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Finite-size, magnetic and chemical-potential effects on first-order phase transitions

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ABSTRACT

We perform a study about effects of an applied magnetic field and a finite chemical potential on the size-dependent phase structure of a first-order transition. These effects are introduced by using methods of quantum fields defined on toroidal spaces, and we study in particular the case of two compactified dimensions, imaginary time and a spatial one (a heated film). It is found that for any value of the applied field, there is a minimal size of the system, independent of the chemical potential, below which the transition disappears.

1. Introduction

On general grounds, systems defined on spaces or spacetimes with some of its dimensions compactified are of interest in several branches of physics, such as statistical, condensed-matter, and particle physics. A development of this kind, which has its roots in the late 1950s, is the systematic approach to quantum field theory at finite temperature [1], as an imaginary-time formalism. In this formalism, the so-called Matsubara prescription states that the momentum conjugate to imaginary time is replaced by frequencies $2n\pi/\beta$ or $2(n+1/2)\pi/\beta$ for bosons or fermions, respectively, corresponding to the period $\beta = T^{-1}$, with T being the temperature. Further developments, as for instance in Refs. [2–5], allowed to give to the imaginary-time approach a topological interpretation. It has been shown that the temperature can be introduced by writing the original theory, formulated in the Euclidean space $\mathbf{R}^4$, in the compactified manifold $\Gamma_4^1 = S^1 \times \mathbf{R}^3$, where the compactified dimension is the imaginary time. The circumference of S^1 is β.

An analogous formalism can be constructed for compactified spatial coordinates, in a D-dimensional Euclidean space. In this case, one can describe systems confined to limited regions of space. This is an idea first advanced in [6] and we are faced with systems defined on spaces with compactified spatial coordinates. One takes then a modified Matsubara prescription in which β is replaced by compactification lengths L_i, $i = 1, \ldots, d$, for each bounded spatial coordinate. As is argued in Ref. [7], this can be interpreted as the system being confined to a d-dimensional parallelepiped embedded in the D-dimensional space. Temperature may be then introduced, as in the Ginzburg–Landau model, through the mass parameter of the Hamiltonian. By taking $D = 3$ and respectively $d = 1, 2, 3$, this can be interpreted as samples of a superconducting material in the form of a film, a wire, or a grain [7].

Since then, progress has been done, in particular to treat jointly spatial compactification [6,7] and the introduction of finite temperature. Recently, in Ref. [8] general algebraic foundations have been presented in this sense, to include concurrently, not only temperature, but also spatial coordinates, in such a way that any set of dimensions of the manifold $\mathbf{R}^D$ can be compactified. One then defines a theory in the topology $\Gamma_D^d = (S^1)^d \times \mathbf{R}^{D-d}$, with $1 \leqslant d \leqslant D$, d being the number of compactified dimensions. Each of these compactified dimensions has the topology of a circle and we refer in general to Γ_D^d as a *toroidal* topology. These ideas, in a simpler form, were already present in Ref. [9] and were applied to the study of spontaneous symmetry breaking/restoration induced by both temperature and spatial boundaries. In their more modern presentation, these methods have been recently employed to investigate several aspects of first and second-order phase transitions in both bosonic and fermionic systems [10–15]. In this framework, here we intend to concurrently study effects of a finite chemical potential and of an applied external magnetic field on the size-dependent phase structure of a first-order transition. Some physical motivations for such a study are given along this introductory section, with several references which testify of the interest of finite-size effects on phase transitions.

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In Ref. [16] the Euclidean large-N Ginzburg–Landau model in D dimensions, d of them being compactified, has been considered. The fixed-point structure of the model is investigated on general grounds, in the presence of an external magnetic field. An infrared-stable fixed point has been found, being independent of the number of compactified dimensions, but for the space dimension D in the range $4 < D < 6$. This could be related to studies of extra-dimension effects in both high and low energy physics [17,18]. In condensed-matter and statistical-physics contexts, as discussed in Refs. [19,20] for systems in bulk form, the fixed point mentioned above should be taken as an indication, not as a demonstration, of a (formal) continuous transition. This has been confirmed for a system in the form of a film in Ref. [21]. The existence of an infrared fixed point in the presence of a magnetic field, as found in Ref. [16], does not assure the (formal) existence of a second-order transition. In any case, for compactified systems under the action of an external magnetic field, as is also the case for systems in bulk form, a phase transition for $D \leqslant 4$, in particular in $D = 4$ or $D = 3$, should not be a second-order one. This furnishes us a motivation to study first-order phase transitions in the presence of a magnetic background, as is done in this Letter. In this sense the present Letter may be seen as an extension including concurrently finite-size, magnetic and chemical potential effects, of previous works on first-order phase transitions that have already been performed for superconducting materials under the form of films and wires [22,23].

There are many potentials that describe first-order transitions both in bulk and film-like systems, for instance, the Halperin–Lubensky–Ma potential, engendered by integrating out gauge-field modes [24,25]. In this Letter, we will remain in a somehow less sophisticated framework and will consider a potential of the Ginzburg–Landau type, $-\lambda\varphi^4 + \eta\varphi^6$ ($\lambda > 0$, $\eta > 0$), which allows that the system undergoes a first-order transition. However, this study will be done in the spirit of an application of the above mentioned developments for field theories defined on toroidal spaces [8], including finite-temperature field theory ideas and compactification of spatial coordinates, not using the Ginzburg–Landau approximation of considering a linear behavior of the mass term of the Hamiltonian with the temperature. We perform a study of concurrent effects of a finite chemical potential and of an applied external magnetic field, on the size-dependent phase structure of a first-order transition. Our main concern will be to analyze the model within a field-theoretical approach, as applied to statistical and condensed-matter physics. We will consider the particular case of two compactified dimensions ($d = 2$), related to finite temperature and one compactified spatial coordinate, with compactification length L. From a condensed-matter physical point of view, we can think of this system as a heated film of thickness L, undergoing a first-order phase transition under the influence of an applied magnetic field.

