The Order Bidual of Lattice Ordered Algebras

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The main topic of the present paper is a systematic investigation of the second order dual \( A'' \) of an Archimedean \( f \)-algebra \( A \) with point separating order dual \( A' \). It is shown that in the case that \( A \) has a unit element, the equality \( A'' = (A')_0 \) holds, where \( (A')_0 \) is the collection of all order continuous linear functionals on \( A' \). It turns out that in general \( (A')_0 \), equipped with the Arens multiplication, is an \( f \)-algebra again. Necessary and sufficient conditions are derived for \( (A')_0 \) to be semiprime and for \( (A')_0 \) to have a unit element with respect to this multiplication.

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1. INTRODUCTION

The principle aim of the present paper is to investigate the behaviour of the Arens multiplication in the second order dual of Riesz algebras (lattice ordered algebras) and, in particular, of \( f \)-algebras.

We recall that for any (associative, but not necessarily commutative) algebra \( A \) a multiplication can be introduced in the second algebraic dual \( A^{**} \) of \( A \) (the so-called Arens multiplication; see, amongst others, [1, 2, 6, 7, and 11]). This is accomplished in three steps: given \( a, b \in A, f \in A^*, \) and \( F, G \in A^{**}, \) we define \( f \cdot a \in A^*, F \cdot f \in A^*, \) and \( F \cdot G \in A^{**} \) by the equations

\[
(f \cdot a)(b) = f(ab) \tag{1}
\]

\[
(F \cdot f)(a) = F(f \cdot a) \tag{2}
\]

\[
(F \cdot G)(f) = F(G \cdot f) \tag{3}
\]
(we adhere to the notations used in [9]). The following identities are easily verified:

\[ f \cdot (ab) = (f \cdot a) \cdot b \]
\[ (F \cdot f) \cdot a = F \cdot (f \cdot a) \]
\[ (F \cdot G) \cdot f = F \cdot (G \cdot f). \]

It follows immediately that \((F \cdot G) \cdot H = F \cdot (G \cdot H)\) for all \(F, G, H \in A^{**}\).

It is straightforward now to show that \(A^{**}\) becomes an associative algebra with respect to the Arens multiplication as defined in (3). Unfortunately, even if the initial multiplication in \(A\) happens to be commutative, the Arens multiplication in \(A^{**}\) need not be. However, it is always true that (after embedding \(A\) canonically in \(A^{**}\)) the Arens multiplication in \(A^{**}\) extends the original one in \(A\). We mention some other properties in relation to the canonical embedding \(\sigma : A \to A^{**}\) (henceforth we shall denote \(\sigma(a)\) by \(a''\) for all \(a \in A\), i.e., \(a''(f) = f(a)\) for all \(f \in A^*\):

(i) if \(A\) has a unit element \(e\), then \(\sigma(e) = e''\) is the multiplicative identity of \(A^{**}\). Note in this connection that if \(A^{**}\) has a unit element \(E\), then \(E \cdot f = f\) for all \(f \in A^*\).

(ii) if \(A\) is commutative, then \(a'' \cdot f = f \cdot a\) for all \(a \in A, f \in A^*\). Indeed, \((a'' \cdot f)(b) = a''(f \cdot b) = (f \cdot b)(a) = f(ba) = f(ab) = (f \cdot a)(b)\) for all \(b \in A\).

So far the generalities. Now, let \(A\) be a (real) Riesz algebra (lattice ordered algebra), i.e., \(A\) is a Riesz space (vector lattice) which is simultaneously an associative algebra with the additional property that \(a, b \in A^+\) (i.e., \(a \geq 0, b \geq 0\)) implies \(ab \in A^+\) (equivalently, \(|cd| \leq |c||d|\) for all \(c, d \in A\)). For notational convenience we denote from now on the first order dual of \(A\) by \(A'\) and the order bidual of \(A\) by \(A''\). We shall show that \(A''\) is a Riesz algebra again with respect to the Arens multiplication. The same holds for \((A')'_n\), the band in \(A''\) of all order continuous linear functionals on \(A'\).

We shall assume throughout this paper that \(A\) is Archimedean and that \(A'\) separates the points of \(A\), in other words,

\[ 0(A') = \{a \in A : f(a) = 0 \text{ for all } f \in A'\} = \{0\}. \]

Recall that an \(f\)-algebra \(A\) is a Riesz algebra which satisfies the extra requirement that \(a \wedge b = 0\) implies

\[ (ac) \wedge b = (ca) \wedge b = 0 \]

for all \(c \in A^+\). By the blanket assumption that \(A\) is Archimedean, \(A\) is
necessarily associative and commutative. One of the major results of this paper is that for any Archimedean $\mathcal{F}$-algebra $A$ with point separating order dual, $(A')'_{n}$ is an Archimedean (and hence commutative!) $\mathcal{F}$-algebra with respect to the Arens multiplication. More generally, a similar result holds for $B'_{n}$ with $B$ a point separating (order) ideal of $A'$. In particular, if $^0(A''_{n}) = \{0\}$, then $(A''_{n})'_{n}$ is an Archimedean $\mathcal{F}$-algebra with respect to the Arens multiplication as well. Furthermore, we shall search for necessary and sufficient conditions in order that $(A')''_{n}$ is semiprime (i.e., the only nilpotent is 0). Finally, we shall derive a necessary and sufficient condition for $(A')''_{n}$ to have a multiplicative unit. It will turn out that this condition much resembles the corresponding condition for Banach algebras (cf. [5, Proposition 28.7]).

The final paragraphs of this introduction are devoted to some motivation for the study of the structure of $(A')'_{n}$. One important reason for considering merely $(A')'_{n}$ instead of the full second order dual $A''$ is that $A''$ and $(A')'_{n}$ coincide if $A$ has a unit element. This rather surprising fact on $\mathcal{F}$-algebras will be shown in Section 3. As a consequence we find that for every Archimedean unital $\mathcal{F}$-algebra $A$ with point separating order dual, $A''$ is an $\mathcal{F}$-algebra with respect to the Arens multiplication. A special case of the above result is that $C(X)'' = (C(X)'')_{n}$ for all topological spaces $X$.

Another motivation for the study of the structure of $(A')'_{n}$ is given in the next paragraphs. The results are stated without proof and are independent of the further contents of the paper. The space $(A')'_{n}$ is in some respect a natural object in relation to $A$. To be more precise, it can be shown that for any Riesz space $L$ with point separating order dual, the space $(L')'_{n}$ is the "largest" Riesz space amongst all Dedekind complete Riesz spaces $K$ with $^0(K'_{n}) = \{0\}$, containing $L$ as a Riesz subspace such that any $0 \leq f \in L'$ has a unique order continuous positive extension $\overrightarrow{f}$ to $K$. Moreover, if we assume in addition that $K$ is equal to the ideal generated by $L$, then $K$ is even uniquely determined by these properties. In fact, we can take $K$ to be the ideal $I(L)$ generated by $L$ in $(L')'_{n}$. Clearly, in the case that $L = A$ with $A$ an $\mathcal{F}$-algebra, it follows from the above mentioned results that $I(A)$ is an $\mathcal{F}$-algebra with respect to the Arens multiplication. Note that all these considerations equally hold if $L'$ is replaced by a point separating band $J$ in $L'$. 

