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The Minimum Modulus of Blaschke Products

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INTRODUCTION

Let D denote the open unit disk in the complex plane. A sequence A in D is called a *Blaschke sequence* provided $\sum_A (1-|a|) < \infty$. Corresponding to such a sequence is the *Blaschke product* B given by

$$B(z) = \prod_A (|a|/a)(a-z)/(1-\bar{a}z) \tag{1}$$

where the term in the product corresponding to a is replaced with z when $a=0$. It is well known that B is an analytic function mapping D into D and having a radial limit of modulus 1 at almost every point of the boundary ∂D. (See [1, Chap. 2].)

For $0<r<1$ let $m(r; B)$ denote the minimum modulus of the Blaschke product B on the circle $|z|=r$. If $m(r; B)$ does not tend to 0 as r increases to 1, then clearly

$$\limsup_{r \to 1} (1-r) \log m(r; B) = 0. \tag{2}$$

M. Heins ([3, Theorem 6.1]; also see [9]) has shown that (2) is valid for all Blaschke products; that is, corresponding to each Blaschke product B there exists a subset Δ of the interval (0, 1) such that 1 is an accumulation point of Δ and

$$(1-r) \log m(r; B) \to 0 \qquad \text{as } r \to 1 \text{ through } \Delta. \tag{3}$$

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Here we shall show that there exists a set Δ that satisfies condition (3) and is very "thick" near 1; more specifically, we shall show that Δ exists so that the complement $C\Delta$ of Δ in (0, 1) is minimally thin at 1, the definition of which is given in Section 1. (This implies that $C\Delta$ has finite logarithmic length, and hence left metric density 0, at 1. See Remark 1.) As applications, we shall improve an identity theorem of A. L. Shaginyan and give a supplement to a result of A. A. Gol'dberg concerning the growth of a meromorphic function whose logarithmic derivative has bounded characteristic.

1. Growth of Blaschke Products

Let E be a subset of D, and let P_ζ be the Poisson kernel at $\zeta \in \partial D$ given by

$$P_\zeta(z) = (1 - |z|^2)/|\zeta - z|^2.$$

If S^+ denotes the class of all nonnegative superharmonic functions in D, then the *reduced function* of P_ζ relative to E (see [4, p. 134]) is given by

$$R_E^{P_\zeta} = \inf\{u \in S^+ : u \geqslant P_\zeta \text{ on } E\}.$$

The set E is said to be *minimally thin* at ζ if $R_E^{P_\zeta} \not\equiv P_\zeta$ on D.

An extended real-valued function u in D is said to have *fine limit* α at a point $\zeta \in \partial D$ if there exists a subset E of D such that E is minimally thin at ζ and $u(z)$ tends to α as z tends to ζ through $D \backslash E$. (L. Naïm [7, p. 219] calls this a *pseudo-limit.*)

Theorem 1. *If B is a Blaschke product, then $(1 - |z|) \log |B(z)|$ has fine limit 0 at every $\zeta \in \partial D$; furthermore, there exists a subset Δ of (0, 1) such that (0, 1)$\backslash\Delta$ is minimally thin at 1 and*

$$(1 - r) \log m(r; B) \to 0 \qquad \text{as } r \to 1 \text{ through } \Delta.$$

Proof. Let $\zeta \in \partial D$ be arbitrary, and set

$$u = (-\log |B|)/P_\zeta.$$

Because $-\log |B|$ is a positive superharmonic function in D, it follows from a result of Naïm ([7, p. 227] with $K(x_0, x) = P_{x_0}(x)$] that u has fine limit

$$\alpha \equiv \inf_{z \in D} u(z)$$

at ζ. Because

$$u(r\zeta) = -[(1-r)\log|B(r\zeta)|]/(1+r) \qquad (0<r<1),$$

it follows from the result of Heins stated in the Introduction that $\alpha = 0$, that is, that u has fine limit 0 at ζ. Then since

$$P_\zeta(z) \leqslant (1-|z|^2)/(1-|z|)^2 < 2/(1-|z|),$$

we have

$$0 < -(1-|z|)\log|B(z)| \leqslant 2u(z).$$

Consequently $(1-|z|)\log|B(z)|$ has fine limit 0 at ζ, and the first part of the theorem is established.

Now consider the Blaschke product B_0 given by

$$B_0(z) = \prod_{\Delta} (|a| - z)/(1-|a|\, z).$$

It is readily shown that

$$|B_0(r)| \leqslant m(r; B) \qquad (0<r<1). \tag{4}$$

By the first part of the theorem, there exists a subset Δ of $(0, 1)$ such that $(0, 1)\backslash\Delta$ is minimally thin at 1 and

$$(1-r)\log|B_0(r)| \to 0 \qquad \text{as } r \to 1 \text{ through } \Delta. \tag{5}$$

Now (4) and (5) yield the second part of the theorem, and the theorem is proved.