We remember that Hamiltonian densities, when taken in the Ginzburg–Landau approximation for temperatures around a given fixed temperature parameter, are currently employed to describe systems (for instance, superconductors) in the absence or the presence of a magnetic background. This has been the case in which this approximation has been employed to perform studies of superconducting films in a magnetic field in Refs. [21,26]. Here, instead of introducing temperature via the mass term, as in the Ginzburg–Landau approximation, we will consider the system in the framework of finite-temperature field theory, with m_0^2 being a fixed squared mass parameter; within this formalism, the model is valid for the whole domain of temperatures, $0 \leqslant T < \infty$.

In this case, we start from the scalar field model described by the following Hamiltonian density in a Euclidean D-dimensional space, including both φ^4 and φ^6 interactions, at zero temperature, in the absence of boundaries and in the presence of an external field (in natural units, $\hbar = c = k_B = 1$):

$$\mathcal{H} = |D_\mu\varphi|^2 + m_0^2|\varphi|^2 - \frac{\lambda}{4}|\varphi|^4 + \frac{\eta}{6}|\varphi|^6. \tag{1}$$

In the above equation, m_0^2 is a *physical* squared mass parameter, $\lambda > 0$ and $\eta > 0$ are, respectively, *physical* quartic and sextic self-coupling constants, all at zero temperature and in the absence of spatial compactification; these quantities are taken as fixed parameters which define the model. Actually, we will define dimensionless quantities in terms of m_0, and only λ and η will be adjustable parameters. The symbol D stands for the covariant derivative, $D_\mu = \partial_\mu - ieA_\mu^{\text{ext}}$, and A_μ^{ext} is an external gauge field.

2. Zero-temperature magnetic effects in the absence of spatial boundaries

In the D-dimensional space with Cartesian coordinates x_1, $x_2, \ldots, x_D$, following Ref. [27], we choose a gauge such that $A^{\text{ext}} = (0, xH, 0, \ldots, 0)$ (to simplify notation we take $x_1 \equiv x$), where H is the applied constant magnetic field, parallel to the $x_3 \equiv z$ axis. In this case, the part of the Hamiltonian $\int d^D r\, \mathcal{H}$ quadratic in φ becomes, after an integration by parts, $-\int d^D r\, \varphi^* \mathcal{D}\varphi$, where the differential operator $\mathcal{D}$ is

$$\mathcal{D} = \nabla^2 - 2i\omega x\partial_y - \omega^2 x^2 - {m_0}^2, \tag{2}$$

with $\omega = eH$ being the cyclotron frequency. Thus the natural basis to expand the field operators is the set of the normalized eigenfunctions of the operator $\mathcal{D}$, the Landau basis,

$$\chi_{\ell,p_y,k}(r) = \frac{1}{\sqrt{2^\ell \ell!}}\left(\frac{\omega}{\pi}\right)^{\frac{1}{4}} e^{ik\cdot\mathcal{Z}} e^{i\omega p_y y} e^{-\omega(x-p_y)^2/2} H_\ell\big[\sqrt{\omega}(x-p_y)\big], \tag{3}$$

where $r = (x, y, \mathcal{Z})$ and H_ℓ are the Hermite polynomials; the corresponding energy eigenvalues are (the subscript ℓ denotes the Landau levels) $E_\ell(k) = k^2 + (2\ell+1)\omega + m_0^2$; k and $\mathcal{Z}$ are conjugate momentum and space $(D-2)$-dimensional vectors, respectively. The free propagator is written as [27]

$$\mathcal{G}(r, r') = \int \frac{d^{D-2}k}{(2\pi)^{D-2}} \int dp_y\, \omega \sum_{\ell=0}^{\infty} \frac{\chi_{\ell,p_y,k}(r)\chi^*_{\ell,p_y,k}(r')}{k^2 + (2\ell+1)\omega + m_0^2}. \tag{4}$$

The non-translational-invariant phase of the propagator (4) can be isolated as in Ref. [27], and we can write

$$\mathcal{G}(r, r'; \omega) = e^{i\omega(x+x')(y-y')/2}\bar{\mathcal{G}}(r - r'; \omega), \tag{5}$$

where $\bar{\mathcal{G}}(r-r';\omega)$ is the translationally invariant part; the momentum-space propagator can be obtained from Eqs. (4) and (5), by inserting Eq. (3) into Eq. (4) and then considering $r = r'$. This will be fully justified at the next section, where we will consider contributions to the effective potential coming from only two kinds of daisy diagrams, for which we need to consider just the coincidence limit $r = r'$.

Then we write

$$\begin{aligned}\mathcal{G}(r, r) &= \int \frac{d^{D-2}k}{(2\pi)^{D-2}} \int_{-\infty}^{+\infty} dp_y \left(\frac{\omega}{\pi}\right)^{\frac{1}{2}} e^{-\omega(x-p_y)^2} \\ &\quad\times \sum_{\ell=0}^{\infty} \frac{1}{2^\ell \ell!}\big[H_\ell(\sqrt{\omega}(x-p_y))\big]^2 \frac{\omega}{k^2 + (2\ell+1)\omega + m_0^2} \\ &\equiv \int \frac{d^{D-2}k}{(2\pi)^{D-2}} \mathcal{G}(k, \omega).\end{aligned} \tag{6}$$

In the above equation, two of the dimensions are taken into account by the introduction of the sum over the Landau levels and the incorporation of the cyclotron frequency; then, by definition, $\mathcal{G}(k,\omega)$ is the free propagator in the remaining $(D-2)$-dimensional momentum space. Using the orthonormality relations for the Hermite polynomials, $\int_{-\infty}^{+\infty} du\, H_n(u) H_m(u) \exp(-u^2) = \sqrt{\pi}\, 2^n n!\, \delta_{nm}$, we obtain straightforwardly the $(D-2)$-dimensional free propagator in momentum space in the presence of a magnetic field,

$$\mathcal{G}(k,\omega) = \sum_{\ell=0}^{\infty} \frac{\omega}{k^2 + (2\ell+1)\omega + m_0^2}; \tag{7}$$

this is to be used in the $(D-2)$-dimensional space, in an entirely analogous manner as in dimension D in the absence of a field.