In illustration of these observations, let (for simplicity) $\Gamma$ be an algebra of subsets of a nonempty point set $X$ and let $A$ be the $\mathcal{F}$-algebra of all (real) step functions with respect to $\Gamma$. Let $J$ be the band of all elementary integrals on $A$ (see [20, Sect. 12]: by an elementary integral we mean the difference of two positive elementary integrals). Obviously, $J$ separates the points of $A$. The ideal $I(A)$ generated by $A$ in $J'_{n}$ is an $\mathcal{F}$-algebra. Note that the elements of $J$ are in a one-one correspondence with the (finite, $\sigma$-additive) signed measures on $\Gamma$ (and hence with the signed measures on the $\sigma$-algebra $A$.
generated by \( I \). Let us denote for any \( 0 \leq f \in J \) the unique extension of \( f \) to \( I(A) \) by \( \tilde{f} \). Write \( N(\tilde{f}) \) for the null ideal of \( \tilde{f} \) in \( I(A) \) and let \( \mu \) be the measure on \( I \) corresponding to \( f \). The Caratheodory extension of \( \mu \) on \( A \) is denoted by \( \tilde{\mu} \). It can be shown that the disjoint complement \( N(\tilde{f})^d \) in \( I(A) \) is Riesz isomorphic to \( L_\infty(\tilde{\mu}) \), so the \( L_\infty(\tilde{\mu}) \)-spaces correspond biunivocally to bands in \( I(A) \), disjoint signed measures corresponding to disjoint bands. The ideal \( I(A) \) can be considered therefore in a way as a universal object for integration theory on \((X, I)\) and \((X, A)\). Hence it seems of some importance that \((A')_n^*\) and \( I(A) \) can be given an \( f \)-algebra structure. Observe however that \( I(A) \) is in general much larger than the space of all bounded \( A \)-measurable functions on \( X \). Notice finally that the case \( J = A' \) corresponds to considering finitely additive signed measures.

A second example of the above described abstract situation is provided by \( A = C(X) \) with \( X \) a compact Hausdorff space. In this situation the band \( J \) of all elementary integrals on \( A \) equals \( A' \).

### 2. SOME PRELIMINARIES

We use the standard works on Riesz spaces \([15; 21]\) as a starting point and we refer to these monographs for unexplained terminology and notation. Throughout the paper we only consider Archimedean Riesz spaces \( L \) with point separating order dual \( L' \). The mapping \( \sigma: L \to (L')_n^* \) assigning to \( a \in L \) the element \( \sigma(a) = a'' \) (with \( a''(f) = f(a) \) for all \( f \in L' \)) is an injective Riesz homomorphism. Moreover, \( \sigma(L) \) is order dense in \( (L')_n^* \) (i.e., \( \sigma(L)^d = \{0\} \) in \( (L')_n^* \)). Hence, after embedding, the order ideal \( I(L) \) generated by \( L \) in \( (L')_n^* \) is strongly order dense in \( (L')_n^* \) (i.e., for every \( 0 < F \in (L')_n^* \) there exists \( 0 < G \in I(L) \) such that \( G < F \); for more information on strongly order denseness, see \([21, \text{Sects. } 79, 109]\).

Recall that the linear mapping \( \pi: L \to L \) is said to be order bounded (denoted by \( \pi \in \mathscr{L}_b(L) \)) whenever \( \pi \) maps order intervals into order intervals; the mapping \( \pi \in \mathscr{L}_b(L) \) is called an orthomorphism if \( a \perp b \) (i.e., \( |a| \wedge |b| = 0 \)) implies \( \pi a \perp b \). The collection \( \text{Orth}(L) \) of all orthomorphisms on \( L \) is, with respect to composition as multiplication, an Archimedean \( f \)-algebra with the identity mapping \( I \) as unit element. Besides

\[
(\pi_1 \vee \pi_2) a - (\pi_1 a) \vee (\pi_2 a), \quad (\pi_1 \wedge \pi_2) a - (\pi_1 a) \wedge (\pi_2 a)
\]

for all \( a \in L^+, \pi_1, \pi_2 \in \text{Orth}(L) \) \([21, \text{Theorems } 140.4 \text{ and } 140.9]\). The centre \( Z(L) \) of \( L \) is, by definition, the order ideal in \( \text{Orth}(L) \) generated by \( I \). Orthomorphisms are order continuous, i.e., if \( \pi \in \text{Orth}(L) \) and \( a_\downarrow \downarrow 0 \) in \( L \), then \( \inf_\uparrow |\pi a_\downarrow| = 0 \). An immediate consequence is that two orthomorphisms \( \pi_1, \pi_2 \in \text{Orth}(L) \), agreeing on some order dense subset \( D \) of \( L \), are the same on the whole of \( L \).
For elementary $f$-algebra theory we refer to [13, 17, and 21]. We list some important properties:

(a) every (Archimedean) $f$-algebra $A$ is automatically associative and commutative.

(b) if $a^2 = 0$, then $ab = 0$ for all $b \in A$.

(c) if $A$ has a unit element, then $A$ is semiprime.

(d) multiplication by positive elements is order continuous, i.e., if $a_t \downarrow 0$ in $A$ and $b \in A^+$, then $a_t b = b a_t \downarrow 0$.

If $A$ is semiprime, then $A$ can be embedded as a Riesz subspace and a ring ideal in Orth($A$) via the mapping $p$ defined by $p(a) = \pi_a$ for all $a \in A$. Here $\pi_a$ denotes the multiplication by $a$. Actually, the mapping $\rho$ is injective if and only if $A$ is semiprime and $\rho$ is bijective if and only if $A$ has a unit element (for details we refer to [4, Sect. 12.3]). In any semiprime $f$-algebra $a \bot b$ is equivalent to $ab = 0$. Moreover, using the formula $\pi\pi_a = \pi_{\pi a}$ ($a \in A, \pi \in \text{Orth}(A)$), it is easily verified that the orthomorphisms on $A$ are precisely the multipliers of $A$ (recall that the linear mapping $\pi: A \to A$ is a multiplier whenever $\pi(ab) = \pi a \cdot b = a \cdot \pi b$ for all $a, b \in A$). By identifying $a \in A$ with $\pi_a \in \text{Orth}(A)$ we embed $A$ in Orth($A$). The formula $\pi\pi_a = \pi_{\pi a}$ shows that the two meanings of $\pi a$ cannot cause confusion. It is an easy matter to verify that $A$ is order dense in Orth($A$). We show that $A$ is even strongly order dense.

**Proposition 2.1.** Let $A$ be an Archimedean semiprime $f$-algebra. Then $A$ is strongly order dense in Orth($A$).

**Proof.** It suffices to show that for every $0 < \pi \in \text{Orth}(A)$ there exists $0 < a \in A$ such that $a \leq \pi$. Since $\pi b > 0$ for some $0 < b \in A$, there exists a natural number $n$ such that $(n\pi b - \pi b^2)^+ > 0$. It follows from

$$(n I - b)(n I - b)^+ \geq 0$$

that

$$b(n I - b)^+ = (nb - b^2)^+ \leq n I(n I - b)^+ \leq n^2 I$$

and so

$$\pi(nb - b^2)^+ = (\pi nb - \pi b^2)^+ \leq n^2 \pi.$$ 

Hence, $a = n^{-2}(\pi nb - \pi b^2)^+$ has the desired property.

**Definition 2.2** (cf. [8, Sect. 1]). The upward directed net $\{b_\tau : \tau \in T\}$ in $A^+$ is called an approximate unit in $A$ whenever $a = \sup_\tau (ab_\tau) = \sup_\tau (b_\tau a)$ for all $a \in A^+$. 

Note that a unit element (if existing) is evidently an approximate unit. Using Proposition 2.1, the proof of the following equivalence is almost straightforward.

**Theorem 2.3** (see also [8, Remark 2.13.1]). The Archimedean $f$-algebra $A$ is semiprime if and only if $A$ has an approximate unit.

**Proof:** If $A$ is semiprime, then $A$ can be embedded as a strongly order dense Riesz subspace and ring ideal in $\text{Orth}(A)$. There exists therefore a directed system $\{b_\tau : \tau \in T\}$ in $A^+$ for which $0 \leq b_\tau \uparrow I$. The equality $a = \sup_\tau (ab_\tau)$ for all $a \in A^+$ is immediate.

Conversely, let $\{b_\tau : \tau \in T\}$ be an approximate unit in $A^+$ and let $a \in A$ be a nilpotent element. As is well known, $a^2 = 0$, so $|a|^2 = 0$ as well. Hence, $|a|b = 0$ for all $b \in A^+$. In particular, $|a|b_\tau = 0$ for all $\tau$. Hence, $|a| = \sup_\tau (|a|b_\tau) = 0$, so $a = 0$.

