σ be a (finite or infinite) projective space of odd dimension $2t - 1$, but now assume that t is at least two. Call a collection, S, of $(t - 1)$-dimensional projective subspaces of Σ a *dual spread* of Σ provided that each $(2t - 2)$-dimensional projective subspace of Σ contains one and only member of S. If Σ is finite, it is easy to see that the class of all spreads of Σ is identical with the class of all dual spreads of Σ. It is not obvious whether the two classes need coincide when Σ is infinite.

4. THE CONSTRUCTION

Let t be a positive integer. What we have to say will be valid for $t = 1$ but will only be new for $t \geqslant 2$.

Let Σ be a projective space of even dimension $2t$, and let Σ' be a fixed projective subspace of Σ of dimension $2t - 1$. Furthermore, let S be a fixed spread of Σ'. We construct a system

$$\pi = \pi(\Sigma, \Sigma', S),$$

(which will turn out to be an affine plane) as follows:

The *points* of π are the points of Σ which are not in Σ'.

The *lines* of π are the t-dimensional projective subspaces of Σ which intersect Σ' in a unique member of S, and are not contained in Σ'.

The *incidence* relation of π is that induced by the incidence relation of Σ.

THEOREM 4.1.   *The system* $\pi = \pi(\Sigma, \Sigma', S)$ *is an affine plane.*

COROLLARY.   *If* $\Sigma = PG(2t, q)$ *then* $\pi = \pi(\Sigma, \Sigma', S)$ *is an affine plane of order* q^t.

REMARK.   If $t = 1$, so that Σ is a projective plane (not necessarily Desarguesian) and Σ' is a line of Σ, then S is the set of all points of Σ' and $\pi(\Sigma, \Sigma', S)$ is isomorphic to the affine plane obtained from Σ by deleting the line Σ' and the point set S. On the other hand, if $t > 1$, then Σ is a Desarguesian space and the construction of π seems to be new.

Proof.   We may assume $t > 1$, so that Σ is a Desarguesian space.

First let P be a point of π and let J be a member of S. Then, since J is a $(t - 1)$-dimensional projective subspace of Σ which is contained in Σ' and since P is a point of Σ which is not contained in Σ', there exists one and only one t-dimensional projective subspace, L, of Σ which contains both J and P.

Moreover, $L \cap \Sigma' = J$. Hence L is a line of π. Thus: *there is one and only one line, L, of π containing a given point P of π and a given member, J, of S.*

Next let L_1, L_2 be two distinct lines of π. If L_1, L_2 contain the same member, J, of S, then, as we have just shown, L_1, L_2 can have no point of π in common. That is: *two distinct lines of π which contain the same member, J, of S are parallel.* Next suppose that $L_i \cap \Sigma' = J_i$ $(i = 1, 2)$ where J_1, J_2 are distinct members of S. In this case, $J_1 \cap J_2$ is empty, and hence $L_1 \cap L_2$ has no point in common with Σ'. However, since L_1, L_2 are t-dimensional projective subspaces of the $(2t)$-dimensional projective space Σ, then $L_1 \cap L_2$ contains at least one point, P, of Σ. And P, not being in Σ', is a point of π. Thus: *two lines of π which contain different members of S have at least one common point of π.* These two facts lead at once to the parallel axiom: *If L is a line of π and if P is a point of π which is not on L, then there exists one and only one line, L', of π which contains P and has no point of π in common with L.*

Finally, let P, Q be distinct points of π. Then the line, PQ, of Σ is not contained in the $(2t - 1)$-dimensional projective subspace Σ' of Σ. Since Σ has dimension $2t$, PQ has a unique point, R, in common with Σ'. Since S is a spread of Σ', R is contained in one and only one member, J, of S. If there exists a line, L, of π which contains P and Q, then L must contain R and hence J. On the other hand, if L is the unique line of π which contains P and J, then L contains R and hence L contains $PR = PQ$. Therefore L contains P and Q. Thus: *if P, Q are two distinct points of π, then P and Q are contained in one and only one line, L, of π.*

In order to complete the proof of Theorem 4.1, we need only show that each line of π contains at least two distinct points of π. But this is sufficiently obvious.