To be more precise, let us remind that, in general, in a D-dimensional non-compactified Euclidean space in the absence of external field, the Feynman amplitude for a diagram G in a scalar field theory has an expression of the form (omitting external constant factors and symmetry coefficients),

$$A_G^{(D)}(\{p\}) = \int \prod_{i=1}^{I} \frac{d^D q_i}{(2\pi)^D} \prod_{i=1}^{I} \frac{1}{q_i^2 + m_0^2} \prod_{v=1}^{V} \delta\Big(\sum_i \epsilon_{vi} q_i\Big), \tag{8}$$

where $\{p\}$ stands for the set of external momenta, V is the number of vertices, I is the number of internal lines and q_i stands for the momentum of each internal line i. The quantity ϵ_{vi} is the *incidence matrix*, which equals 1 if the line i arrives at the vertex v, -1 if it starts at v, and 0 otherwise. Performing the integrations over the internal momenta leads to a choice of independent loop-momenta $\{k_\alpha\}$ and we get

$$A_G^{(D)}(\{p\}) = \int \prod_{\alpha=1}^{L} \frac{d^D k_\alpha}{(2\pi)^D} \prod_{i=1}^{I} \frac{1}{q_i^2(\{p\},\{k_\alpha\}) + m_0^2}, \tag{9}$$

where L is the number of independent loops. The momentum q_i is a *linear* function of the independent internal momenta k_α and of the external momenta $\{p\}$.

This means that, *taking into account all Landau levels*, calculations of a generic Feynman amplitude for a *daisy* diagram can be performed in the $(D-2)$-dimensional space using Eq. (7), in an entirely similar way as in the absence of the external field, *i.e*, performing, from Eq. (9) in momentum space, for the momentum integrations over the independent momenta $k_\alpha, \alpha = 1,2,\ldots,L$, and for the set of propagators corresponding to the internal lines $i, i = 1,2,\ldots,I$, the replacements

$$\begin{aligned}
&\int \prod_{\alpha=1}^{L} \frac{d^D k_\alpha}{(2\pi)^D} \;\to\; \int \prod_{\alpha=1}^{L} \frac{d^{D-2} k_\alpha}{(2\pi)^{D-2}}, \\
&\prod_{i=1}^{I} \frac{1}{q_i^2(\{p\},\{k_\alpha\}) + m_0^2} \\
&\quad\to \prod_{i=1}^{I} \sum_{\ell=0}^{\infty} \frac{\omega}{q_i^2(\{p\},\{k_\alpha\}) + (2\ell+1)\omega + m_0^2}.
\end{aligned} \tag{10}$$

This gives, for a generic Feynman amplitude of a daisy diagram, after taking into account the applied magnetic field, an expression of the form of an integral in the remaining $(D-2)$-dimensional momentum space,

$$\begin{aligned}
A_G^{(D)}(\{p\},\omega) = \int \prod_{\alpha=1}^{L} \frac{d^{D-2} k_\alpha}{(2\pi)^{D-2}} \prod_{i=1}^{I} \\
\times \left[\sum_{\ell=0}^{\infty} \frac{\omega}{q_i^2(\{p\},\{k_\alpha\}) + (2\ell+1)\omega + m_0^2} \right].
\end{aligned} \tag{11}$$

3. Effective potential at finite temperature and chemical potential, in the presence of boundaries, under the action of an external field

We consider the system under the influence of an external field, at temperature β^{-1}, and we compactify one of the spatial coordinates (say, x) with compactification length L. As is argued in Ref. [7], this can be considered as a heated system confined to a region of space delimited by a pair of parallel planes (a film of thickness L). As already noticed, under these conditions the system makes sense for dimensions $D \geqslant 4$. Taking into account the prescriptions (10) in a generic dimension, we use $(D-2)$-dimensional Cartesian coordinates $\mathcal{Z} = (\tau, x, \mathcal{W})$, where τ corresponds formally to the imaginary-time (inverse-temperature) coordinate, x to a spatial coordinate and $\mathcal{W}$ is a $(D-4)$-dimensional vector. The momentum conjugate to $\mathcal{Z}$ is $k = (k_\tau, k_x, \mathcal{Q})$, $\mathcal{Q}$ being a $(D-4)$-dimensional vector in momentum space. Then we follow the method described in Ref. [8] in the particular case $d = 2$, to treat jointly finite temperature and compactification of one spatial coordinate. This amounts to perform a double Matsubara prescription, one in imaginary time, as is done in finite-temperature field theory, and an analogous one in the x coordinate. We also consider a chemical potential μ associated to the thermal reservoir. Therefore, the Feynman rules should be modified according to

$$\begin{aligned}
&\int \frac{dk_\tau}{2\pi} \;\to\; \frac{1}{\beta} \sum_{n_\tau=-\infty}^{\infty}, \qquad k_\tau \;\to\; \frac{2n_\tau \pi}{\beta} - i\mu, \\
&\int \frac{dk_x}{2\pi} \;\to\; \frac{1}{L} \sum_{n_x=-\infty}^{\infty}, \qquad k_x \;\to\; \frac{2n_x \pi}{L},
\end{aligned} \tag{12}$$

where L is the size of the system and we remind that attention must be paid to the conditions in Eq. (10).

