Positive Einstein metrics with small $L^{n/2}$-norm of the Weyl tensor

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Abstract: A gravitational analogue is given of Min-Oo's gap theorem for Yang-Mills fields.

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Introduction

In this note we prove

**Theorem 1.** Let $M$ be a compact oriented $n$-manifold ($n = 2m \geq 4$) with non-vanishing Euler characteristic $\chi(M)$ and let $g$ be a (Riemannian) positive Einstein metric on $M$ with Weyl curvature $W$. Then there is a constant $\varepsilon > 0$, depending only upon $n$ and $\chi(M)$, such that if $\|W\|_{L^{n/2}} < \varepsilon$, then $W = 0$ (and so $M$ is isometric to a quotient of $S^n$ with the standard metric).

**Remark.** The hypothesis $\chi(M) \neq 0$ is unnecessary in dimension 4 since in that case it is implied by the Einstein condition (cf. [1, Section 6.31]).

**Notation.** Recall (cf. for example [1, Sections G–H]) that the Riemann tensor $R$ may be decomposed algebraically as

$$R = W + \Phi \otimes g + (s/2m(n-1))g \otimes g$$

(1)

where $W$ is the Weyl tensor (the totally trace-free part of $R$), $(n-2)\Phi$ is the trace-free part of the Ricci tensor $r$, $s$ is the scalar curvature and $\otimes$ is the Kulkarni-Nomizu product of symmetric 2-tensors (i.e. tensor product followed by the projection $\{4$-tensors$\} \rightarrow \{4$-tensors with Riemann tensor symmetries$\}$).

The Einstein condition is $\Phi = 0$; an Einstein metric is called 'positive' if $s > 0$. (This makes sense because the Einstein condition and the Bianchi identity imply that $s$ is constant.)
The notation for norms that is used in this paper is standard: all norms are those defined by the given Einstein metric $g$.

**Outline of the Proof.** The idea is to use the Bianchi identity and a Sobolev inequality as in Min-Oo’s analogous result [7] for Yang-Mills fields. In our situation, however, the Sobolev ‘constant’ $\sigma$ depends upon $g$ and so it is necessary to obtain a universal lower bound under the hypothesis the $g$ is a positive Einstein metric and $\|W\|_{L^{n/2}}$ is small. One can refer to the work of Croke [4] and Li [5] for a lower bound on $\sigma$ in terms of $s$ and the volume $V$ of $M$. Then an application of the Gauss-Bonnet theorem [3] shows that if $\|W\|_{L^{n/2}}$ is sufficiently small, and $\chi(M) \neq 0$, then $V$ and $s$ cannot both be small. In this way, the desired lower bound on $\sigma$ is obtained.

**Remark.** The $L^{n/2}$-norm of the Weyl tensor is an invariant of the conformal class of $g$; in particular it is invariant under any homothety $g \mapsto cg$ ($c$ a positive constant). Since the Einstein condition and the sign of $s$ are also invariant under such homotheties, we may suppose that $g$ is normalized so that

$$V = 1$$

without affecting the value of $\varepsilon$ in Theorem 1.

Finally we warn the reader that $A, B, C$ etc. will denote constants (i.e. functions only of the dimension $n$ and the Euler characteristic $\chi(M)$). The values of such constants may vary from line to line.

**The proof of Theorem 1**

*Estimation of the Sobolev constant in terms of $V$ and $s$*

Geometric estimates of the Sobolev constants begin with the isoperimetric constant

$$\alpha = \inf \frac{\text{Vol}(\partial \Omega)^n}{\text{Vol}(\Omega)^{n-1}}$$

where the inf runs over all open submanifolds $\Omega \subset M$ with sufficiently regular boundary $\partial \Omega$ and volume $\leq V/2$ [2]. Li [5, Lemma 2], for example, proves that for all $f \in L^2_{1}$,

$$\|\nabla f\|_{L^2}^2 \geq A \alpha^{2/n} \left( \|f\|_{L^p}^2 - V^{-2/n} \|f\|_{L^2}^2 \right)$$

(2)

($p = 2n/(n-2)$). On the other hand Croke [4, Theorem 13] has given a lower bound for $\alpha$ in terms of a lower bound $(n - 1)k$ of the Ricci tensor and the volume and diameter of $M$,

$$\alpha \geq A (V/I)^{n+1},$$

(3)

where

$$I = \int_0^d (k^{-1/2} \sin(k^{1/2}t))^{n-1} dt.$$
Positive Einstein metrics 271

(The integrand has a standard interpretation if \( k \leq 0 \), but we shall not need it.)