In the case of the Corollary, we need only compute the number, n, of 1-dimensional vector spaces of a $(t + 1)$-dimensional vector space, L, over $GF(q)$, which are not in a specified t-dimensional vector subspace, J, of L. Clearly

$$n(q - 1) = q^{t-1} - q^t ,$$

whence $n = q^t$. Thus, if $\Sigma = PG(2t, q)$ each line of $\pi = \pi(\Sigma, \Sigma', q)$ has precisely $n = q^t$ distinct points. This proves the Corollary.

We may imbed the affine plane $\pi = \pi(\Sigma, \Sigma', S)$ in a projective plane π^* in the familiar manner. Since each member, J, of S corresponds to a class of parallel lines of π, namely those containing J, we adjoin each such J to π as a "point at infinity." And we adjoin the spread, S, to π as a "line at infinity". Hence the corresponding projective plane π^* has a perfectly concrete representation in terms of our construction.

If (as will turn out to be the case) some of our planes π are not Desarguesian (for $t > 1$), the main advantage of the present construction is that it exhibits

nonDesarguesian projective planes in the realm of classical Desarguesian projective geometry. In particular, the various planes π may be related to the group of all collineations of Σ. There is, however, a practical question which now must be answered: *How extensive is the present construction?* In order to give a complete answer we must relate our work to the known theory of projective planes, and for this purpose we must make rather more use of coordinates than we would choose under other circumstances.

5. AN AFFINE REPRESENTATION OF SPREADS

Let $t \geqslant 2$ be a positive integer and let Σ' be a projective space of odd dimension $2t - 1$. Then, in the sense of Section 2, $\Sigma' = \Sigma(W/F)$ where F is a skew-field and W is a $(2t)$-dimensional vector space over F as a ring of left operators. We shall study the spreads of Σ' which contain a specified $(t - 1)$-dimensional projective subspace of Σ'. This subspace we shall designate by $J(\infty)$.

Since $J(\infty)$ is a t-dimensional vector subspace of the $(2t)$-dimensional vector space W over F, we may choose an arbitrary basis

$$e_1, \quad e_2, \quad \cdots, \quad e_t$$

of $J(\infty)$ over F and complete this to a basis of W over F by adjoining t additional basis elements

$$e_1', \quad e_2', \quad \cdots, \quad e_t'.$$

Thus

$$J(\infty) = \{e_1, e_2, \cdots, e_t\}, \tag{5.1}$$

$$W = \{e_1, e_2, \cdots, e_t, e_1', e_2', \cdots, e_t'\}. \tag{5.2}$$

Once the bases have been chosen, we define a one-to-one mapping $x \to x'$ of $J(\infty)$ upon a (t-dimensional vector) subspace of W by the following rule: If

$$x = \sum_{i=1}^{t} x_i e_i \tag{5.3}$$

where the x_i are in F, then

$$x' = \sum_{i=1}^{t} x_i e_i'. \tag{5.4}$$

Next suppose that J is any $(t - 1)$-dimensional projective subspace of Σ'. Then J is a t-dimensional vector subspace of W. A necessary and sufficient condition that $J(\infty)$ and J have no point in common is that $J(\infty) \cap J$ consist of the zero vector alone:

$$J(\infty) \cap J = \{0\}, \tag{5.5}$$

or, equivalently, that

$$J(\infty) + J = W. \tag{5.6}$$

We note that the equivalent conditions (5.5), (5.6) will hold precisely when each w in W has a unique representation $w = x + y$ where x is in $J(\infty)$ and y is in J. Equivalently, we must have, for each $i = 1, 2, \cdots, t$,

$$e_i' = - x_i + y_i$$

where the x_i are in $J(\infty)$ and the y_i constitute a basis of J. Moreover, if x_i is in $J(\infty)$, then

$$x_i = \sum_{j=1}^{t} x_{ij} e_j$$

where the x_{ij} are in F. As a consequence, there corresponds to each J skew to $J(\infty)$ a unique matrix $X = (x_{ij})$ of t rows and columns with elements in F such that $J = J(X)$ where

$$J(X) = \{x_1 + e_1', x_2 + e_2', \cdots, x_t + e_t'\} \tag{5.7}$$

and

$$x_i = \sum_{j=1}^{t} x_{ij} e_j \qquad (i = 1, 2, \cdots, t). \tag{5.8}$$

Next let X, Y be two distinct t by t matrices over F, and set $Z = X - Y$. The intersection of