We consider in principle corrections to the mass,

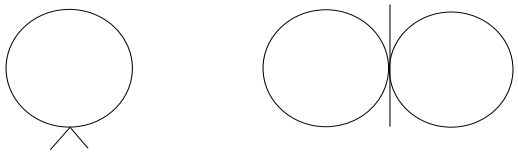
$$m^2(\beta,\mu,L,\omega) = m_0^2 + \Sigma(\beta,\mu,L,\omega), \tag{13}$$

and coupling constants, $\lambda(\beta,\mu,L,\omega) = \lambda_0 + \Pi(\beta,\mu,L,\omega)$ and $\eta(\beta,\mu,L,\omega) = \eta_0 + \Xi(\beta,\mu,L,\omega)$. Then, a free-energy density of the Ginzburg–Landau type can be constructed,

$$\mathcal{F} = \mathcal{F}_0 + A\,|\varphi_0|^2 + B|\varphi_0|^4 + C|\varphi_0|^6, \tag{14}$$

where $A = m^2(\beta,\mu,L,\omega)$, $B = -\lambda(\beta,\mu,L,\omega)/4$ and $C = \eta(\beta,\mu,L,\omega)/6$ and where φ_0 is the vacuum expectation value of the field, $\varphi_0 = \langle 0|\varphi|0\rangle$, the classical field. For the sake of simplicity we will consider only corrections to the mass, the fixed coupling constants λ and η will be taken as the physical ones.

Our analysis starts from the effective potential, which is related to the physical mass through a renormalization condition. In principle, the effective potential is obtained, following the analysis introduced in Ref. [28], as an expansion in the number of loops in Feynman diagrams. Accordingly, to the free propagator and to the no-loop (tree) diagrams for both couplings, radiative corrections are added, with increasing number of loops. Thus, at the 1-loop approximation, we get the infinite series of 1-loop diagrams with all numbers of insertions of the φ^4 vertex (two external legs in each vertex), plus the infinite series of 1-loop diagrams with all numbers of insertions of the φ^6 vertex (four external legs in each vertex), plus the infinite series of 1-loop diagrams with all kinds of mixed numbers of insertions of φ^4 and φ^6 vertices. Analogously, we should include all those types of insertions in diagrams with two loops, etc. This is an extremely hard task; instead of undertaking this computation, in our approximation we restrict ourselves to the lowest terms in the loop expansion. The renormalization condition giving the physical mass then reduces considerably the number of relevant Feynman diagrams, if we restrict ourselves to



(a) Tadpole diagram (b) Shoestring diagram

Fig. 1. Contributions to the effective potential.

first-order terms in both coupling constants. In this case, just two diagrams need to be considered in this approximation: a tadpole graph with the φ^4 coupling (one loop) and a "shoestring" graph with the φ^6 coupling (two loops), as depicted in Fig. 1. No diagram with both couplings occur. The effects of temperature, finite size, chemical potential and magnetic field appear from the treatment of the loop integrals.

The gap equation we are seeking is given by the condition in which the physical squared mass is defined as the second derivative of the effective potential $U(\varphi_0)$ with respect to the classical field $|\varphi_0|$, taken at zero value,

$$\left.\frac{\partial^2 U(\varphi_0)}{\partial|\varphi_0|^2}\right|_{|\varphi_0|=0} = m^2, \tag{15}$$

where we remind that m is the *physical* mass. In our case, we will have a β, L, μ and ω-dependent squared mass, $m^2 = m^2(\beta, L, \mu, \omega)$.

Within our approximation, we do not take into account the thermal and boundary corrections for the interaction coupling constants. As already stated, they were considered as physical quantities when they were written in the Hamiltonian at the starting point, being fixed parameters of the model.

3.1. The tadpole contribution

At the one-loop approximation, the contribution from the diagram with only one $|\varphi_0|^4$ vertex (the *tadpole*) to the effective potential, in the presence of a magnetic field, is obtained from the one-loop contribution to the zero-temperature effective potential in unbounded space, as an adaptation of the expression in Ref. [28], taking into account the modified propagator in Eq. (7),

$$U_1(\varphi_0) = \sum_{s=1}^{\infty} \frac{(-1)^{s+1}}{2s}\left[\frac{\omega\,(-\lambda)|\varphi_0|^2}{2}\right]^s \times \int \frac{d^{D-2}k}{(2\pi)^{D-2}}\left[\sum_{\ell=0}^{\infty}\frac{1}{[k^2+m_\ell^2(\omega)]^s}\right], \tag{16}$$

where we introduce the notation $m_\ell^2(\omega) \equiv m_0^2 + (2\ell+1)\omega$. As the parameter s counts the number of φ^2 insertions on the loop, the tadpole contribution comes from only the $s=1$ term of the sum in Eq. (16). However, due to analytic continuations that will be made in the following, the value of $s=1$ will be taken only at the end of the calculation.

After changing variables in the integral, $k_i/2\pi \to k_i$, and putting $a_\tau = 1/\beta^2$, $a_x = 1/L^2$, the one-loop contribution to the effective potential carrying temperature and finite-size effects is obtained using Eq. (12), as a generalization of Eq. (16),

$$U_1(\varphi_0,\beta,L,\mu,\omega) = \sum_{s=1}^{\infty}\frac{(-1)^{s+1}}{2s}\left[\frac{\omega\,(-\lambda)|\varphi_0|^2}{2}\right]^s \frac{1}{\beta L}\frac{1}{(4\pi^2)^s} \times \sum_{\ell=0}^{\infty}\sum_{n_\tau,n_x=-\infty}^{\infty}\int \frac{d^{D-4}\,\mathcal{Q}}{[\mathcal{Q}^2 + a_\tau(n_\tau - \frac{i\beta}{2\pi}\mu)^2 + a_x n_x^2 + c_\ell^2]^s},$$

where $c_\ell^2 = m_\ell^2(\omega)/4\pi^2$.