In our situation \( (r = (s/n)g > 0) \), we obtain a simple lower bound for \( \alpha \) as follows. Taking \( k > 0, d \leq \pi k^{-1/2} \) by Myers’ theorem [8] and so

\[
I \leq \int_0^{\pi k^{-1/2}} k^{(1-n)/2} \, dt = \pi k^{-n/2}.
\]

Combining with (2) and (3), we obtain

**Proposition 2.** Let \( M \) be a compact Riemannian manifold with unit volume and suppose \( r \geq (n - 1)k > 0 \). Then the Sobolev inequality may be written

\[
\|\nabla f\|_{L^2}^2 \geq A k^{n+1} \left( \|f\|_{L^p}^2 - \|f\|_{L^2}^2 \right).
\]

In particular, if \( M \) is Einstein, (5) holds for all \( k \) such that

\[
0 < k \leq s/n(n - 1).
\]

**Remark.** The inequality (5) holds with the same constant if \( f \) is replaced by any tensor \( T \). This follows from Kato’s inequality

\[
|\nabla |T| | \leq |\nabla T|
\]

which holds at all points where \( |T| \neq 0 \).

**A lower bound on s from the Gauss-Bonnet formula**

On an oriented \( n \)-manifold \( M \), the Gauss-Bonnet formula ([3], [6, p. 311]) states

\[
\chi(M) = \int_M \text{Pf}(R/2\pi).
\]

Here ‘Pf’ denotes the Pfaffian of a skew endomorphism: all we shall need to know about it is that it is a homogeneous polynomial of degree \( m = n/2 \).

On substituting (1), with \( \Phi = 0 \), into (7), we obtain, since \( s \) is constant,

\[
\chi(M) = \sum_{r=0}^{\infty} P_r(W) s^{n-r}
\]

where \( P_r(W) \) is the integral over \( M \) of a universal polynomial of degree \( r \) in \( W \). The constant \( P_0 \) can be discovered by applying (8) to the standard metric on \( S^n \), for which \( W = 0 \) and \( s = n(n - 1) \):

\[
2 = P_0 \cdot \text{Vol}(S^n)[n(n - 1)]^{n/2}.
\]

In particular, \( P_0 \) is a positive constant. To estimate the other terms in (8), suppose as before that \( V = 1 \). Then

\[
|P_r(W)| \leq B\|W\|_{L^r}^r \leq B\|W\|_{L^m}^r
\]
where the last step uses Hölder’s inequality. Substituting into (8),

$$|\chi(M)| \leq P_0|s|^m + B \sum_{r=1}^{m} \| W \|^r_{L^m} s^{m-r}.$$  \hfill (9)

It is now straightforward to prove the following.

**Proposition 3.** Let $M$ satisfy the hypotheses of Theorem 1 and suppose also that $V = 1$. Then there are constants $\varepsilon_1 > 0$ and $B_1 > 0$, depending only upon $n$ and $\chi(M)$, such that if $\| W \|_{L^{n/2}} < \varepsilon_1$ then $s > B_1$.

**Proof.** With $P_0$ and $B$ as in (9), choose $\varepsilon_1 > 0$ so that for each $r = 1, \ldots, m$, $B \| W \|^r_{L^m} < \frac{1}{4} |\chi(M)|$. Then if $s \leq \frac{1}{2}$, (9) gives

$$|\chi(M)| \leq P_0|s|^m + \frac{1}{4} |\chi(M)| \sum_{r=1}^{m} 2^{r-m} \leq P_0|s|^m + \frac{1}{2} |\chi(M)|.$$  

Hence the result with $B_1 = \min \left( \frac{1}{2}, \left( \frac{|\chi(M)|}{2P_0} \right)^{1/m} \right)$. \hfill \Box

**Remark.** As in the remark following the statement of Theorem 1, we point out the hypothesis $\chi(M) \neq 0$ is implied by the Einstein condition if $n = 4$ and is therefore redundant in the statement of Proposition 3 in that case.