For later purposes we introduce a notion in $f$-algebras which stems from integration theory (see [20, Sect. 17]; cf. also [14, Theorems 3.5 and 6.5]).

**Definition 2.4.** Let $A$ be an Archimedean, semiprime $f$-algebra. It is said that $A$ satisfies the (Daniell-) Stone condition whenever $a \land I \in A^+$ for all $a \in A^+$ (we consider $A$ embedded in $\text{Orth}(A)$).

**Theorem 2.5.** Any uniformly complete, semiprime $f$-algebra $A$ satisfies the Stone condition.

**Proof.** Since $\text{Orth}(A)$ is a uniformly complete $f$-algebra with unit element $I$, the inverse $(a \lor I)^{-1}$ exists in $\text{Orth}(A)$ for all $a \in A^+$ by [13, Theorem 3.4]. On account of $a = aI = (a \land I)(a \lor I)$, we have $a \land I = (a \lor I)^{-1}a$. Hence, $A$ being a ring ideal in $\text{Orth}(A)$, we deduce that $a \land I \in A$.

Finally we mention a result that is concerned with homomorphism of $f$-algebras and that will be used several times in what follows. If $A, B$ are Archimedean $f$-algebras with unit elements $e_A$ and $e_B$ respectively, and $T: A \to B$ is a positive linear mapping with the additional property that $Te_A = e_B$, then $T$ is an algebra homomorphism if and only if $T$ is a Riesz homomorphism. For the proof we refer to [14, Corollary 5.5].

3. WHEN $A''$ EQUALS $(A^{'})'_o$

The main topic of the present section is the proof that $A'' = (A^{'})'_o$ holds for any Archimedean $f$-algebra $A$ with unit element. We recall a definition.
DEFINITION 3.1. The Archimedean Riesz space $L$ is said to have the order continuity property whenever every positive linear mapping from $L$ into an Archimedean Riesz space is order continuous.

Note that the order continuity property implies that $L' = L'_{\text{Arch}}$. Furthermore, the order continuity property is equivalent to the fact that every uniformly closed ideal in $L$ is a band (for details, see [21, Sect. 83]). We first need a Lemma.

**Lemma 3.2.** Let $A$ be an Archimedean $\mathfrak{f}$-algebra with unit element $e$. Then the inequality

$$(a - ne)^+ \leq n^{-2}a^3$$

holds for all $a \in A^+$ ($n = 1, 2, \ldots$).

**Proof.** Fix a natural number $n$ and take $a \in A^+$. Since $(a - ne)^+ \land (ne - a)^+ = 0$, the defining property of $\mathfrak{f}$-algebras implies

$$(a - ne)^+ \land (ne - a)^+ (ne + a) = 0,$$

i.e., $(a - ne)^+ \land (n^2e - a^2)^+ = 0$. Therefore, $(a - ne)^+ \cdot (n^2e - a^2)^+ = 0$, as $A$ is semiprime. Hence $(a - ne)^+(n^2e - a^2) \leq 0$ and thus

$$n^2(a - ne)^+ \leq a^2(a - ne)^+ \leq a^3.$$

We are now in a position to prove the main result of this section.

**Theorem 3.3.** For any Archimedean $\mathfrak{f}$-algebra $A$ with unit element $e$, the order dual $A'$ has the order continuity property.

**Proof:** Let $M$ be an Archimedean Riesz space and let $T$ be a positive linear mapping from $A'$ into $M$. Suppose that $f_0 \geq f_\tau \downarrow 0$ in $A'$ and that $T(f_\tau) \geq d \geq 0$ in $M$. Now $f_\tau \downarrow 0$ in $A'$ implies that $f_\tau(e) \downarrow 0$. The net $\{f_\tau\}$ contains therefore a subsequence $\{f_n\}_{n=1}^\infty$ such that $0 \leq f_n(e) \leq n^{-3}$ ($n = 1, 2, \ldots$). Lemma 3.2 yields that for all $a \in A^+$

$$0 \leq f_n(a) = f_n(a \land ne) + f_n((a - ne)^+)$$

$$\leq nf_n(e) + f_n((a - ne)^+) \leq n^{-2}(1 + f_0(a^3))$$

$(n = 1, 2, \ldots)$ and so

$$\sum_{n=1}^\infty f_n(a) < \infty$$

for all $a \in A^+$. Define $g_n = \sum_{k=1}^n f_k$ $(n = 1, 2, \ldots)$. By the above,
\[ \sup_n g_n(a) < \infty \] for all \( a \in A^+ \). Introduce \( g_0 : A^+ \to \mathbb{R} \) by \( g_0(a) = \sup_n g_n(a) \) for all \( a \in A^+ \). It is easily verified that \( g_0 \) is additive on the positive cone \( A^+ \) of \( A \) and thus \( g_0 \) extends to a positive linear functional on the whole of \( A \), denoted by \( g_0 \) again. Actually, \( 0 \leq g_n \uparrow g_0 \) in \( A' \). It follows from \( 0 \leq \sum_{k=1}^n f_k \leq g_0 \) that \( 0 \leq \sum_{k=1}^n T(f_k) \leq T(g_0) \) for all \( n \). By hypothesis, \( T(f_k) \geq d \geq 0 \) for all \( k \) and hence \( 0 \leq nd \leq T(g_0) \) \((n = 1, 2, \ldots)\). Since \( M \) is Archimedean, we deduce that \( d = 0 \). It is shown that \( T(f_t) \downarrow 0 \) and the proof is complete.

**Corollary 3.4.** For any Archimedean unital \( f \)-algebra \( A \) we have \( A'' = (A')' \), i.e., \( (A')' = \{ (A')_a \} = \{0 \} \).

Note that \( A' \) is order separable in the above situation (as \( e'' \) is a strictly positive linear functional on \( A' \)). The equality \( A'' = (A')' \) does not hold for arbitrary \( f \)-algebras. By way of example, take \( A = l_1 \) (which is an \( f \)-algebra with respect to the coordinatewise multiplication). The order dual \( A' = l_\infty \) does not have the order continuity property, as \( (A')_a \not\subsetneq A'' \). We end this section with some remarks about the \( C(X) \)-case.

**Remark 3.5.** (i) Theorem 3.3 is more or less obvious in the case that \( A = C(X) \) with \( X \) a compact Hausdorff space. Indeed, \( A \) is now an \( AM \)-space, so the \( AL \)-space \( A' \) has order continuous norm (see, e.g., \([21, \text{Theorem 118.2}])\) and hence \( A' \) has the order continuity property (see \([21, \text{Ex. 105.20}])\).

(ii) The fact that \( C(X)'' = (C(X)')' \) holds for arbitrary topological spaces \( X \) can be obtained also as follows. Without violating the generality it may be assumed that \( X \) is realcompact (otherwise pass from \( X \) to \( vX \), the Hewitt real compactification of \( X \)). Denote by \( M_\circ(X) \) the collection of all regular Borel measures on \( X \) with compact support. A theorem, due to Hewitt (see \([12, \text{Theorem 17}; \text{or [10, Sect. 4]}])\) states that \( C(X)' = M_\circ(X) \circ \). It is easily verified that \( M_\circ(X)' = (M_\circ(X))_a \circ \). Combining the two equalities we find the desired result.

4. The \( f \)-Algebra \( (A')_a \)

We plunge into the matter with a basic theorem.

**Theorem 4.1.** Let \( A \) be a Riesz algebra. If \( F, G \in A'' \), then \( F \cdot G \in A'' \) and \( |F \cdot G| \leq |F| \cdot |G| \). In other words, \( A'' \) is a Riesz algebra with respect to the Arens multiplication. The same holds for \( (A')_a \).
Proof. The proof of the following steps is straightforward and therefore omitted:

\[ |f \cdot a| \leq |f| \cdot |a|, \quad f \cdot a \in \mathcal{A}' \quad \text{for all } a \in \mathcal{A}, f \in \mathcal{A}' \quad (1) \]

\[ |F \cdot f| \leq |F| \cdot |f|, \quad F \cdot f \in \mathcal{A}' \quad \text{for all } f \in \mathcal{A}', F \in \mathcal{A}'' \quad (2) \]

\[ |F \cdot G| \leq |F| \cdot |G|, \quad F \cdot G \in \mathcal{A}'' \quad \text{for all } F, G \in \mathcal{A}'' \quad (3) \]

The last inequality implies immediately that \( F \cdot G \geq 0 \) whenever \( 0 \leq F, G \in \mathcal{A}'' \), so \( \mathcal{A}'' \) is a Riesz algebra.