$$J(X) + J(Y)$$

with $J(\infty)$ is spanned by the t vectors

$$z_i = \sum_{i=1}^{t} z_{ij} e_j \qquad (i = 1, 2, \cdots, t).$$

Hence

$$J(X) + J(Y) = W \Leftrightarrow X - Y \text{ is nonsingular.} \tag{5.9}$$

Next let us observe the spaces $J(0)$, $J(I)$ corresponding to the zero matrix, 0 and the identity matrix, I:

$$J(0) = \{e_1', e_2', \cdots, e_t'\}$$

$$J(I) = \{e_1 + e_1', e_2 + e_2', \cdots, e_t + e_t'\} \tag{5.10}$$

Clearly each two of $J(\infty)$, $J(0)$, $J(I)$ are disjoint. We wish to establish the following converse: *If $J(\infty)$, L, M are three mutually disjoint $(t-1)$-dimensional projective subspaces of Σ', and if $e_1, \cdots, e_t$ is a preassigned basis of $J(\infty)$ over F, there exists one and only set of t additional basis vectors $e_1', \cdots, e_t'$ of W over F such that $L = J(0)$, $M = J(I)$.* We see this as follows: Since L, M are disjoint, then

$$L \cap M = \{0\}, \qquad L + M = V.$$

Hence, for each $i = 1, 2, \cdots, t$,

$$e_i = -e_i' + b_i$$

for unique elements e_i', b_i in L and M respectively. Since $L(\infty)$ and L are disjoint, the t elements

$$b_i = e_i + e_i'$$

must constitute a basis of M over F. Since $L(\infty)$ and M are disjoint, the t elements e_i' must constitute a basis of L over F; in addition, the $2t$ elements e_i, e_i' must constitute a basis of W over F. This is the promised proof.

At this point it should be clear that, in considering spreads of Σ', there is no loss of generality, in terms of the above notation, in limiting attention to spreads containing $J(\infty)$, $J(0)$ and $J(I)$. Such a spread, S, must have the following properties:

(i) S contains $J(\infty)$.

(ii)   Each member of S, other than $J(\infty)$, has the form $J(X)$, where X belongs to a collection, $\mathscr{C}$, of matrices of t rows and columns with elements in F, subject to the following conditions:

(iia) $\mathscr{C}$ contains the zero matrix, 0, and the identity matrix, I.

(iib)   If X, Y are distinct matrices in $\mathscr{C}$, then the matrix $X - Y$ is non-singular.

(iic)   To each ordered pair of elements a, b of $J(\infty)$ with $a \neq 0$ there corresponds a (unique) matrix X in $\mathscr{C}$ such that $a^X = b$.

In view of the preceding discussion, we need only explain the condition (iic) and, in particular the notation a^X. First, if $X = (x_{ij})$ and if

$$a = \sum_{i=1}^{t} a_i e_i \tag{5.11}$$

where the a_i are in F, then

$$a^X = \sum_{i=1}^{t} \sum_{j=1}^{t} a_i x_{ij} e_j . \tag{5.12}$$

As a consequence, for each t by t matrix X with elements in F, the mapping $a \to a^X$ is a linear transformation of $J(\infty)$ over F.

Now we may explain (iic) as a maximality condition. Conditions (i) and (iia) are merely normalization conditions. Condition (iib) merely ensures that no two members of S have a common point. As we shall see, (iic) has precisely the effect of ensuring that each point of Σ is contained in at least one member (and hence in exactly one member) of S. Consider a point of Σ', that is, a 1-dimensional subspace $\{w\}$ of W over F, where, of course, w is a nonzero element of W. In terms of the mapping defined by (5.3), (5.4), $w = b + a'$ for a unique pair of elements a, b of $J(\infty)$. If $a = 0$ then w is in $J(\infty)$. If $a \neq 0$ then we want w to be in $J(X)$ for some (unique) X in $\mathscr{C}$. However, in view of (5.7), w will be in $J(X)$ precisely when (assuming (5.11))

$$b = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

or (in view of (5.8), (5.12)) precisely when $a^X = b$.

In the section which follows we shall use the present discussion to exhibit the connection between spreads and the so-called Veblen-Wedderburn systems. For this reason we shall not bother to give examples at this point.

6. Affine Coordinates for π

Let t be a positive integer, $t \geqslant 2$, let Σ be a $(2t)$-dimensional projective space, let Σ' be a $(2t - 1)$-dimensional projective subspace of Σ, and let S be a spread of Σ'. We wish to introduce affine coordinates for the affine plane $\pi = \pi(\Sigma, \Sigma', S)$ defined in Section 4.