**Completion of proof**

On an Einstein manifold, the Bianchi identities reduce to

\begin{align*}
\nabla_a W_{bcde} + \nabla_b W_{cade} + \nabla_c W_{abde} &= 0, \\
\nabla^a W_{abcd} &= 0, \\
\nabla_a s &= 0.
\end{align*}  \hfill (10, 11, 12)

Of course (11) follows from (10) by contraction and use of the symmetries of $W$, but we will find it convenient to regard (10) and (11) as separate equations, bundle-valued versions of the equations

$$d\omega = 0, \quad d^*\omega = 0$$  \hfill (13)

for a 2-form $\omega$. Recall that information about the solution space of (13) can sometimes be obtained by reducing to the single equation

$$\Delta \omega = (dd^* + d^*d)\omega = 0$$

and using the Weitzenböck formula

$$dd^* + d^*d = \nabla^*\nabla + \text{curvature terms.}$$

Our strategy is to analyze (10) and (11) in exactly the same way.
In this way, we obtain
\[-\nabla^a \nabla_a W_{bcde} + [\nabla^a, \nabla_b] W_{acde} - [\nabla^a, \nabla_c] W_{abde} = 0. \tag{14}\]
The first commutator term may be written in terms of curvature as follows:
\[[\nabla^a, \nabla_b] W_{acde} = R^a_{\ b} p^p W_{pecde} + R^a_{\ bc} p^p W_{apde} \tag{15}\]
\[\quad + R^a_{\ bd} p^p W_{acpe} + R^a_{\ be} p^p W_{acdp}.\]
The first term on the RHS is just \((s/n)W_{bcde}\) by the Einstein conditions; and using (1) in the remaining terms on the RHS we get terms quadratic in \(W\), which we shall lump together and denote by \(\{W, W\}\), and three other terms in \(s\) and \(W\):
\[\frac{s}{n(n - 1)} (W_{cbde} + W_{dcbe} + W_{ecdb}).\]
This expression is identically zero, however, because of the symmetries
\[W_{ecdb} = W_{bdce} \quad \text{and} \quad W_{cbde} + W_{dcbe} + W_{bdce} = 0.\]
The second curvature term in (14) is obtained from the first by interchanging the indices \(b\) and \(c\), so (14) reduces to an equation of the form
\[\nabla^* \nabla W + (2s/n)W = \{W, W\}. \tag{16}\]
The proof of Theorem 1 is now completed as follows. The inner product of (16) with \(W\) yields
\[\|\nabla W\|_{L^2}^2 + (2s/n)\|W\|_{L^2}^2 \leq C\|W\|_{L^3}^3 \leq C\|W\|_{L^{n/2}}^2 \|W\|_{L^p}^2, \tag{17}\]
where the second step uses Hölder’s inequality and, as before, \(p = 2n/(n - 2)\). Combining with the Sobolev inequality of Proposition 2, we find
\[C\|W\|_{L^{n/2}}^2 \|W\|_{L^p}^2 \geq Ak^{n+1} \|W\|_{L^p}^2 + (2s/n - Ak^{n+1})\|W\|_{L^2}^2 \tag{18}\]
for every \(k\) which satisfies (6):
\[0 < k \leq s/n(n - 1).\]
To complete the proof it suffices to choose \(k > 0\) so that both (6) and the inequality
\[2s/n - Ak^{n+1} \geq 0 \tag{19}\]
are satisfied. The existence of such a \(k\) is guaranteed by Proposition 3. For if \(\|W\|_{L^{n/2}} < \varepsilon_1\), then \(s > B_1 > 0\) and choosing \(k\) so that
\[0 < k < \min \left(\frac{B_1}{n(n - 1)}, \frac{(2B_1/na)^{1/(n+1)}}{1/(n+1)}\right)\]
makes (18) and (19) hold. Then (18) yields
\[C\|W\|_{L^{n/2}} \geq Ak^{n+1}\]
if \(W\) is not identically zero. This yields Theorem 1 with
\[\varepsilon = \min(Ak^{n+1}/C, \varepsilon_1).\]
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References