It remains to show that \((\mathcal{A}')_n \) is a Riesz algebra. To this end, it is sufficient to prove that \( 0 \leq F, G \in (\mathcal{A}')_n \) implies \( F \cdot G \in (\mathcal{A}')_n \). Let \( h_\tau \downarrow 0 \) be a decreasing net in \( \mathcal{A}' \). Evidently, \( h_\tau \cdot a \downarrow 0 \) for all \( a \in \mathcal{A}^+ \). Since \( G \) is order continuous, we derive

\[ (G \cdot h_\tau)(a) = G(h_\tau \cdot a) \downarrow 0 \]

for all \( 0 \leq a \in \mathcal{A} \), so \( G \cdot h_\tau \downarrow 0 \) in \( \mathcal{A}' \) as well. Finally, the order continuity of \( F \) implies that

\[ (F \cdot G)(h_\tau) = F(G \cdot h_\tau) \downarrow 0, \]

which yields the desired result.

Let \( \mathcal{A} \) be an (as agreed upon) Archimedean \( f \)-algebra with point separating order dual \( \mathcal{A}' \). Recall that the adjoint mapping \( \pi' \) of an orthomorphism \( \pi \in \text{Orth}(\mathcal{A}) \) is defined by \( (\pi'f)(a) = f(\pi a) \) for all \( a \in \mathcal{A}, f \in \mathcal{A}' \). It is a familiar result from the theory of orthomorphisms that \( \pi' \in \text{Orth}(\mathcal{A}') \) [21, Theorem 142.10]. Observe now that

\[ (\pi'_a f)(b) = f(\pi_a b) = f(ab) = (f \cdot a)(b) \]

for all \( b \in \mathcal{A} \). Hence, \( \pi'_a f = f \cdot a \) for all \( f \in \mathcal{A}' \) (and each \( a \in \mathcal{A} \) separately). The mapping \( f \mapsto f \cdot a \) is therefore a member of \( \text{Orth}(\mathcal{A}') \) for each \( a \in \mathcal{A} \). In the next lemma we generalize this observation to the elements of \( (\mathcal{A}')_n \).

**Lemma 4.2.** For every \( F \in (\mathcal{A}')_n \) the mapping \( v_F : \mathcal{A}' \to \mathcal{A}' \) defined by \( v_F(f) = F \cdot f \) for all \( f \in \mathcal{A}' \) satisfies \( v_F \in \text{Orth}(\mathcal{A}') \).

**Proof.** Without loss of generality we may assume that \( F \geq 0 \). Notice first that the lemma is evidently true in the case that \( F = a'' \) for some \( a \in \mathcal{A} \). Indeed,

\[ v_{a/core}(f) = a'' \cdot f = f \cdot a = \pi'_a(f) \]

for all \( f \in \mathcal{A}' \), so \( v_{a/core} = \pi'_a \in \text{Orth}(\mathcal{A}') \) by the above remark.
Let $I(A)$ be the ideal generated by $A$ in $(A')''$, i.e.,

$$I(A) = \{ G \in (A')' : |G| \leq b'' \text{ for appropriate } b \in A^+ \}.$$ 

If $F \in I(A)$, say $0 \leq F \leq a''$ for some $a \in A^+$, then it follows from $0 \leq v_F \leq v_{a''}$ and $v_{a''} \in \text{Orth}(A')$ that $v_F \subseteq \text{Orth}(A')$.

Finally, take $0 \leq F \in (A')''$ arbitrary. Since $I(A)$ is strongly order dense in $(A')''$, there exists an upwards directed set $\{ G_{\tau} : \tau \in \mathcal{T} \}$ in $I(A)$ such that $0 \leq G_{\tau} \uparrow F$. It is clear that $G_{\tau} \cdot f \uparrow F \cdot f$ for all $f \in A'$ and so $0 \leq v_{G_{\tau}} \uparrow v_F$ in $\mathcal{L}_b(A')$. By the result of the preceding paragraph, $v_{G_{\tau}} \in \text{Orth}(A')$ for all $\tau$. We may conclude that $v_F \in \text{Orth}(A')$, as $\text{Orth}(A')$ is a band in $\mathcal{L}_b(A')$ (see [21, Corollary 142.9]). The lemma is proved.

Before showing that $(A')''$ is an $f$-algebra we need another lemma. Recall that the carrier $C_F$ of an element $F \in (A')''$ is by definition $N^d_F$, where $N_F$ is the null ideal of $F$, i.e.,

$$N_F = \{ f \in A' : |F|(|f|) = 0 \}.$$ 

Evidently, the order continuity of $F$ implies that $N_F$ is a band. By Theorem 90.6 of [21], $F, G \in (A')''$ are disjoint (i.e., $F \perp G$) if and only if

$$C_F \cap C_G = C_{|F| \wedge |G|} = \{0\}.$$

**Lemma 4.3.** If $F, G \in (A')''$, then

(i) $C_{F \cdot G} \subseteq C_F$

(ii) $C_{G \cdot F} \subseteq C_F$.

**Proof:** (i) The orthomorphism $v_G \in \text{Orth}(A')$ is band preserving, so $v_G(N_F) \subseteq N_F$. Hence, $G \cdot f = v_G(f) \in N_F$ for all $f \in N_F$, in other words $(F \cdot G)(f) = F(G \cdot f) = 0$ for all $f \in N_F$. This proves the inclusion $N_F \subseteq N_{F \cdot G}$, i.e., $C_{F \cdot G} \subseteq C_F$.

(ii) The proof of this inclusion follows the same lines. Indeed, the band preserving property of $\pi'_F$ implies that $\pi'_F(f) = f \cdot a \in N_F$ for all $f \in N_F$. Hence, $(F \cdot f)(a) = F(f \cdot a) = 0$ for all $a \in A$ and all $f \in N_F$, and so $F \cdot f = 0$ for all $f \in N_F$. It follows that $(G \cdot F)(f) = G(F \cdot f) = G(0) = 0$ for all $f \in N_F$. Hence, $N_F \subseteq N_{G \cdot F}$ or, equivalently, $C_{G \cdot F} \subseteq C_F$. The proof is complete.

We are now in a position to prove the main theorem of this section.

**Theorem 4.4.** For any Archimedean $f$-algebra $A$ which satisfies $^0(A') = \{0\}$, the space $(A')''$ is an Archimedean (and hence commutative) $f$-algebra with respect to the Arens multiplication.
Proof. It is shown in Theorem 4.1 that \((A')_n\) is a Riesz algebra. Suppose now that \(F \wedge H = 0\) in \((A')_n\), or, equivalently, that \(C_F \cap C_H = \{0\}\). By Lemma 4.3, \(C_{F \cdot G} \subset C_F\) for all \(0 \leq G \in (A')_n\). Therefore, \(C_{F \cdot G} \cap C_H = \{0\}\) and hence \((F \cdot G) \wedge H = 0\). Analogously, it is shown that \(G \cdot F \wedge H = 0\) for all \(0 \leq G \in (A')_n\) and we are through.

A combination of Corollary 3.4 and Theorem 4.4 yields the following result.

Corollary 4.5. For any Archimedean unital \(f\)-algebra \(A\) with point separating order dual, \(A''\) is an \(f\)-algebra with respect to the Arens multiplication.

We conclude this section with some remarks. Theorem 4.4 can be generalized with little effort in the following way: if \(B\) is an order ideal in \(A'\) and \(0_B = \{0\}\), then \(B'_n\) is an \(f\)-algebra with respect to the Arens multiplication (note that \(A\) can still be embedded in \(B'_n\)). In particular, \((A'_n)^'_n\) is an \(f\)-algebra as soon as \(0(A'_n)^'_n = \{0\}\).

