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Positive Einstein metrics with small $L^{n/2}$-norm of the Weyl tensor

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Abstract: A gravitational analogue is given of Min-Oo's gap theorem for Yang-Mills fields.

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Introduction

In this note we prove

Theorem 1. *Let M be a compact oriented n-manifold ($n = 2m \geqslant 4$) with non-vanishing Euler characteristic $\chi(M)$ and let g be a (Riemannian) positive Einstein metric on M with Weyl curvature W. Then there is a constant $\varepsilon > 0$, depending only upon n and $\chi(M)$, such that if $\|W\|_{L^{n/2}} < \varepsilon$, then $W = 0$ (and so M is isometric to a quotient of S^n with the standard metric).*

Remark. The hypothesis $\chi(M) \neq 0$ is unnecessary in dimension 4 since in that case it is implied by the Einstein condition (cf. [1, Section 6.31]).

Notation. Recall (cf. for example [1, Sections G–H]) that the Riemann tensor R may be decomposed algebraically as

$$R = W + \Phi \owedge g + (s/2n(n-1))g \owedge g \tag{1}$$

where W is the Weyl tensor (the totally trace-free part of R), $(n-2)\Phi$ is the trace-free part of the Ricci tensor r, s is the scalar curvature and $\owedge$ is the Kulkarni-Nomizu product of symmetric 2-tensors (i.e. tensor product followed by the projection {4-tensors} $\to$ {4-tensors with Riemann tensor symmetries}).

The Einstein condition is $\Phi = 0$; an Einstein metric is called 'positive' if $s > 0$. (This makes sense because the Einstein condition and the Bianchi identity imply that s is constant.)

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The notation for norms that is used in this paper is standard: all norms are those defined by the given Einstein metric g.

Outline of the Proof. The idea is to use the Bianchi identity and a Sobolev inequality as in Min-Oo's analogous result [7] for Yang-Mills fields. In our situation, however, the Sobolev 'constant' σ depends upon g and so it is necessary to obtain a universal lower bound under the hypothesis the g is a positive Einstein metric and $\|W\|_{L^{n/2}}$ is small. One can refer to the work of Croke [4] and Li [5] for a lower bound on σ in terms of s and the volume V of M. Then an application of the Gauss-Bonnet theorem [3] shows that if $\|W\|_{L^{n/2}}$ is sufficiently small, and $\chi(M) \neq 0$, then V and s cannot both be small. In this way, the desired lower bound on σ is obtained.

Remark. The $L^{n/2}$-norm of the Weyl tensor is an invariant of the conformal class of g; in particular it is invariant under any homothety $g \mapsto cg$ (c a positive constant). Since the Einstein condition and the sign of s are also invariant under such homotheties, we may suppose that g is normalized so that

$$V = 1$$

without affecting the value of ε in Theorem 1.

Finally we warn the reader that A, B, C etc. will denote constants (i.e. functions only of the dimension n and the Euler characteristic $\chi(M)$). The values of such constants may vary from line to line.

The proof of Theorem 1

Estimation of the Sobolev constant in terms of V and s

Geometric estimates of the Sobolev constants begin with the isoperimetric constant

$$\alpha = \inf \frac{\mathrm{Vol}(\partial\Omega)^n}{\mathrm{Vol}(\Omega)^{n-1}}$$

where the inf runs over all open submanifolds $\Omega \subset M$ with sufficiently regular boundary $\partial\Omega$ and volume $\leqslant V/2$ [2]. Li [5, Lemma 2], for example, proves that for all $f \in L^2_1$,

$$\|\nabla f\|^2_{L^2} \geqslant A\alpha^{2/n}\left(\|f\|^2_{L^p} - V^{-2/n}\|f\|^2_{L^2}\right) \tag{2}$$

($p = 2n/(n-2)$). On the other hand Croke [4, Theorem 13] has given a lower bound for α in terms of a lower bound $(n-1)k$ of the Ricci tensor and the volume and diameter of M,

$$\alpha \geqslant A(V/I)^{n+1}, \tag{3}$$

where

$$I = \int_0^d \left(k^{-1/2}\sin(k^{1/2}t)\right)^{n-1} dt.$$

(The integrand has a standard interpretation if $k \leqslant 0$, but we shall not need it.)