As in Section 2 we represent Σ in the form $\Sigma(V/F)$ where F is a skewfield and V is a $(2t + 1)$-dimensional vector space over F as a ring of left operators. Then $\Sigma' = \Sigma(W/F)$ where W is a $(2t)$-dimensional vector subspace of V over F. Without loss of generality we may give a special role to some (arbitrarily chosen) ordered triple, $J(\infty)$, $J(0)$, $J(I)$, of distinct members of the spread S. Then we may assume that, in the notation of Section 5, S consists

of $J(\infty)$ and other members $J(X)$, $X \in \mathscr{C}$, where $\mathscr{C}$ is a collection of t by t matrices with properties (iia), (iib), (iic). Here W has a basis of $2t$ elements e_i, e_i' $(i = 1, 2, \cdots, t)$, and we need only add a single element, e^*, of V which is not in W in order to get a basis of V.

We observe that, in terms of the notation of Section 5, each point of π, that is, each 1-dimensional vector space of V over F which is not in W, has a unique basis element of the form

$$y + x' + e^*$$

where x, y are in $J(\infty)$. Thus we may speak of the point (x, y) of π where we define

$$(x, y) = \{y + x' + e^*\} \tag{6.1}$$

for every ordered pair x, y of elements of $J(\infty)$.

A line of π, that is, a $(t + 1)$-dimensional vector space of V over F which intersects W in a member J of S, has the form

$$J + (x, y) = J + \{y + x' + e^*\}$$

provided (x, y) is one of its points. These lines may be divided into two types:

(I) *Lines $x = a$.* If a is in $J(\infty)$, the point (x, y) of π lies on the line

$$J(\infty) + (a, 0) = J(\infty) + \{a' + e^*\}$$

if and only if $x = a$.

(II) *Lines $y = x^M + b$.* If b is in $J(\infty)$ and if $J(M)$ is in S, the point (x, y) lies on the line

$$J(M) + (0, b) = J(M) + \{b + e^*\}$$

if and only if $y - b + x'$ is in $J(M)$; that is, if and only if

$$y - b = x^M.$$

Now we have specified all the points and all the lines of π by coordinates and equations, respectively. For purposes of comparison we wish to go slightly further and introduce a coordinate ring $(R, +, \cdot)$. To begin with, we take $R = J(\infty)$ and we define addition, $+$, in R to be the addition in $J(\infty)$ (as a subspace of V). To specify multiplication in R we must specialize a non-zero element of $R = J(\infty)$ or, equivalently, we must pick a *unit point* of π. We pick the unit point

$$I = (1, 1) = \{1 + 1' + e^*\} \tag{6.2}$$

where 1 is any fixed nonzero element of $R = J(\infty)$. Then we define multiplication $(\cdot)$ in $R = J(\infty)$ as follows: To each x in R there corresponds a unique matrix $T(x)$ in $\mathscr{C}$—or a unique member $J(T(x))$ of S—such that

$$1^{T(x)} = x. \tag{6.3}$$

And we define

$$yx = y \cdot x = y^{T(x)} \tag{6.4}$$

for all y, x in R. It is now easy to verify that $(R, +, \cdot)$ has the following properties:

(i) $(R, +)$ *is an abelian group with zero, 0.*

(ii) $(R, \cdot)$ *is a groupoid.*

(iii) *If R^* is the set of nonzero elements of R, then $(R^*, \cdot)$ is a loop with identity element 1.*

(iv) $(x + y) z = xz + yz$ *for all x, y, z, in R.*

(v) *If a, b, c are elements of R, with $a \neq b$, there exists one and only element x of R such that $xa = xb + c$.*

The axioms (i) through (v) characterize a so-called Veblen-Wedderburn system (or quasi field). See, for example, refs. *3, 4,* and *7.* However, in the present case we have additional properties. Obviously

$$f(x + y) = fx + fy \tag{6.5}$$

and

$$(fx)\, y = f(xy) \tag{6.6}$$

for every f in the skew field F and for all x, y in R. In addition, if 1 is the identity element of R, we see from (6.6) that

$$(f1)\, x = fx$$

for all f in F, x in R, and from (6.5), (6.6) that

$$(f + g)\, 1 = f1 + g1, \qquad (fg)\, 1 = (f1)(g1)$$

for all f, g in F. Consequently, the mapping

$$f \to f1$$

is an operator isomorphism of F upon a skew field $F1$ which is a subsystem of $(R, +, \cdot)$. Therefore we may imbed F as a subskew field of $(R, +, \cdot)$ with properties (6.5), (6.6) by making the identification $f = f1$ for every f in F. At this point we need a known lemma:

LEMMA 6.1.   *Let $(R, +, \cdot)$ be a Veblen-Wedderburn system and let F be the set of all elements f in R which satisfy (6.5), (6.6) for all x, y in R. Then the subsystem $(F, +, \cdot)$ of $(R, +, \cdot)$ is a skew field.*

DEFINITION.   We shall call the skew field F of Lemma 6.1 the *leftoperator skewfield* of the Veblen-Wedderburn system $(R, +, \cdot)$.

Proof.   With each element x of R we associate a mapping, $R(x)$, of R, the *right multiplication* by x, defined by

$$yR(x) = yx \tag{6.7}$$

for all y in R. In view of axiom (iv), each $R(x)$ is an endomorphism of the abelian group $(R, +)$. By axiom (iii), if a, b are elements of R with $a \neq 0$, there exists a unique x in R such that $aR(x) = b$. Hence the set $\mathscr{R}$ of right multiplications of R is an irreducible set of endomorphisms of $(R, +)$. Therefore, by Schur's Lemma, the centralizer, $\mathscr{R}^*$, of $\mathscr{R}$ in the ring of all endomorphisms of $(R, +)$, is a skew field. An endomorphism, θ, of $(R, +)$ is in $\mathscr{R}^*$ if and only if

$$(y\theta) x = (yx) \theta \tag{6.8}$$

for all y, x in R. Setting $y = 1$ in (6.8), we get

$$x\theta = fx \tag{6.9}$$

for all x, where $f = l\theta$. From (6.9) in (6.8), we get

$$(fy) x = f(yx)$$

for all y, x in R. That is, (6.6) holds. In addition, since θ is an endomorphism of $(R, +)$, then (6.5) holds. Conversely, if f satisfies (6.5), (6.6) and if θ is defined by (6.9), then θ is an endomorphism of $(R, +)$ which satisfies (6.8). If, further, g satisfies (6.5), (6.6) and if φ is defined by

$$x\varphi = gx,$$

then

$$x(\theta + \varphi) = x\theta + x\varphi = fx + gx = (f + g) x$$

and

$$x(\theta\varphi) = (x\theta) \varphi = g(fx) = (gf) x.$$

Consequently the system $(F, +, \cdot)$ is a skew field antiisomorphic to the skew field $\mathscr{R}^*$.

Now we may add our crucial axiom:

(vi)   *The Veblen-Wedderburn system* $(R, +, \cdot)$ *is a finite-dimensional vector space over its left-operator skew-field.*

We note that, although the skew field F from which we started is not necessarily the full left-operator skew field of $(R, +, \cdot)$, nevertheless, $(R, +)$ must, a fortiori, have finite dimension over its left-operator skew field.

Axiom (vi) raises a question to which the answer is probably unknown: *Has every Veblen-Wedderburn system finite dimension over its left-operator skew field?*

7. TRANSLATION PLANES

For an axiomatic characterization of translation planes, see ref. *4*. We need merely say here that a projective plane π^* is a translation plane with respect to one of its lines, L, if and only if the corresponding affine plane π, obtained from π^* by deleting L and its points, can be coordinatized by a Veblen-Wedderburn system. It will be convenient here to speak of the affine plane π as an affine translation plane. Now we may state a theorem:

THEOREM 7.1.   *Every affine plane $\pi(\Sigma, \Sigma', S)$, constructed as in Section 4, is a translation plane. Conversely, if π is an affine translation plane with a coordinating Veblen-Wedderburn system which is finite dimensional over its leftoperator skew field, then π is isomorphic to at least one plane $\pi(\Sigma, \Sigma', S)$.*

COROLLARY.   *Every finite affine translation plane is isomorphic to at least one plane $\pi(\Sigma, \Sigma', S)$.*

Proof. We need only concern ourselves with the second sentence of Theorem 7.1. Suppose then that π is coordinatized by a Veblen-Wedderburn system $(R, +, \cdot)$. Suppose also that $(R, +, \cdot)$ has a subsystem, $F = (F, -, \cdot)$, such that F is a skew field contained in (but not necessarily equal to) the left-operator skew field of $(R, +, \cdot)$ and such that R is a t-dimensional vector space over F, where t is a positive integer. It is to be understood that the points of π are ordered pairs (x, y), $x, y \in R$, and that the lines of π belong to two types: (I) the lines $x = a$, one for each a in R; (II) the lines $y = xm + b$, one for each ordered pair of elements, m, b in R. We shall omit the verification that π is indeed an affine plane.