5. The Connection Between Orth\((A)\) and \((A')_n\)

A well-known theorem due to Synnatschke (see [19 or 21, Theorem 115.2 and Corollary 115.3]) reads as follows: if \(L\) and \(M\) are Riesz spaces, \(M\) is Dedekind complete, \(0(M'_n) = \{0\}\) and, for any \(T \in \mathcal{L}_n(L, M)\), the restriction of the adjoint mapping of \(T\) to \(M'_n\) is denoted by \(T'\), then the mapping \(\gamma: \mathcal{L}_n(L, M) \to \mathcal{L}_n(M'_n, L')\) defined by \(\gamma(T) = T'\) is a Riesz homomorphism. The following proposition states a more or less similar result for orthomorphisms.

Proposition 5.1. Let \(L\) be an Archimedean Riesz space with \(0(L') = \{0\}\) and let the mapping \(\gamma: \text{Orth}(L) \to \text{Orth}(L')\) be defined by \(\gamma(\pi) = \pi'\) for all \(\pi \in \text{Orth}(L)\). Then \(\gamma\) is an injective algebra homomorphism and Riesz homomorphism.

Proof. The mapping \(\gamma\) is obviously injective, positive, and satisfies \(\gamma(I) = I'\) (where \(I'\) denotes the identity mapping on \(L'\)). Moreover, \(\text{Orth}(L')\) is commutative and so

\[
\gamma(\pi_1 \pi_2) = (\pi_1 \pi_2)' = \pi_2' \pi_1' = \pi_1' \pi_2' = \gamma(\pi_1) \gamma(\pi_2)
\]

for all \(\pi_1, \pi_2 \in \text{Orth}(L')\). By Corollary 5.5 of [14], cited at the end of Section 2, \(\gamma\) is a Riesz homomorphism as well, and we are done.

As before, we denote for any \(F \in (A')_n\) the mapping \(f \mapsto F \cdot f\) by \(v_f\), and it
is shown in Lemma 4.2 that \( v_F \) is an orthomorphism of \( A' \). Following Maté (see [16]) we shall refer to \( v_F \) as to the orthomorphism represented by \( F \). Let us consider the mapping \( v: (A')_n' \rightarrow \text{Orth}(A') \), defined by \( v(F) = v_F \) for all \( F \in (A')_n' \), more closely. Note that for any \( \pi \in \text{Orth}(A') \) and any \( a \in A \) we have \( \pi(f \cdot a) = (\pi f) \cdot a \), as the mapping \( f \rightarrow f \cdot a \) is an orthomorphism of \( A' \), commuting with \( \pi \). In particular we get for the orthomorphism \( v_F \) represented by \( F \in (A')_n' \) that \( v_F(f \cdot a) = (v_F f) \cdot a \), i.e., \( F \cdot (f \cdot a) = (F \cdot f) \cdot a \) for all \( a \in A, f \in A' \) and \( F \in (A')_n' \). All the preparations have been made now for the proof of the next theorem.

**Theorem 5.2** (cf. [16, Theorem 1]). Let \( A \) be an Archimedean \( f \)-algebra with point separating order dual. Then the mapping \( v \) is an algebra and Riesz homomorphism from \( (A')_n' \) into \( \text{Orth}(A') \). Moreover, \( v \) is onto if and only if \( (A')_n' \) has a unit element, and in that case \( v \) is injective.

**Proof.** Clearly, \( v \) is linear and positive. If \( F, G \in (A')_n' \), then we have

\[
((F \cdot G) \cdot f)(a) = (F \cdot G)(f \cdot a) = F(G \cdot (f \cdot a)) = F(G \cdot f)(a) = F((G \cdot f)(a) = F((G \cdot f)(a) = (F \cdot (G \cdot f)(a)
\]

for all \( a \in A \) and all \( f \in A' \). Hence, \( v_{F \cdot G} = v_F v_G \) for all \( F, G \in (A')_n' \), i.e., \( v \) is an algebra homomorphism. Now suppose that \( F \wedge G = 0 \) in \( (A')_n' \). Then \( v(F \cdot G) = v(F) v(G) = 0 \) in \( \text{Orth}(A') \), as \( F \cdot G = 0 \). However, \( \text{Orth}(A') \) is a unital and hence semiprime \( f \)-algebra, so \( v(F) \wedge v(G) = 0 \). We may conclude that \( v \) is also a Riesz homomorphism.

Now assume that \( v \) is surjective. Then there exists an element \( E \) in \( (A')_n' \) such that \( E \cdot f = f \) for all \( f \in A' \). For any \( G \in (A')_n' \) we find that

\[
(G \cdot E) f = G(F \cdot f) = G(f)
\]

for all \( f \in A' \), so \( G \cdot E = G \). The element \( E \) is therefore the unit element of \( (A')_n' \). Furthermore, if \( F \in (A')_n' \) satisfies \( v_F = 0 \), then \( F(f) = F(E \cdot f) = (F \cdot E) f = E(F \cdot f) = E(v_F(f)) = 0 \) for all \( f \in A' \), so \( F = 0 \). This shows that \( v \) is injective.

Conversely, suppose that \( (A')_n' \) has a unit element \( E \). Observe that \( E \cdot f = f \) for all \( f \in A' \). Indeed,

\[
(E \cdot f)(a) = a''(E \cdot f) = (a'' \cdot E)(f) = a''(f) = f(a)
\]

for all \( a \in A \). Let \( \pi \in \text{Orth}(A') \) be given and let \( \pi' \) denote the restriction of the adjoint of \( \pi \) to \( (A')_n' \), so \( \pi' \in \text{Orth}(A')_n' \). Put \( F = \pi'(E) \). We claim that \( \pi = v_F \). Indeed, for each \( f \in A' \) and all \( a \in A \)

\[
(v_F(f'))(a) = (\pi'(E) \cdot f')(a) = (\pi' E)(f \cdot a) = E(\pi(f \cdot a)) = E(\pi f \cdot a) = (E \cdot \pi f')(a) = (\pi f')(a),
\]

where \( f' \) is the restriction of \( f \) to \( (A')_n' \).
which proves the claim. Hence, \( \nu \) is onto which takes care of this implication. Notice for later purposes that if \( \nu \) is surjective (and hence bijective) the above element \( F \) satisfies \( F = \nu^{-1}(\pi) = \pi'(E) \).

**Remark 5.3.** The mapping \( \nu \) in the above theorem is in general not injective. In fact, the kernel of \( \nu \) is equal to the set \( N((A')_n') \) of all nilpotents in \((A')_n'\). Since \((A')_n'\) is an \( f \)-algebra, \( N((A')_n') \) is a band and

\[
N((A')_n') = \{ F \in (A')_n' : F^2 = F \cdot F = 0 \}.
\]

In order to prove that \( N((A')_n') \) equals the kernel of \( \nu \), suppose first that \( F \in (A')_n' \) and that \( \nu_F = 0 \). It follows that \( F^2(f) = F(F \cdot f) = F(0) = 0 \) for all \( f \in A' \), so \( F \in N((A')_n') \). Conversely, the fact that \( F \in N((A')_n') \) implies that \( F \cdot G = G \cdot F = 0 \) for all \( G \in (A')_n' \). In particular we find that \( (F \cdot f)(a) = a''(F \cdot f) = (a'' \cdot F)(f) = 0 \) for all \( a \in A, f \in A' \) and hence \( \nu_F = 0 \).

In the next section we shall present a complete description of \( N((A')_n') \). In the case that \((A')_n'\) has a unit element, a combination of Proposition 5.1 and Theorem 5.2 yields an embedding of \( \text{Orth}(A) \) into \((A')_n'\), both as a Riesz space and as an algebra. The details follow in the next theorem. For any \( \pi \in \text{Orth}(A) \) we denote the restriction of the adjoint mapping of \( \pi' \) to \((A')_n'\) by \( \pi'' \).