In our situation ($r = (s/n)g > 0$), we obtain a simple lower bound for α as follows. Taking $k > 0$, $d \leqslant \pi k^{-1/2}$ by Myers' theorem [8] and so

$$I \leqslant \int_0^{\pi k^{-1/2}} k^{(1-n)/2} dt = \pi k^{-n/2}. \tag{4}$$

Combining with (2) and (3), we obtain

Proposition 2. *Let M be a compact Riemannian manifold with unit volume and suppose $r \geqslant (n-1)k > 0$. Then the Sobolev inequality may be written*

$$\|\nabla f\|_{L^2}^2 \geqslant A k^{n+1} \left(\|f\|_{L^p}^2 - \|f\|_{L^2}^2 \right). \tag{5}$$

In particular, if M is Einstein, (5) *holds for all k such that*

$$0 < k \leqslant s/n(n-1). \tag{6}$$

Remark. The inequality (5) holds with the same constant if f is replaced by any tensor T. This follows from Kato's inequality

$$\big|\nabla |T|\big| \leqslant |\nabla T|$$

which holds at all points where $|T| \neq 0$.

A lower bound on s from the Gauss-Bonnet formula

On an oriented n-manifold M, the Gauss-Bonnet formula ([3], [6, p. 311]) states

$$\chi(M) = \int_M \mathrm{Pf}(R/2\pi). \tag{7}$$

Here 'Pf' denotes the Pfaffian of a skew endomorphism: all we shall need to know about it is that it is a homogeneous polynomial of degree $m = n/2$.

On substituting (1), with $\Phi = 0$, into (7), we obtain, since s is constant,

$$\chi(M) = \sum_{r=0}^{m} P_r(W) s^{m-r} \tag{8}$$

where $P_r(W)$ is the integral over M of a universal polynomial of degree r in W. The constant P_0 can be discovered by applying (8) to the standard metric on S^n, for which $W = 0$ and $s = n(n-1)$:

$$2 = P_0 \cdot \mathrm{Vol}(S^n)[n(n-1)]^{n/2}.$$

In particular, P_0 is a positive constant. To estimate the other terms in (8), suppose as before that $V = 1$. Then

$$|P_r(W)| \leqslant B \|W\|_{L^r}^r \leqslant B \|W\|_{L^m}^r$$

where the last step uses Hölder's inequality. Substituting into (8),

$$|\chi(M)| \leqslant P_0|s|^m + B\sum_{r=1}^{m} \|W\|_{L^m}^r s^{m-r}. \tag{9}$$

It is now straightforward to prove the following.

Proposition 3. *Let M satisfy the hypotheses of Theorem 1 and suppose also that $V = 1$. Then there are constants $\varepsilon_1 > 0$ and $B_1 > 0$, depending only upon n and $\chi(M)$, such that if $\|W\|_{L^{n/2}} < \varepsilon_1$ then $s > B_1$.*

Proof. With P_0 and B as in (9), choose $\varepsilon_1 > 0$ so that for each $r = 1, \ldots, m$, $B\|W\|_{L^m}^r < \frac{1}{4}|\chi(M)|$. Then if $s \leqslant \frac{1}{2}$, (9) gives

$$|\chi(M)| \leqslant P_0|s|^m + \tfrac{1}{4}|\chi(M)|\sum_{r=1}^{m} 2^{r-m} \leqslant P_0|s|^m + \tfrac{1}{2}|\chi(M)|.$$

Hence the result with $B_1 = \min\left(1/2, (|\chi(M)|/2P_0)^{1/m}\right)$. □

Remark. As in the remark following the statement of Theorem 1, we point out the hypothesis $\chi(M) \neq 0$ is implied by the Einstein condition if $n = 4$ and is therefore redundant in the statement of Proposition 3 in that case.

Completion of proof

On an Einstein manifold, the Bianchi identities reduce to

$$\nabla_a W_{bcde} + \nabla_b W_{cade} + \nabla_c W_{abde} = 0, \tag{10}$$

$$\nabla^a W_{abcd} = 0, \tag{11}$$

$$\nabla_a s = 0. \tag{12}$$

Of course (11) follows from (10) by contraction and use of the symmetries of W, but we will find it convenient to regard (10) and (11) as separate equations, bundle-valued versions of the equations

$$d\omega = 0, \qquad d^*\omega = 0 \tag{13}$$

for a 2-form ω. Recall that information about the solution space of (13) can sometimes be obtained by reducing to the single equation

$$\Delta\omega = (dd^* + d^*d)\omega = 0$$

and using the Weitzenböck formula

$$dd^* + d^*d = \nabla^*\nabla + \text{curvature terms}.$$

Our strategy is to analyze (10) and (11) in exactly the same way.