Remark 1. Because the set $C\Delta \equiv (0, 1)\backslash\Delta$ lies on the radius to 1 and is minimally thin at 1, it follows from a result of H. L. Jackson [6, Theorem 5] that $C\Delta$ has *finite logarithmic length* at 1, that is, that

$$\int_{C\Delta} (1-r)^{-1}\, dr < \infty.$$

(Jackson's proof is for the right half-plane, but the conformal mapping $T(z) = (1-z)/(1+z)$ readily transfers the result to D.)

Remark 2. C. Tanaka [10, Theorem 1] has proved: *Let* $a_1, a_2, \ldots$ *be the zeros of the Blaschke product* B, *and let* $r_1, r_2, \ldots$ *be a sequence of real numbers that decrease to* 0 *and satisfy the inequality*

$$\sum_{n=1}^{\infty} (1-|a_n|)/r_n^2 < \infty. \tag{6}$$

If Q is the set of those points in D that lie outside each of the pseudo-hyperbolic disks

$$\lambda(z, a_n) \equiv |z - a_n|/|1 - \bar{a}_n z| < r_n \qquad (n = 1, 2, \ldots),$$

then $(1 - |z|) \log |B(z)| \to 0$ *as* $|z| \to 1$ *through* Q. To see that Tanaka's result does not imply Theorem 1, we consider an example.

If $a_n = 1 - 1/n^2$, then $\lambda(a_n, a_{n+1}) < 1/n$ for each n. Also, to satisfy condition (6) we must have $r_n > 1/n$ for all sufficiently large n. That is, Q necessarily omits a terminal segment of the radius to 1; but this implies that the complement of Q is *not* minimally thin at 1.

2. Theorems of Heins and Shaginyan

If f is a meromorphic function in D, the Nevanlinna proximity function for f is given by

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{it})| \, dt \qquad (0 < r < 1),$$

and we set

$$m(1, f) = \lim_{r \to 1} m(r, f)$$

if the limit exists, which it will if f is of bounded characteristic.

Shaginyan [8] considered a proper subclass C of the class of all *analytic* functions of bounded characteristic in D, and for that subclass he has proved: *If Γ is a Jordan arc in D except for one endpoint at* 1 *and if f is in C with*

$$(1 - |z|) \log |f(z)| \to -\infty \qquad \text{as } z \to 1 \text{ through } \Gamma,$$

then $f \equiv 0$. A stronger statement can be made in the case when f is bounded. Heins [3, Theorems 7.1 and 7.2] proved: *If f is a bounded analytic function in D with*

$$(1 - r) \log m(r; f) \to -\infty \qquad \text{as} \quad r \to 1,$$

then $f \equiv 0$. (See [5] for an alternate proof.) We give the following extension.

THEOREM 2. *If f is a meromorphic function* ($\not\equiv 0$) *of bounded characteristic in D, then there exists a subset Δ of* $(0, 1)$ *such that* $(0, 1) \backslash \Delta$ *is minimally thin at* 1 *and*

$$\liminf_{r \to 1; r \in \Delta} (1 - r) \log m(r; f) \geqslant -2 m(1, 1/f). \tag{7}$$

Proof. As is well known, we may write $f = (B_1/B_2)g$ where B_1 and B_2 are Blaschke products (or $\equiv 1$) and g is a zero-free analytic function of bounded characteristic in D. We note that

$$\log|f| \geqslant \log|B_1| + \log|g| \qquad \text{in } D. \tag{8}$$

Let z_r be a point on the circle $|z| = r$ with $|f(z_r)| = m(r; f)$, and let Δ_1 be the subset of $(0, 1)$ guaranteed by Theorem 1 for B_1. (If $B_1 \equiv 1$, take $\Delta_1 = (0, 1)$.) Then because $|B_1(z_r)| \geqslant m(r; B_1)$ we have

$$(1 - r)\log|B_1(z_r)| \to 0 \qquad \text{as } r \to 1 \text{ through } \Delta_1.$$

This and inequality (8) yield

$$\liminf_{r\to 1; r\in\Delta_1} (1 - r)\log|f(z_r)| \geqslant \liminf_{r\to 1; r\in\Delta_1} (1 - r)\log|g(z_r)|. \tag{9}$$

Because $1/g$ is analytic and zero free in D, $\log|g|^{-1}$ is harmonic in D; hence for $|z| < R < 1$,

$$2\pi \log|g(z)|^{-1} = \int_0^{2\pi} P(z; \mathrm{Re}^{it}) \log|g(Re^{it})|^{-1}\, dt$$

where $P(z; Re^{it}) = (R^2 - |z|^2)/|Re^{it} - z|^2$. Because

$$0 < P(z; Re^{it}) \leqslant (R + |z|)/(R - |z|) < 2/(R - |z|)$$

and $\log x \leqslant \log^+ x$ for $x > 0$, we have

$$\pi \log|g(z)|^{-1} < (R - |z|)^{-1} \int_0^{2\pi} \log^+ |g(Re^{it})|^{-1}\, dt.$$