**THEOREM 5.4.** Let \( A \) be Archimedean \( f \)-algebra with point separating order dual such that \((A')_n'\) possesses a unit element \( E \). The mapping \( \Phi : \text{Orth}(A) \to (A')_n' \) defined by \( \Phi(\pi) = \pi''E \) is an injective algebra and Riesz homomorphism. Moreover, the diagram

\[
\begin{array}{ccc}
\text{Orth}(A) & \xrightarrow{\rho} & A \\
\downarrow & & \downarrow \sigma \\
(A')_n' & \xrightarrow{\Phi} & \Phi
\end{array}
\]

is commutative.

**Proof.** Consider the mapping \( \gamma : \text{Orth}(A) \to \text{Orth}(A') \) and \( \nu : (A')_n' \to \text{Orth}(A') \) as defined in Proposition 5.1 and Theorem 5.2, respectively. The mapping \( \gamma \) is an injective algebra and Riesz homomorphism, whereas \( \nu \) is an \( f \)-algebra isomorphism. We claim that \( \Phi = \nu^{-1} \circ \gamma \). Indeed,

\[
\Phi(\pi) = \nu^{-1}(\gamma(\pi)) = \nu^{-1}(\pi') = \pi''(E)
\]

according to the remark made at the end of the proof of Theorem 5.2. It
follows immediately that $\phi$, being the composition of two injective algebra and Riesz homomorphisms, is itself an injective algebra and Riesz homomorphism.

Before proceeding it should be observed that the Archimedean unital $f$-algebra $(\mathcal{A})'_n$ is semiprime, so $\mathcal{A}$ is semiprime as well, as $\mathcal{A}$ is embeddable in $(\mathcal{A})'_n$. The mapping $\rho$ is therefore injective and embeds $\mathcal{A}$ canonically in $\text{Orth}(\mathcal{A})$.

It remains to show that the above diagram is commutative, in other words that $\sigma = \Phi \cdot \rho$, where $\sigma$ is the canonical embedding of $\mathcal{A}$ into $(\mathcal{A})'_n$. Fix an element $a \in \mathcal{A}$. For all $f \in \mathcal{A}'$ we have

\[
(\pi''(E)(f)) = E(\pi'_a f) = E(f \cdot a) = E(a'' \cdot f) = (E \cdot a'')(f) = a''(f),
\]

so $a'' = \pi''(E)$, i.e., $\sigma(a) = \Phi(\pi_a) = (\Phi \circ \rho)(a)$.

Since this holds for all $a \in \mathcal{A}$, we are through.

6. The Band of Nilpotents in $(\mathcal{A})'_n$

The set $N(\mathcal{A})$ of all nilpotents in an Archimedean $f$-algebra $\mathcal{A}$ is a band [21, Theorem 142.5]. Observe that $a \in N(\mathcal{A})$ if and only if $a^2 = 0$. The principal projection property in $\mathcal{A}$ is a sufficient condition that $N(\mathcal{A})$ be a projection band [17, Proposition 10.2]. If $\mathcal{A}$ has a unit element, then $N(\mathcal{A}) = \{0\}$.

This section is devoted to a description of $N((\mathcal{A})'_n)$, the band of all nilpotents in $(\mathcal{A})'_n$. Since $(\mathcal{A})'_n$ is Dedekind complete, $N((\mathcal{A})'_n)$ is even a projection band.

As agreed upon, $\mathcal{A}$ is an Archimedean $f$-algebra and $^0(\mathcal{A}') = \{0\}$. Let $J$ be the order ideal generated by all products of $\mathcal{A}$, i.e.,

\[
J = \{a \in \mathcal{A} : |a| \leq bc \text{ for some } b, c \in \mathcal{A}^+\}.
\]

The order ideal $J$ is obviously also a ring ideal in $\mathcal{A}$. If $\mathcal{A}$ is in addition uniformly complete and semiprime, then, by the results of [3, Sect. 4], the elements of $J$ are itself products if and only if $\mathcal{A}$ has property $(\ast)$ (or equivalently, the multiplicative decomposition property). The annihilator

\[
J^0 = \{f \in \mathcal{A}' : f(a) = 0 \text{ for all } a \in J\}
\]

is a projection band in $\mathcal{A}'$ [21, Theorem 88.4]. Note that the element $f \in (\mathcal{A})'$ is a member of $J^0$ if and only if $f(ab) = 0$ for all $a, b \in \mathcal{A}^+$. 
**Lemma 6.1.** If $F \in (A')_n^d$, then $F \in N((A')_n^d)$ if and only if $C_F \subseteq J^0$.

**Proof.** Without loss of generality we may assume that $F \geq 0$. Suppose first that $F \in N((A')_n^d)$, i.e., that $F^2 = 0$ and take $0 \leq f \in C_F$. Since the mapping $v_F$ is band preserving, it follows that $F \cdot f \in C_F$. On the other hand, $F^2(f) - F(F \cdot f) = 0$, so $F \cdot f \in N_F$. Combining these two results we find $F \cdot f = 0$. Furthermore, $\pi'_F$ is band preserving for all $a \in A^+$ and so $f \cdot a \in C_F$. It follows from $F(f \cdot a) = (F \cdot f)(a) = 0$ that $f \cdot a \in N_F$ and hence $f \cdot a = 0$. This implies that $F(ab) = (f \cdot a) b = 0$ for all $a, b \in A^+$ and thus $f \in J^0$.

Conversely, suppose that $C_F \subseteq J^0$. If $0 \leq f \in N_F$, then $F \cdot f \in N_F$ and so $F^2(f) = F(F \cdot f) = 0$. If $0 \leq f \in C_F$, then follows from the hypothesis that $f(ab) = (f \cdot a) b = 0$ for all $a, b \in A^+$, i.e., $f \cdot a = 0$ for all $a \in A^+$. This implies that $(F \cdot f)a = F(f \cdot a) = 0$ for all $0 \leq a \in A$ and hence $F \cdot f = 0$. The equality $F^2(f) = 0$ is immediate. Since $A' = N_F \oplus C_F$, we may conclude that $F^2(f) = 0$ for all $f \in A'$, i.e., $F^2 = 0$. The proof is complete.

As observed before, $N((A')_n^d)$ is a projection band in $(A')_n^d$, so

$$(A')_n^d = N((A')_n^d) \oplus N((A')_n^d)^d$$

is the decomposition of $(A')_n^d$ in a nilpotent part and a semiprime part. The next theorem gives a complete description of both parts.

**Theorem 6.2.** (i) $N((A')_n^d) = \{(J^0)^d\}^0$,

(ii) $N((A')_n^d)^d = (J^0)^0$,

where the second annihilator is taken in $(A')_n^d$.

**Proof.** (i) $N((A')_n^d) = \{F \in (A')_n^d: F^2 = 0\} = \{F \in (A')_n^d: C_F \subseteq J^0\} = \{F \in (A')_n^d: (J^0)^d \subseteq N_F\} = \{(J^0)^d\}^0$, where we use Lemma 6.1.

(ii) Since $A' = J^0 \oplus (J^0)^d$, we find $(A')_n^d = (J^0)^0 \oplus \{(J^0)^d\}^0$. Part (i) implies immediately that $N((A')_n^d)^d = (J^0)^0$.

The following corollary is evident.

**Corollary 6.3.** $(A')_n^d$ is semiprime if and only if $J^0 = \{0\}$.

The factorization property in $A$ (i.e., for every $a \in A$ there exists $b, c \in A$ such that $a = bc$; see [3, Sect. 4]) is thus a sufficient condition that $(A')_n^d$ be semiprime. Notice that in the uniformly complete case the former condition is equivalent to the property "square root closed."
7. When \((A^\prime)^n\) Is Semiprime

As before, \(A\) is an Archimedean \(f\)-algebra and \(\theta(A^\prime) = \{0\}\). We search for necessary and sufficient conditions that the \(f\)-algebra \((A^\prime)^n\) be semiprime. Since \(A\) is embeddable in \((A^\prime)^n\), it is quite reasonable therefore to assume beforehand that \(A\) is semiprime. Hence, we shall impose on \(A\) throughout this section the extra condition that \(A\) is semiprime.