The integral in the previous equation is calculated by a dimensional-regularization formula [29], so that the one-loop contribution to the effective potential can be put into the form

$$U_1(\varphi_0;\beta,L,\mu,\omega) = \sum_{s=1}^{\infty}\frac{(-1)^{s+1}}{2s}\left[\frac{\omega\,(-\lambda)|\varphi_0|^2}{2}\right]^s \frac{1}{\beta L}\frac{\pi^{(D-4)/2}}{(4\pi^2)^s}\frac{\Gamma(s-\frac{D-4}{2})}{\Gamma(s)} \times \sum_{\ell=0}^{\infty}\sum_{n_\tau,n_x=-\infty}^{\infty}\left[a_\tau\left(n_\tau - \frac{i\beta}{2\pi}\mu\right)^2 + a_x n_x^2 + c_\ell^2\right]^{(D-4)/2-s}. \tag{17}$$

The double sum in Eq. (17) may be recognized as one of the inhomogeneous Epstein–Hurwitz zeta functions [30,32], which gives to the one-loop contribution to the effective potential the expression,

$$U_1(\varphi_0;\beta,L,\mu,\omega) = \frac{1}{\beta L}\sum_{s=1}^{\infty} f(D,s)\left[\frac{\omega\,(-\lambda)|\varphi_0|^2}{2}\right]^s \times \sum_{\ell=0}^{\infty} Z_2^{c_\ell^2}\left(s-\frac{D-4}{2}; a_\tau, a_x; b_\tau, b_x\right), \tag{18}$$

where $b_\tau = i\beta\mu/2\pi$, $b_x = 0$, and

$$f(D,s) = \frac{\pi^{(D-4)/2}}{(4\pi^2)^s}\frac{(-1)^{s+1}}{2s\Gamma(s)}\Gamma\left(s-\frac{D-4}{2}\right). \tag{19}$$

The zeta functions can be analytically continued to the whole s-plane, leading to an expression for $Z_2^{c^2}$ of the general form,

$$Z_2^{c^2}\left(\nu;\{a_j\};\{b_j\}\right) = \frac{\pi\,|c|^{2-2\nu}\,\Gamma(\nu-1)}{\Gamma(\nu)\sqrt{a_1 a_2}} + \frac{4\pi^\nu |c|^{1-\nu}}{\Gamma(\nu)\sqrt{a_1 a_2}} \times \left[\sum_{j=1}^{2}\sum_{n_j=1}^{\infty}\cos(2\pi n_j b_j)\left(\frac{n_j}{\sqrt{a_j}}\right)^{\nu-1} K_{\nu-1}\left(\frac{2\pi c n_j}{\sqrt{a_j}}\right) + 2\sum_{n_1,n_2=-\infty}^{\infty}\cos(2\pi n_1 b_1)\cos(2\pi n_2 b_2)\left(\sqrt{\frac{n_1^2}{a_1}+\frac{n_2^2}{a_2}}\right)^{\nu-1} \times K_{\nu-1}\left(2\pi c\sqrt{\frac{n_1^2}{a_1}+\frac{n_2^2}{a_2}}\right)\right], \tag{20}$$

where $K_{\nu-1}(z)$ are modified Bessel functions of the second kind. For us, $a_1 = a_\tau, a_2 = a_x, b_1 = b_\tau, b_2 = b_x = 0$ and $\nu = s - (D-4)/2$. The first term in Eq. (20) is singular for even $D \geqslant 4$ and will be suppressed by a regularization procedure. This procedure is known as the zeta-function regularization and is well established, being largely employed for a long time in the context of the Casimir effect (see for instance [31]); mathematical foundations for this method are for instance in [32].

Let us remark that the physical zero-temperature coupling constants in the absence of boundaries λ and η have dimensions respectively, of $(\text{mass})^{4-D}$ and $(\text{mass})^{6-2D}$. We define the dimensionless coupling constants, λ', η'; we also define the reduced temperature t, reduced chemical potential γ, reduced inverse length of the system ξ, and the reduced magnetic field δ,

$$\lambda' = \frac{\lambda}{m_0^{4-D}}, \qquad \eta' = \frac{\eta}{m_0^{6-2D}}, \qquad t = \frac{T}{m_0}, \qquad \xi = \frac{L^{-1}}{m_0}, \qquad \gamma = \frac{\mu}{m_0}, \qquad \delta = \frac{\omega}{m_0^2}, \tag{21}$$

in such a way that we have, for any dimension D, the set of dimensionless parameters λ', η', t, γ, ξ and δ.