Letting R tend to 1, we obtain

$$(1 - |z|)\log|g(z)| \geqslant -2\mathscr{m}(1, 1/g) \qquad (z \in D). \tag{10}$$

Since $\mathscr{m}(1, 1/g) = \mathscr{m}(1, 1/f)$, the desired inequality follows from (9) and (10), and the proof is complete.

3. A Result of Gol'dberg

Let F be a meromorphic function in D, and let ξ be a point in D. If $0 < \rho < 1 - |\xi|$ and A_ρ denotes the sequence of poles of F in the disk $|z - \xi| < \rho$, then the *characteristic function of F relative to ξ* is given by

$$T_\xi(\rho, F) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |F(\xi + \rho e^{it})|\, dt + \log\left(\prod_{A_\rho} (\rho/|p - \xi|)\right)$$

with the usual modification if $F(\xi) = \infty$. Also we let

$$T_\xi(1 - |\xi|, F) = \lim_{\rho \to 1 - |\xi|} T_\xi(\rho, F).$$

The first result of this section represents a slight improvement of a result of Gol'dberg [2, Theorem 3] concerning the growth of the maximum modulus $M(r; f)$ of a meromorphic function f. To obtain the statement of Gol'dberg's result, simply replace "is minimally thin" with "has finite logarithmic length" in the statement of Theorem 3 below.

THEOREM 3. *If f is a meromorphic function* ($\not\equiv 0$) *in D with $T_0(1, f'/f) < \infty$, then there exists a subset Δ of* $(0, 1)$ *such that* $(0, 1)\backslash\Delta$ *is minimally thin at* 1 *and*

$$\limsup_{r \to 1; r \in \Delta} (1 - r) \log |\log M(r; f)| < \infty; \tag{11}$$

if f is analytic in D, then (11) *is valid with "$r \in \Delta$" deleted.*

Proof. Since f/f' is also of bounded characteristic in D, it follows from Theorem 2 that there exist a positive constant K and a subset Δ of $(0, 1)$ such that $(0, 1)\backslash\Delta$ is minimally thin at 1 and

$$(1 - r) \log m(r; f/f') > -K \qquad (r \in \Delta).$$

Thus

$$|f'(z)/f(z)| < \exp\left(\frac{K}{1 - r}\right) \qquad (|z| = r \in \Delta). \tag{12}$$

Because f'/f is of bounded characteristic in D, there exists a real number α such that f has no zero or pole on the open radial segment to $e^{i\alpha}$ and f'/f has a finite radial limit at $e^{i\alpha}$; hence there exists an $a \in (0, 1)$ such that

$$|f'(re^{i\alpha})/f(re^{i\alpha})| < \exp\left(\frac{K}{1 - r}\right) \qquad (a \leqslant r < 1). \tag{13}$$

Because of (12), f has no zero or pole on the circle $|z| = r$ for each $r \in \Delta$. Now choose any $r \in \Delta\backslash(0, a)$ and any point z in D that satisfies $|z| = r$. Let γ be the path traversed when a point is moved radially from $ae^{i\alpha}$ to $re^{i\alpha}$ and then along the circle $|\xi| = r$ to z. A single-valued branch of $\log f$ may be defined in a neighborhood of γ, and we have

$$\log f(z) - \log f(ae^{i\alpha}) = \int_{\gamma} (f'/f).$$

Considering real parts and using (12) and (13), we deduce that

$$|\log(|f(z)|/|f(ae^{i\alpha})|)| < (2\pi + 1) \exp\left(\frac{K}{1-r}\right).$$

This implies (11).

Now suppose f is analytic in D. Also suppose $M(r; f)$ is unbounded; otherwise there is nothing to prove. Because $(0, 1)\backslash\Delta$ has finite logarithmic length at 1 (see Remark 1 of Section 1), there exists a positive constant C such that for each $r \in (0, 1)$ there exists an $r' \in \Delta$ for which $r' > r$ and $(1 - r) < C(1 - r')$. Consequently

$$(1 - r) \log |\log M(r; f)| < C(1 - r') \log |\log M(r'; f)|$$

when $M(r; f) > e$, and this completes the proof.

We now prove a supplement to Theorem 3.