If \((A^\prime)^n\) is semiprime, then the \(f\)-algebra \((A^\prime)^n\) can be embedded in \(\text{Orth}((A^\prime)^n)\) via the mapping \(\pi'\) that assigns to every \(F \in (A^\prime)^n\) the orthomorphism \(\pi_F\) (where \(\pi_F(G) = F \cdot G\) for all \(G \in (A^\prime)^n\)). Moreover, \(\text{Orth}(A)\) can be embedded in \(\text{Orth}((A^\prime)^n)\) by means of the mapping \(\kappa\) defined by \(\kappa(\pi) = \pi''\), where \(\pi''\) denotes the restriction of the second adjoint of \(\pi\) to \((A^\prime)^n\). As in the previous sections, \(\rho\) denotes the canonical embedding of \(A\) into \(\text{Orth}(A)\) (i.e., \(\rho(a) = a^\alpha\) for all \(a \in A\)) and \(\sigma\) is the embedding of \(A\) in \((A^\prime)^n\) (i.e., \(\sigma(a) = a^\alpha\) for all \(a \in A\)). Consider the following diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{\rho} & \text{Orth}(A) \\
\sigma \downarrow & & \kappa \\
(A^\prime)^n & \xrightarrow{\rho'} & \text{Orth}((A^\prime)^n)
\end{array}
\]

A moment's reflection shows that \(\pi_a^\alpha = \pi_a''\) for all \(a \in A\), in other words, the diagram is commutative. Since we embed \(\text{Orth}(A)\) in \(\text{Orth}((A^\prime)^n)\), we identify the unit elements, so we shall denote the unit element of \(\text{Orth}((A^\prime)^n)\) (i.e., the identity mapping on \((A^\prime)^n\)) by \(I\) again. After accomplishing all embeddings, the elements \(a, \pi_a, a^\alpha,\) and \(\pi_a^\alpha = \pi_a''\) are identified for all \(a \in A\).

We introduce the following concept.

**Definition 7.1.** The upward directed net \(\{a_\tau\}\) in \(A^+\) is said to be a \(\sigma(A, A')\)-approximate unit (or weak approximate unit) if \(f(b) = \sup_{\tau} f(a_\tau b)\) for all \(b \in A^+\) and all \(f \in (A^\prime)^+\).

It is easily verified that any \(\sigma(A, A')\)-approximate unit in \(A\) is an approximate unit. If \(A\) has a \(\sigma(A, A')\)-approximate unit, then \(A\) is necessarily semiprime by Theorem 2.3. The terminology "\(\sigma(A, A')\)-approximate" or "weak approximate" is of course on the analogy of the notations used in connection to the weak topology.

We are now in a position to prove the main theorem of this section. We have to impose, however, on \(A\) an extra condition, namely the Stone condition. On first sight this may seem rather artificial, but the reader must keep in mind that by Theorem 2.5 every uniformly complete semiprime \(f\)-algebra satisfies the Stone condition. Moreover, the Stone condition is strictly weaker than uniform completeness. By way of example, an \(f\)-algebra of step functions with respect to a ring of subsets of a nonempty point set
satisfies the Stone condition, but is in general not uniformly complete. The Stone condition is from the point of view of integration theory a very natural one.

**Theorem 7.2.** Let $A$ an Archimedean semiprime $f$-algebra with point separating order dual such that $A$ satisfies in addition the Stone condition. Then the following are equivalent.

(i) $(A')_n^\prime$ is semiprime.

(ii) $A$ has a weak approximate unit.

**Proof.** (ii) $\Rightarrow$ (i). Let $J$ be the order ideal generated by all products of $A$. By Corollary 6.3, it suffices to prove that $J^0 = \{0\}$. To this end, take $0 \leq f \in A'$ and suppose that $f(c) = 0$ for all $c \in J$. There exists a net $\{a_\tau\}$ in $A^+$ such that $f(b) = \sup_\tau f(a_\tau, b)$ for all $b \in A^+$. But $a_\tau, b \in J$ for all $\tau$, so $f(a_\tau, b) = 0$ for all $\tau$. It follows that $f(b) = 0$. This holds for all $b \in A^+$, so $f = 0$ and we are finished. Note that we do not use the Stone condition in this implication.

(i) $\Rightarrow$ (ii). We carry out all embeddings and identifications as described in the diagram preceding Definition 7.1. The order ideal $I(A)$ generated by $A$ in $(A')_n^\prime$ is strongly order dense. Hence, since $(A')_n^\prime$ is strongly order dense in Orth$((A')_n^\prime)$, the order ideal $I(A)$ is strongly order dense in Orth$((A')_n^\prime)$ as well. There exists therefore a net $\{F_\tau: \tau \in T\}$ in $I(A)$ such that $0 \leq F_\tau \uparrow I$ in Orth$((A')_n^\prime)$. The directed system $\{F_\tau\}_{\tau \in T}$ is therefore an approximate unit in $(A')_n^\prime$. Since $F_\tau \in I(A)$, there exists $a_\tau \in A^+$ such that $0 \leq F_\tau \leq a_\tau$ for all $\tau$. Hence,

$$0 \leq F_\tau = F_\tau \wedge I \leq a_\tau \wedge I \uparrow I$$

in Orth$((A')_n^\prime)$. By the Stone condition and the commutativity of the diagram, for appropriate $b_\tau \in A^+$.

8. **The Existence of a Unit Element in $(A')_n^\prime$**

Let $A$ be an Archimedean $f$-algebra with point separating order dual. It has been shown before that if $A$ has a unit element $e$, then $e''$ is the multiplicative identity of the $f$-algebra $(A')_n^\prime$. The present section is devoted to finding not only sufficient but also necessary conditions for the presence of a unit element in $(A')_n^\prime$ with respect to the Arens multiplication. If $(A')_n^\prime$
has a unit element, then \((A')_n^0\) and hence \(A\) are semiprime. We shall assume therefore throughout this section that \(A\) is, in addition, semiprime.

An account of corresponding problems in Banach algebras (where the second norm dual is given the Arens multiplication) can be found in the book by Bonsall and Duncan [5, Sect. 28]. They introduce the notion of a bounded (right, two-sided) approximate identity in a Banach algebra. We shall do the same for \(f\)-algebras.

Since \(A\) is, by hypothesis, semiprime, \(A\) can be embedded in Orth\((A)\) and according to Theorem 2.3,

\[ [0, I] \cap A = \{ a \in A : 0 \leq a \leq I \} \]

is an approximate unit of \(A\).

**DEFINITION 8.1.** The approximate unit \([0, I] \cap A\) is said to be \(\sigma(A, A')\)-bounded if

\[ M_f = \sup \{ f(a) : a \in [0, I] \cap A \} < \infty \]

for all \(0 \leq f \in A'\). More generally, an approximate unit \(\{a_\tau : \tau \in T\}\) in \(A\) is called \(\sigma(A, A')\)-bounded whenever \(\sup_\tau f(a_\tau) < \infty\) for all \(0 \leq f \in A'\).

Observe that \([0, I] \cap A\) is \(\sigma(A, A')\)-bounded if and only if all approximate units in \(A\) are \(\sigma(A, A')\)-bounded.

Suppose now that \((A')_n^0\) possesses a unit element \(E > 0\). We can embed Orth\((A)\) in \((A')_n^0\), as described in Theorem 5.4 and we therefore identify \(\Phi(I) = \Phi(E) = E\) with \(I\). If \(a \in [0, I] \cap A\), then we have that \(0 \leq a'' \leq E\) in \((A')_n^0\), so

\[ 0 \leq f(a) = a''(f) \leq E(f) \]

for all \(0 \leq f \in A'\). This implies that

\[ M_f = \sup \{ f(a) : a \in [0, I] \cap A \} \leq E(f) < \infty \]

for all \(f \in A'\) \(^+\), in other words, \([0, I] \cap A\) is \(\sigma(A, A')\)-bounded. The importance of this property lies in the fact that conversely for any \(f\)-algebra \(A\) satisfying the Stone condition, the \(\sigma(A, A')\)-boundedness of \([0, I] \cap A\) implies that \((A')_n^0\) has a unit element.