It will be convenient to speak of R as a t-dimensional vector space over F. Next we introduce a second vector space R', isomorphic to R over F, but having only the zero vector in common with R. Then we define the vector space

$$W = R + R',$$

the direct sum of R and R', whose zero element coincides with that of R and R'. We shall understand that the mapping

$$x \rightarrow x'$$

is an isomorphism of R upon R' over F. Then W is a $(2t)$-dimensional vector space over F. We define

$$J(\infty) = R$$

and, for each element m of R, we define $J(m)$ to be the set of all vectors of form

$$xm + x'$$

where x ranges over R. Then $J(m)$ is a t-dimensional subspace of W over F and

$$J(\infty) \cap J(m) = \{0\}$$

for each m. Indeed, $xm + x'$ is in $J(\infty) = R$ if and only if $x = 0$. Similarly,

$$J(m) \cap J(k) = \{0\} \qquad \text{if} \qquad m \neq k,$$

for if

$$xm + x' = yk + y'$$

then

$$xm - yk = (y - x)' = 0$$

whence $y = x$ and

$$0 = xm - xk.$$

Since $m \neq k$, the latter equation has the unique solution $x = 0$. Next, let

$$w = y + x' \neq 0$$

be an arbitrary nonzero element of W. If (and only if) $x = 0$, w is in $J(\infty) = R$. On the other hand, if $x \neq 0$, there is one and only one m in R such that $xm = y$; and, for this (and only this) choice of m, w is in $J(m)$.

Consequently, if S is the collection consisting of $J(\infty)$ and the $J(m)$, $m \in R$, then S is (in vector form) a spread.

Next we introduce a 1-dimensional vector space $\{e^*\}$ over F, having only 0 in common with W, and define

$$V = W + \{e^*\},$$

where the direct sum is understood. Thus V is a $(2t + 1)$-dimensional vector space over F.

At this point, we define $\Sigma = \Sigma(V/F)$ to be the usual $(2t)$-dimensional projective space, take $\Sigma' = \Sigma(W/F)$ to be the corresponding $(2t - 1)$-dimensional projective subspace of Σ, and use the spread S, just defined, as a spread of Σ'. It is now a simple matter to verify that the affine plane $\pi(\Sigma, \Sigma', S)$ is isomorphic to the plane π from which we started. This completes the proof of Theorem 7.1.

Clearly we have made very little use, in the foregoing proof, of the fact that R has finite dimension t over F. However, without this restriction, the projective space Σ is infinite dimensional, and so is Σ'. And now we see the problem: How do we define the concept of a spread of an infinite-dimensional projective space?

For specific examples of Veblen-Wedderburn systems—and hence for examples of spreads—see ref. *4*.

8. SOME GEOMETRIC EXAMPLES

In order to avoid giving the impression that the only way to construct spreads is by using Veblen-Wedderburn systems, we shall sketch briefly, without proof, part of a geometric theory developed by Bruck for spreads of $PG(3, q)$.

We recall that a spread of $PG(3, q)$ is a set of $1 + q^2$ skew lines of $PG(3, q)$.

If A, B, C are three distinct skew lines, the set, $\mathscr{R}'$, of transversals to A, B, C, consists of $q + 1$ skew lines. Each transversal to three members of $\mathscr{R}'$ is a transversal to all the members of $\mathscr{R}'$. The set, $\mathscr{R} = \mathscr{R}(A, B, C)$, consisting of the $q + 1$ transversals to $\mathscr{R}'$, may be called the *regulus* containing A, B, C, and $\mathscr{R}'$ may be called the *opposite regulus* to $\mathscr{R}$. The points of the lines of $\mathscr{R}$ are the same as the points of the lines of $\mathscr{R}'$; and these $(q + 1)^2$ points constitute a *doubly-ruled quadric*, $\mathscr{Q} = \mathscr{Q}(A, B, C)$, with $\mathscr{R}$ and $\mathscr{R}'$ as its two reguli.