In terms of the dimensionless quantities above, after suppression of the singular term, putting s equal to 1, and using the symmetry property of Bessel functions $K_\alpha(z) = K_{-\alpha}(z)$, the tadpole contribution to the effective potential is given by

$$\widetilde{U}_1(\varphi_0; t, \xi, \gamma, \delta) = -\frac{\lambda'\delta m_0^2|\varphi_0|^2}{2(2\pi)^{\frac{D-2}{2}}}\mathcal{K}(t, \xi, \gamma, \delta), \tag{22}$$

where

$$\begin{aligned}
&\mathcal{K}(t, \xi, \gamma, \delta)\\
&= \sum_{\ell=0}^{\infty}\Bigg[\sum_{n=1}^{\infty}\cosh\left(\frac{\gamma n}{t}\right)\left(\frac{t\sqrt{1+(2\ell+1)\delta}}{n}\right)^{\frac{D-4}{2}}\\
&\quad\times K_{\frac{D-4}{2}}\left(\frac{n\sqrt{1+(2\ell+1)\delta}}{t}\right)\\
&\quad+\sum_{n=1}^{\infty}\left(\frac{\xi\sqrt{1+(2\ell+1)\delta}}{n}\right)^{\frac{D-4}{2}}K_{\frac{D-4}{2}}\left(\frac{n\sqrt{1+(2\ell+1)\delta}}{\xi}\right)\\
&\quad+2\sum_{n_1,n_2=1}^{\infty}\cosh\left(\frac{\gamma n_1}{t}\right)\left(\frac{\sqrt{1+(2\ell+1)\delta}}{\sqrt{\frac{n_1^2}{t^2}+\frac{n_2^2}{\xi^2}}}\right)^{\frac{D-4}{2}}\\
&\quad\times K_{\frac{D-4}{2}}\left(\sqrt{\frac{n_1^2}{t^2}+\frac{n_2^2}{\xi^2}}\sqrt{1+(2\ell+1)\delta}\right)\Bigg].
\end{aligned}\tag{23}$$

Notice that in dimension $D = 4$, the above expression is well-defined for the reduced chemical potential restrained to the domain $0 \leqslant \gamma < 1$. Indeed, using an asymptotic formula for large values of the argument z of the Bessel function, $K_0(z) \approx \sqrt{(\pi/2z)}\exp(-z)$, with $z = (n/t)\sqrt{1+(2\ell+1)\delta}$, we can see that, for large values of n, the argument of the first sum in Eq. (23) has the asymptotic form, for arbitrary values of the reduced applied field,

$$\begin{aligned}
&f_n(t, \gamma, \delta)\\
&\approx \frac{\sqrt{\pi t}}{\sqrt{2n\sqrt{1+(2\ell+1)\delta}}}\frac{1}{2}\Bigg[\exp\left(-\frac{n(\sqrt{1+(2\ell+1)\delta}-\gamma)}{t}\right)\\
&\quad+\exp\left(-\frac{n(\sqrt{1+(2\ell+1)\delta}+\gamma)}{t}\right)\Bigg].
\end{aligned}\tag{24}$$

The second term inside the square brackets of the above equation does not present any problem for the convergence of the sum over n for all values of $\gamma \geqslant 0$, but the first one implies that the sum over n can be convergent only if $0 \leqslant \gamma < \sqrt{1+(2\ell+1)\delta}$. In order to include arbitrarily small values of δ, we should restrain γ to the domain $0 \leqslant \gamma < 1$. A similar argument applies for the last term in Eq. (23).

3.2. The shoestring contribution

The two-loop shoestring diagram contribution to the effective potential is obtained using again the Matsubara-modified Feynman rule prescription for the compactified dimensions. In the absence of boundaries, at zero temperature, and not submitted to the action of an external field, the shoestring diagram contribution is simply given by the product, with the proper coefficients, of two tadpoles,

$$\widetilde{U}_2(\varphi_0) = \frac{\eta|\varphi_0|^2}{16}\left[\sum_{\ell=0}^{\infty}\int\frac{d^{D-2}q}{(2\pi)^{D-2}}\frac{1}{\mathbf{q}^2+m_\ell^2(\omega)}\right]^2. \tag{25}$$

Then, after steps analogous to those which have been done for $\widetilde{U}_1$, we have

$$\widetilde{U}_2(\varphi_0; t, \xi, \gamma, \delta) = \frac{\eta'\delta^2 m_0^2|\varphi_0|^2}{4(2\pi)^{D-2}}\big[\mathcal{K}(t, \xi, \gamma, \delta)\big]^2. \tag{26}$$

3.3. Critical temperature

We now take $m^2(t, \xi, \gamma, \delta) \equiv m'^2(t, \xi, \gamma, \delta)$ as *dimensionless*, measured in units of m_0^2. It is obtained from the condition (15) by using Eq. (21), that is, with the dimensionless coupling constants λ', η' and in terms of the reduced temperature, inverse size, chemical potential and external field. At the first order in the coupling constants λ' and η' it is given by

$$m'^2(t, \xi, \gamma, \delta) = \frac{\partial^2}{\partial|\varphi_0|^2}\widetilde{U}(\varphi_0; t, \xi, \gamma, \delta)\Big|_{|\varphi_0|=0}, \tag{27}$$

where $\widetilde{U} = \widetilde{U}_0 + \widetilde{U}_1 + \widetilde{U}_2$ and $\widetilde{U}_0$ stands for the tree-level approximation. Then, from Eqs. (22), (23), (26), and (27), we have

$$\begin{aligned}
m'^2(t, \xi, \gamma, \delta) &= 1 - \frac{\lambda'\delta}{(2\pi)^{\frac{D-2}{2}}}\mathcal{K}(t, \xi, \gamma, \delta)\\
&\quad+\frac{\eta'\delta^2}{2(2\pi)^{D-2}}\big[\mathcal{K}(t, \xi, \gamma, \delta)\big]^2.
\end{aligned}\tag{28}$$

As the temperature is lowered, the system approaches the symmetry-breaking region. Taking the full equation (28), with $\eta' > 0$ and $\lambda' > 0$, there is a possibility that the system undergoes a first-order phase transition. Besides these conditions, it is required that the minimum values of the free-energy density given by Eq. (14),

$$\begin{aligned}
\mathcal{F} &= \mathcal{F}_0 + m'^2(t, \xi, \gamma, \delta)\,|\varphi_0|^2 - \lambda'(\xi, \gamma, \delta)|\varphi_0|^4\\
&\quad+\eta'(t, \xi, \gamma, \delta)\,|\varphi_0|^6,
\end{aligned}\tag{29}$$

which occur for φ_0 satisfying $\eta'|\varphi_0|^5 - \lambda'|\varphi_0|^3 + m'^2|\varphi_0| = 0$, should be equal to $\mathcal{F}_0$, which can be fixed as zero without loss of generality; this leads to the critical condition,

$$m'^2(t_c, \xi, \gamma, \delta) = 3\big(\lambda'\big)^2/32\eta', \tag{30}$$

where the mass term is given by the full expression, Eq. (28), containing mass corrections at the first-order in λ' and η'.