In this way, we obtain

$$-\nabla^a \nabla_a W_{bcde} + [\nabla^a, \nabla_b] W_{acde} - [\nabla^a, \nabla_c] W_{abde} = 0. \tag{14}$$

The first commutator term may be written in terms of curvature as follows:

$$\begin{aligned} [\nabla^a, \nabla_b] W_{acde} = {} & R^a{}_{ba}{}^p W_{pcde} + R^a{}_{bc}{}^p W_{apde} \\ & + R^a{}_{bd}{}^p W_{acpe} + R^a{}_{be}{}^p W_{acdp}. \end{aligned} \tag{15}$$

The first term on the RHS is just $(s/n)W_{bcde}$ by the Einstein conditions; and using (1) in the remaining terms on the RHS we get terms quadratic in W, which we shall lump together and denote by $\{W, W\}$, and three other terms in s and W:

$$\frac{s}{n(n-1)} (W_{cbde} + W_{dcbe} + W_{ecdb}).$$

This expression is identically zero, however, because of the symmetries

$$W_{ecdb} = W_{bdce} \quad \text{and} \quad W_{cbde} + W_{dcbe} + W_{bdce} = 0.$$

The second curvature term in (14) is obtained from the first by interchanging the indices b and c, so (14) reduces to an equation of the form

$$\nabla^* \nabla W + (2s/n) W = \{W, W\}. \tag{16}$$

The proof of Theorem 1 is now completed as follows. The inner product of (16) with W yields

$$\|\nabla W\|_{L^2}^2 + (2s/n)\|W\|_{L^2}^2 \leqslant C\|W\|_{L^3}^3 \leqslant C\|W\|_{L^{n/2}} \|W\|_{L^p}^2, \tag{17}$$

where the second step uses Hölder's inequality and, as before, $p = 2n/(n-2)$. Combining with the Sobolev inequality of Proposition 2, we find

$$C\|W\|_{L^{n/2}} \|W\|_{L^p}^2 \geqslant Ak^{n+1} \|W\|_{L^p}^2 + (2s/n - Ak^{n+1}) \|W\|_{L^2}^2 \tag{18}$$

for every k which satisfies (6):

$$0 < k \leqslant s/n(n-1).$$

To complete the proof it suffices to choose $k > 0$ so that both (6) and the inequality

$$2s/n - Ak^{n+1} \geqslant 0 \tag{19}$$

are satisfied. The existence of such a k is guaranteed by Proposition 3. For if $\|W\|_{L^{n/2}} < \varepsilon_1$, then $s > B_1 > 0$ and choosing k so that

$$0 < k < \min\left(B_1/n(n-1), (2B_1/nA)^{1/(n+1)}\right)$$

makes (18) and (19) hold. Then (18) yields

$$C\|W\|_{L^{n/2}} \geqslant Ak^{n+1}$$

if W is not identically zero. This yields Theorem 1 with

$$\varepsilon = \min(Ak^{n+1}/C, \varepsilon_1).$$

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References

[1] A.L. Besse, *Einstein Manifolds* (Springer, Berlin, 1987).
[2] E. Bombieri, Theory of minimal surfaces and a counter-example to the Bernstein conjecture in higher dimensions, Lecture notes, Courant Institute of Mathematics, 1970.
[3] S.S. Chern, A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds, *Ann. of Math.* **45** (1944) 747–752.
[4] C.B. Croke, Some isoperimetric inequalities and eigenvalue estimates, *Ann. Scient. Éc. Norm. Sup.* **13** (1980) 419–435.
[5] P. Li, On the Sobolev constant and the p-spectrum of a compact Riemannian manifold, *Ann. Scient. Éc. Norm. Sup.* **13** (1980) 451–469.
[6] J.W. Milnor and J.D. Stasheff, *Characteristic Classes* (Princeton University Press, Princeton, 1974).
[7] M. Min-Oo, An L^2-isolation theorem for Yang-Mills fields, *Composito Mathematica* **47** (1982) 153–163.
[8] S. Myers, Riemannian manifolds with positive mean curvature, *Duke Math. J.* **8** (1941) 401–404.