A spread, S, of $PG(3, q)$ will be called *regular* provided that, for every three (necessarily skew) lines A, B, C belonging to S, S contains every line of the regulus $\mathscr{R}(A, B, C)$. Note that if D is one of the $q^2 - q$ lines which is in S but not in $\mathscr{R}(A, B, C)$, then D is skew to every line of $\mathscr{R}(A, B, C)$. It may be show that if A, B, C are three skew lines and if D is any line skew to every member of $\mathscr{R}(A, B, C)$, there is one and only one regular spread S containing A, B, C and D. As a consequence, *three skew lines A, B, C are contained in precisely $(q^2 - q)/2$ distinct regular spreads, and each two of these spreads have precisely $\mathscr{R}(A, B, C)$ in common.*

The case $q = 2$ is exceptional. Every spread of $PG(3, 2)$ is regular; and three skew lines A, B, C of $PG(3, 2)$ are contained in precisely one spread. In the rest of the discussion we assume $q > 2$.

Let S be a spread of $PG(3, q)$, $q > 2$, which happens to contain a regulus $\mathscr{R}$. If S' is derived from S by replacing the regulus $\mathscr{R}$ by the opposite regulus $\mathscr{R}'$, then S' is a spread. Moreover, *if one of S, S' is regular, the other is not.* It seems reasonable to conjecture that every spread may be obtained from a regular spread by iteration of the process of replacing a regulus by the opposite regulus. The conjecture is correct for $q = 3$.

Next suppose that $\mathscr{R}$, $\mathscr{R}'$ are opposite reguli of $PG(3, q)$, belonging to a quadric $\mathscr{Q}$. It may be shown that *if S, S' are regular spreads containing $\mathscr{R}$, $\mathscr{R}'$ respectively, then $S \cap S'$ consists of precisely two lines A, B, namely of a pair of conjugate nonsecants of the quadric $\mathscr{Q}$.*

Again, let A, B be two arbitrarily chosen lines of a regular spread S of $PG(3, q)$. It may be shown that *the remaining $q^2 - 1$ lines of S are partitioned into $q - 1$ disjoint reguli $\mathscr{R}_1$, $\mathscr{R}_2$, $\cdots$, $\mathscr{R}_{q-1}$ uniquely defined by the requirement that, for each i, A, B are conjugate nonsecants to the quadric $\mathscr{Q}_i$ with reguli $\mathscr{R}_i$, $\mathscr{R}_i'$.* As a consequence, by combining A, B with one of $\mathscr{R}_i$, $\mathscr{R}_i'$ for each i, we get 2^{q-1} distinct spreads, many of which are nonregular for q large. If $q = 3$, there are 4 spreads, of which 2 are regular and 2 are nonregular.

The analytic formulation of the above remarks is quite interesting. We use the notation of Section 6, with $F = GF(q)$ and $t = 2$, except that, to emphasize the fact that we are dealing with lines, we use L instead of J. Then

$$L(\infty) = \{e_1, e_2\}, \qquad L(0) = \{e_1', e_2'\}, \tag{8.1}$$

$$L(X) = \{x_{11}e_1 + x_{12}e_2 + e_1', x_{21}e_1 + x_{22}e_2 + e_2'\}, \tag{8.2}$$

where X is a matrix of two rows and columns over $GF(q)$.

If $\mathscr{R}$ is the regulus determined by $L(\infty)$, $L(0)$, $L(I)$, then $\mathscr{R}$ consists of $L(\infty)$ and the lines $L(aI)$ crorresponding to the scalar matrices aI, $a \in GF(q)$. Each regular spread containing $\mathscr{R}$ consists of $L(\infty)$ and lines $L(T)$ where T ranges over q^2 matrices forming a field isomorphic to $GF(q^2)$. Equivalently, T ranges over q^2 matrices of form

$$aI + bX, \qquad a, b \in GF(q)$$

where X is a fixed (but arbitrarily chosen) irreducible matrix. Another type of spread consists of $\mathscr{R}$ and $q^2 - q$ lines $L(Y)$ where Y ranges over the $q^2 - q$ distinct conjugates

$$P^{-1}XP$$

of a fixed (but arbitrarily chosen) irreducible matrix X. The latter spread is not regular, but becomes regular when $\mathscr{R}$ is replaced by R'.