The solution of Eq. (30) gives the reduced critical temperature t_c as a function of the reduced inverse size, chemical potential and applied field, $t = t_c(\xi, \gamma, \delta)$.

4. Magnetic and chemical potential effects on the size-dependent phase structure: Comments and conclusions

We fix ourselves in dimension $D = 4$. This corresponds to a heated film under the influence of an external field. In order to perform a qualitative analysis of the phase structure of the model, we take for the coupling constants the numerical values $\lambda' = 0.5$ and $\eta' = 0.05$. Our objective is to investigate the interplay of the simultaneous influences of a finite chemical potential and of an applied magnetic field on the critical temperature as a function of the size of the system.

Let us remind that an effect of the external field is of breaking the translational symmetry on two of the space dimensions, x and y, leaving a $(D-2)$-dimensional translationally invariant subspace. Nevertheless, our system remains defined on a D-dimensional space, although it is not globally translationally invariant. On the other hand, the general formalism of field theories in toroidal topologies is constructed for translationally invariant spaces. This has as a consequence that, if we want to introduce,

in the framework of field theories defined on toroidal spaces [8], finite temperature with chemical potential and finite-size effects, we should compactify two of the dimensions remaining in the $(D-2)$-dimensional subspace, in such a way that the theory will be valid for dimensions $D \geqslant 4$. In the case of dimension $D = 4$, the dimensions of the whole space are $\{x, y, z, \tau\}$. We take one of these dimensions, τ, corresponding, after compactification, to inverse temperature, the three others being spatial dimensions; then compactification of τ and of the z-coordinate (with compactification length L) makes our system, embedded in *three* spatial dimensions, have the form of a heated film of *finite* thickness L, under the influence of an applied magnetic field. Moreover, as we will see below, for each value of the (reduced) applied field, δ, the thickness of the film has a lower bound, $L_0(\delta)$, sustaining the transition, below which the transition disappears.

One may speculate on the physics below $L_0(\delta)$. Due to the symmetries of the problem in the two-dimensional limit achieved by taking $L \to 0$, it would be expected that the system could have the conditions for a Berezinsky–Kosterlitz–Thouless (BKT) transition. Actually, numerical calculations show that the equation for the critical temperature has no solution for $L < L_0(\delta)$, and so we are unable, within our formalism, to investigate this range of thicknesses. This means, in particular, that we cannot take the $L \to 0$ limit, in order to verify whether a BKT transition occurs.

In a simpler situation of a first-order transition in the absence of an applied magnetic field, a similar result was found by some of us in Ref. [23] in the context of a Ginzburg–Landau model. In this case, we have obtained an analytical expression for the critical temperature of a superconducting film, as function of its thickness. We found that our predicted curve for the critical temperature is in a relatively good agreement with experimental data, particularly for small film thicknesses. Both the theoretical curve and the experimental data suggest the existence of a minimal allowed thickness, below which no transition occurs.

In Fig. 2 we show the reduced critical temperature as a function of the reduced inverse size of the system, for several values of the reduced chemical potential at a fixed value of the reduced applied field, $\delta = 1.5$. We find, in particular, that there exists a minimal size of the system, L_0 (corresponding to a maximal reduced inverse size $\xi_0 \approx 3.20$), sustaining the existence of the transition. This minimal allowed size appears to be *independent* of the value of the chemical potential.

In Fig. 3 we plot the reduced critical temperature as a function of the reduced inverse size of the system, for several couples of values (γ, δ). We take for such couples the values $(0, 0.3)$, $(0.5, 0.3)$, and $(0.9, 0.3)$; $(0, 1.5)$, $(0.5, 1.5)$, and $(0.9, 1.5)$. We can infer from this figure that the pattern of Fig. 2 for $\delta = 0.5$, is reproduced for all values of the reduced applied field. For each couple of values (γ, δ) of the (reduced) chemical potential and applied field, there exists a minimal allowed size of the system, $L_0(\gamma, \delta)$ [corresponding to a maximal reduced inverse size $\xi_0(\gamma, \delta)$], below which there is no transition.

From Fig. 3 we can also see that the minimal allowed size of the system, $L_0(\gamma, \delta)$ [or the maximal allowed value of the reduced inverse size, $\xi_0(\gamma, \delta)$], is *independent* of the chemical potential for both values of the reduced applied field, $\delta = 0.3$ and $\delta = 1.5$. Actually, this conclusion is valid for all values of δ. This is not a trivially expected feature, but we can prove it by finding the solutions for $\xi_0(\gamma, \delta)$ directly from Eqs. (23) and (30) considering the limit $t \to 0$. Indeed, it should be noted that for $\xi = \xi_0(\gamma, \delta)$, the symmetry-breaking region disappears completely and we have a null critical temperature. Then $\xi_0(\gamma, \delta)$ is obtained by solving Eq. (30) for $t = 0$ using an argument similar to one that was used above to determine the allowed range of values of the reduced chemical potential. For $t \to 0$, we use again the asymptotic formula for large values of the argument of the Bessel function,