THEOREM 4. *Suppose f is a meromorphic function in D, and suppose there exist a finite constant T and a number $R \in (0, 1)$ such that*

$$T_{\xi}(1 - R, f'/f) < T \qquad \textit{whenever } |\xi| = R.$$

If the poles of f'/f form a Blaschke sequence, then there exists a subset Δ of $(0, 1)$ such that $(0, 1)\backslash\Delta$ is minimally thin at 1 and

$$\limsup_{r \to 1; r \in \Delta} (1 - r) \log |\log M(r; f)| < \infty; \tag{14}$$

if f is analytic, then (14) is valid with "$r \in \Delta$" deleted.

Proof. Set $F = f'/f$. Suppose $|\xi| = R$ and $0 < \rho < 1 - R$ with F finite at ξ and on the circle $C_\rho : |z - \xi| = \rho$. Let D_ρ be the disk $|z - \xi| < \rho$. For each $p \in D_\rho$ let G_p be a conformal mapping of D_ρ onto D with $G_p(p) = 0$. Define the function

$$H = F\left(\prod_{A_\rho} G_p\right)\bigg/\left(\prod_{B_\rho} G_p\right)$$

where A_ρ and B_ρ are the sequences of poles and zeros of F in D_ρ.

Let $z = \xi + \sigma e^{i\theta}$ with $0 < \sigma < \rho$ and $F(z) \neq \infty$. Because $|H| = |F|$ on C_ρ, the Poisson–Jensen formula for H is

$$2\pi \log |H(z)| = \int_0^{2\pi} P(\sigma e^{i\theta}; \rho e^{it}) \log |F(\xi + \rho e^{it})| \, dt$$

where again $P(\sigma e^{i\theta}; \rho e^{it}) = (\rho^2 - \sigma^2)/|\rho e^{it} - \sigma e^{i\theta}|^2$. Because

$$0 < P(\sigma e^{i\theta}; \rho e^{it}) \leqslant (\rho + \sigma)/(\rho - \sigma) < 2/(\rho - \sigma)$$

and $\log x \leqslant \log^+ x$ for $x > 0$, we have

$$\pi \log |H(z)| \leqslant (\rho - \sigma)^{-1} \int_0^{2\pi} \log^+ |F(\xi + \rho e^{it})| \, dt.$$

Then since

$$\int_0^{2\pi} \log^+ |F(\xi + \rho e^{it})| \, dt \leqslant 2\pi T_\xi(\rho, F) < 2\pi T,$$

we have

$$\log |H(z)| \leqslant 2T(\rho - \sigma)^{-1}. \tag{15}$$

Also, it is evident that

$$\log |H(z)| \geqslant \log |F(z)| + \log \left| \prod_{A_\rho} G_p(z) \right| . \tag{16}$$

Now let B denote the Blaschke product that corresponds to the sequence consisting of all the poles of F in D. (Recall that the poles of F are simple and occur at the zeros and poles of f.) Because

$$|G_p(w)| = 1 > |p - w|/|1 - \bar{p}w| \qquad \text{for } p \in D_\rho \text{ and } w \in C_\rho,$$

the maximum modulus principle gives

$$|G_p(w)| \geqslant |p - w|/|1 - \bar{p}w| \qquad \text{for } p, w \in D_\rho;$$

thus we have

$$\log \left| \prod_{A_\rho} G_p(z) \right| \geqslant \log \left| \prod_{A_\rho} (p - z)/(1 - \bar{p}z) \right| \geqslant \log |B(z)|. \tag{17}$$

Conditions (15) to (17) imply that

$$\log |F(z)| \leqslant 2T(\rho - \sigma)^{-1} - \log |B(z)|.$$

Letting ρ tend to $1-R$ we obtain

$$\log|F(z)| \leqslant 2T[1-(R+\sigma)]^{-1} - \log|B(z)|.$$

In particular, if $\xi = Re^{i\psi}$ and $z = re^{i\psi}$ with $R < r < 1$, then $\sigma = r - R$ and

$$\log|F(re^{i\psi})| \leqslant 2T(1-r)^{-1} - \log|B(re^{i\psi})| \qquad (R < r < 1). \tag{18}$$

(Note that by continuity (18) is valid for all real ψ.)

Using (18) and applying Theorem 1, we obtain a subset Δ of $(0, 1)$ such that $(0, 1)\backslash\Delta$ is minimally thin at 1 and

$$(1-r)\log|F(re^{i\psi})| < 2T+1 \qquad (r \in \Delta). \tag{19}$$

Also, if B has a radial limit of modulus 1 at the point $e^{i\alpha}$, then by (18) there exists an $a \in (R, 1)$ such that

$$(1-r)\log|F(re^{i\alpha})| < 2T+1 \qquad (a \leqslant r < 1). \tag{20}$$

Now consider the portion of the proof of Theorem 3 that follows display (13). The remainder of the proof of Theorem 4 can be obtained from it by replacing K with $2T+1$, (12) with (19), and (13) with (20).

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