Of the two types of spread just described, the regular spread corresponds to a Veblen-Wedderburn system which is a field, and hence corresponds

(according to our construction) to a Desarguesian plane. On the other hand, the nonregular spread corresponds to a Veblen-Wedderburn system which is a Hall system, and hence corresponds (according to our construction) to a Hall plane.

Returning again to the regular spread defined above, we may verify that the $q - 1$ reguli $\mathscr{R}_1$, $\mathscr{R}_2$, $\cdots$, $\mathscr{R}_{q-1}$ of the spread, with respect to which $L(\infty), L(0)$ are conjugate nonsecants, consist, for each i, of the lines $L(aI + bX)$ such that the determinant

$$| aI + bX |$$

has a constant value $d_i \neq 0$. Clearly, if q is even, each $\mathscr{R}_i$ has a unique line in common with $\mathscr{R}$, while, if q is odd, half of the $\mathscr{R}_i$ have a unique line in common with $\mathscr{R}$ and the other half are disjoint from $\mathscr{R}$.

So far we have no satisfactory formulation of a geometric theory of spreads of $(2t - 1)$-dimensional projective space for the case $t > 2$.

9. A CONNECTION WITH A PROCEDURE OF OSTRUM

In a series of papers, of which we shall mention only two, Ostrum [5, 6] has developed a procedure for constructing from a given plane π of finite order n^2 a "conjugate" plane π' which is symmetrically related to π but usually has different properties. We explain the procedure in its affine form:

Let π be an affine plane of finite order n^2 possessing a collection, K, made up of all the lines of some $n + 1$ parallel classes of lines of π, such that, to every pair, P, Q, of distinct points of π which are joined in π by a line, PQ, in K, there corresponds an affine subplane, $(PQ)'$, of π with the following properties: (i) $(PQ)'$ contains P and Q; (ii) $(PQ)'$ has order n; (iii) the lines of $(PQ)'$ are all in K. (There can be at most one subplane $(PQ)'$ with properties (i), (ii), (iii).) Let K' be the collection consisting of the subplanes $(PQ)'$. Let π' be the system obtained from π by retaining the points of π and the lines of π which are not in K but replacing the lines PQ in K by the subplanes $(PQ)'$ in K'—and using the latter as lines. Then π' is an affine plane of order n^2. Moreover, K' consists of all the lines of some $n + 1$ parallel classes of π'; and K is a collection of affine subplanes of order n of π', one for each pair, P, Q of distinct points of π' such that the line $(PQ)'$ of π' is in K'.

Now let us apply Ostrum's procedure to an affine translation plane $\pi = \pi(\Sigma, \Sigma', S)$ of order $n^2 = q^2$ constructed by the method of Section 4 from the projective 4-space $\Sigma = PG(4, q)$. Here $\Sigma' = PG(3, q)$ is a projective 3-space of Σ and S, a spread of Σ', consists of $q^2 + 1$ skew lines of Σ'. We recall that the points of π are the points of Σ which are not in Σ' and that the lines of π are the planes of Σ which are not in Σ' but meet Σ' in a line of S.

First, consider any plane, T, of Σ which is not in Σ' and does not meet Σ' in a line of S. Then T meets Σ' in a line, L, which is not in S. To L corresponds a set, C, of $q + 1$ lines of S, one through each point of L. Let (T) be the system consisting of the q^2 points of T which are not in Σ' and of the $q(q + 1)$ planes of Σ containing a line of C and at least one point of (T). It is easy to see that (T) is an affine subplane of π of order q. Hence π has affine subplanes of order q in rich profusion.

Next suppose that the spread S happens to contain a regulus, $\mathscr{R}$. In the sense of Ostrum's procedure, let K be the collection consisting of all planes of Σ which are not in Σ' and which meet Σ' in a line of $\mathscr{R}$. Then K consists of all the lines of some $q + 1$ parallel classes of lines of π. And—in view of the preceding paragraph—Ostrum's procedure amounts, in this case, to changing the spread S by replacing the regulus $\mathscr{R}$ by the opposite regulus $\mathscr{R}'$. In particular, the conjecture about spreads of $PG(3, q)$ mentioned in Section 8 could be rephrased as a conjecture that, by iteration of Ostrum's procedure, any translation plane obtainable by our construction from $PG(4, q)$ could be derived from a Desarguesian plane.

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