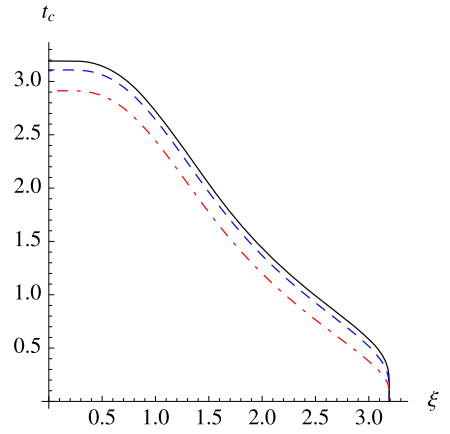


Fig. 2. Reduced critical temperature as a function of the reduced inverse size of the system for dimension $D = 4$, for the value of the reduced magnetic field $\delta = 1.5$. We fix $\lambda' = 0.5$ and $\eta' = 0.05$ and take for the reduced chemical potential the values $\gamma = 0.0$ (full line), 0.5 (dashed line) and 0.9 (dot-dashed line).

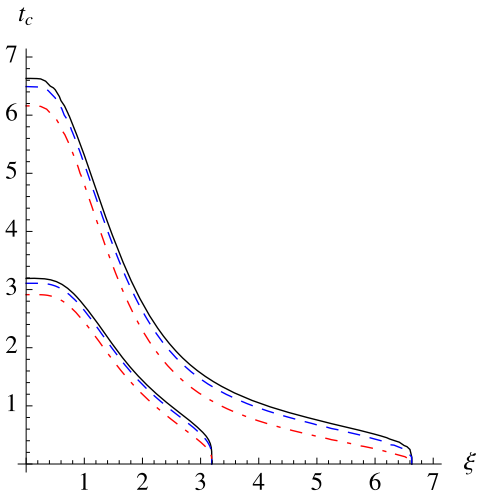


Fig. 3. The same as in Fig. 2 for several couples of values (γ, δ). We take for them the values $(0.0, 0.3)$, $(0.5, 0.3)$ and $(0.9, 0.3)$ (respectively, full, dashed and dot-dashed lines in the right set of curves); $(0, 1.5)$, $(0.5, 1.5)$ and $(0.9, 1.5)$ (respectively, full, dashed and dot-dashed lines in the left set of curves).

$K_0(z) \approx \sqrt{(\pi/2z)}\exp(-z)$, for $t \to 0$, so that the argument of the first sum between square brackets in Eq. (23), for small temperatures is formally the same as in Eq. (24), *i.e.*, in this case, the sum can be written as $\sum_{n=0}^{\infty} f_n(t, \gamma, \delta)$. Taking into account the condition $0 \leqslant \gamma < 1$, this sum vanishes in the limit $t \to 0$. A similar argument applies for the last term in Eq. (23). Therefore, in the limit $t \to 0$, only the second term in Eq. (23) survives, in such a way that all dependence coming from the chemical potential drops out. Consequently, the resulting solution of Eq. (30), for ξ in this case, $\xi = \xi_0(\gamma, \delta) = \xi_0(\delta)$, does not have any influence from the chemical-potential magnitude.

The same kind of "mathematical phenomenon" is found in the absence of an external field for both first- and second-order phase transitions [10,11]. As explicitly stated by some of us in Ref. [10], what appears to happen is that for zero temperature the behavior of the physical system having the minimal (finite) size collapses to the one corresponding to a zero chemical potential, as is the case of a Bose–Einstein distribution. In the presence of a magnetic field, for each value of δ, there is a limiting smallest size of the system, $L_{\min}(\delta)$, corresponding to a largest reduced inverse size $\chi_{\max}(\delta)$, over which the first-order transition described by

the adopted model, ceases to exist. In other words, in the presence of a magnetic field, we find the same kind of "mathematical phenomenon", of the collapsing of the system into a Bose–Einstein distribution for the minimal allowed film thickness; the main difference is that the minimal allowed size for the system is now dependent on the intensity of the applied field; the larger the field is, the larger is the minimal allowed size of the system.

Moreover, let us remind that with our choice of gauge for $D = 4$, $A = (0, xH, 0, 0)$, the applied field lies on a direction perpendicular to the film. In this case, we see from Fig. 3 that, for a higher applied field, the minimal allowed thickness of the film is larger, that is, thinner films cannot be made for stronger values of the applied field. On the other hand, let us consider any film thickness such that the transition can exist for both values of the applied field ($0 < \xi < 3.20$ in the figure). We see in this case that the critical temperature is lower for higher applied field, *i.e.*, the applied field goes against the transition. This behavior for a system in the form of a film is in agreement with the observed behavior for systems in bulk form, that is, the applied field tends to destroy the (for instance, superconducting) transition. In other words, the tendency of the applied field to destroy the phase transition is a common feature for materials in bulk form and for films, independently of its thickness. However, the lowering of the critical temperature for a given thickness and applied field depends on the density of the material, in such a way that for higher values of the chemical potential, the material "resists" less to the destruction of the transition by the magnetic background.

As an overall conclusion, we can say that some of the above results seem *a priori* somehow unexpected, such as the independency of the minimal size of the system (the minimal film thickness) on density and the fact that for higher applied fields, the minimal allowed thicknesses of the film are larger; actually, they are a direct consequence of considering effects coming from the finite size of the system. Other results, such as the decreasing of the critical temperature as the magnetic field intensity grows, go along the lines of known features of superconducting materials in bulk, under the influence of a magnetic background. In any case, the results found in this Letter suggest that magnetic and finite-size effects with finite chemical potential are relevant for bounded systems and significantly changes the phase structure with respect to the one for the system in bulk form. In particular, these actors lead to the appearance of a minimal allowed size of the system, for each value of the applied field, which is independent of the chemical potential. On the other side, there are other aspects in agreement with some observations for materials in bulk form.

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