Before proving the main result of this section, we need a proposition.

**PROPOSITION 8.2.** Let \(A\) be an Archimedean semiprime \(f\)-algebra such that \(0(A') = \{0\}\) and \([0, I] \cap A\) is \(\sigma(A, A')\)-bounded. Then \((A')_n^0\) is semiprime.

**Proof.** It is here that Corollary 6.3 comes into play again. It is sufficient to show that \(J_0^0 = \{0\}\), where \(J\) denotes the order ideal generated by all products of \(A\). Let \(0 \leq a \in A\) and \(0 \leq f \in J_0^0\). Since \((I - na)(I - na)^+ \geq 0\) in Orth\((A)\), it is easily deduced that

\[ 0 \leq na - n(a \wedge na^2) \leq I \quad (n = 1, 2, \ldots). \]
Hence,

\[ 0 \leq nf'(a) - nf'(a \wedge na^2) \leq M_f < \infty \quad (n = 1, 2, \ldots). \]

However, \( a \wedge na^2 \in J \), so \( f(a \wedge na^2) = 0 \) \((n = 1, 2, \ldots)\). Now it follows from

\[ 0 \leq nf(a) \leq M_f \quad (n = 1, 2, \ldots) \]

that \( f(a) = 0 \). This holding for all \( a \in A^+ \), we derive \( f = 0 \). The desired result follows.

We are now in a position to prove the main theorem.

**Theorem 8.3.** Let \( A \) be an Archimedean semiprime \( f \)-algebra with point separating order dual such that \( A \) satisfies in addition the Stone condition. Then the following are equivalent:

(i) \((A')'_n\) has a unit element.

(ii) \([0, I] \cap A\) is \(\sigma(A, A')\)-bounded.

**Proof.** (i) \(\Rightarrow\) (ii). This has been observed already in the remarks preceding the above proposition. Note that the Stone condition is not needed for this implication.

(ii) \(\Rightarrow\) (i). According to Proposition 8.2, \((A')'_n\) is semiprime. We adhere to the identifications made in Section 7. By Theorem 7.2, there exists a weak approximate unit \( \{b_\tau : \tau \in T\} \) in \( A \) such that \( 0 \leq b_\tau \uparrow 1 \) in \( \text{Orth}((A')'_n) \). Hence, \( E(f) = \sup f(b_\tau) < \infty \) for all \( 0 \leq f \in A' \). This defines an additive function \( E \) on the positive cone \((A')^+_n\) of \( A' \), denoted by \( E \) again. Since \( b_\tau \in (A')'_n, b_\tau \uparrow E \) in \( A'' \) and \((A')'_n\) is a band in \( A'' \), we derive that \( E \in (A')'_n \). Summarizing, we have found that \( 0 \leq b_\tau \uparrow 1 \) in \( \text{Orth}(A')'_n \) and \( 0 \leq b_\tau \uparrow E \) in \((A')'_n \). It follows that \( E = 1 \), as \((A')'_n\) is strongly order dense in \( \text{Orth}(A')'_n \). Hence, \( E \) is the unit element of \((A')'_n\) and the proof is complete.

A combination of Theorems 7.2 and 8.3 together with the fact that every Archimedean unital \( f \)-algebra is semiprime, yields (under the same conditions for \( A \) as in Theorem 8.3) the following corollary.

**Corollary 8.4.** If \([0, I] \cap A\) is \(\sigma(A, A')\)-bounded, then \( A \) possesses a weak approximate unit.

For a better understanding of the notion of \(\sigma(A, A')\)-boundedness we present some equivalences. We maintain in the remainder of this section the conditions imposed on the \( f \)-algebra \( A \) in Theorems 7.2 or 8.3.

Introduce in \( A \) the order ideal \( A_h \) of all bounded elements of \( A \), i.e.,

\[ A_h = \{ a \in A : |ab| \leq n_a |b| \text{ for all } b \in A \} \]
(where \( n_a \) is a natural number depending on \( a \)). Since \( A \) is semiprime, \( A_b \) can be described equally as

\[
A_b = \{ a \in A : |a| \leq n_a I \text{ in Orth}(A) \}.
\]

For every \( f \subseteq A' \) we shall denote henceforth the restriction of \( f \) to \( A_b \) by \( f_b \).

**Theorem 8.5.** Under the same assumptions for \( A \) as before, the following are equivalent.

(i) \([0, I] \cap A \) is \( \sigma(A, A') \)-bounded.

(ii) Each \( f \in (A')^+ \) has a positive extension to Orth\( (A) \).

(iii) For every \( f \in (A')^+ \) the restriction \( f_b \) to \( A_b \) has a positive extension to \( Z(A) \).

**Proof.** (i) \( \Rightarrow \) (ii). By Theorem 8.3, \((A')_n^+\) has a unit element \( E > 0 \) with respect to the Arens multiplication. According to Theorem 5.4, Orth\( (A) \) can be embedded in \((A')_n^+\). The elements of \( A' \) act on \((A')_n^+\) as order continuous linear functionals (in fact, \( A' \) is perfect, i.e., \( A' = ((A')^+_n)_{\sigma} \); see [21, Sect. 110]. The restriction of such an element to Orth\( (A) \) is the desired extension.

(ii) \( \Rightarrow \) (iii). Evident.

(iii) \( \Rightarrow \) (i). We have to show that the supremum

\[
M_f = \sup \{ f(a) : a \in [0, I] \cap A \}
\]

is finite for all \( f \in (A')^+ \). Denote the extension of \( f_b \) to \( Z(A) \) by \( f_b \) again. Since \([0, I] \cap A \subseteq Z(A)\), we find

\[
0 \leq f(a) = f_b(a) \leq f_b(I)
\]

for all \( a \in [0, I] \cap A \). Hence, \( M_f \leq f_b(I) < \infty \) for all \( f \in (A')^+ \) and we are done.

Since \( Z(A) \) is an Archimedean Riesz space possessing \( I \) as a strong order unit, one can introduce in \( Z(A) \) the so-called \( I \)-uniform norm:

\[
\| \pi \| = \inf \{ k : k \geq 0, |\pi| \leq kI \}
\]

(see [15, Theorem 62.4]). Observe that \( |\pi| \leq \| \pi \| I \). The centre \( Z(A) \) is a normed Riesz space with respect to this norm. Furthermore, if \( A \) is in addition uniformly complete, then so is \( Z(A) \), i.e., \((Z(A), \| \cdot \|)\) is a Banach lattice.
By the Stone condition, \(a \in A\) implies \(|a| \wedge I \in A\). Define
\[
d(a, b) = \| |a| \wedge I\|
\]
for all \(a, b \in A\). Obviously \(d\) is a translation invariant metric in \(A\) (note that \(|a - b| \wedge I = 0\) implies that \(a = b\), as \(I\) is a weak order unit in Orth(\(A\))). Moreover,
\[
p(a) = d(a, 0) = \| |a| \wedge I\|
\]
defines a Riesz pseudonorm in \(A\) \([21, \text{Ex. 100.19}]\) which satisfies
\[
|a| \wedge I \leq p(a) I
\]
for all \(a \in A\).

Before presenting another characterization of the \(\sigma(A, A')\)-boundedness of \([0, I] \cap A\) we first need a lemma.

**Lemma 8.6.** Let \(A\) be a Riesz space with weak order unit \(e > 0\), and let \(a \in L^+\) satisfy \(a \wedge e \leq \varepsilon e\) for some \(0 < \varepsilon < 1\). Then \(a \leq \varepsilon e\).

**Proof.** Since \((a \wedge e - \varepsilon e)^+ = ((a - \varepsilon e) \wedge (1 - \varepsilon) e)^+ = 0\), we get \((a - \varepsilon e)^+ \wedge (1 - \varepsilon) e = 0\). Hence, \((a - \varepsilon e)^+ = 0\), as \(e\) is a weak order unit. The desired result follows.

**Theorem 8.7.** Under the same conditions for \(A\) as in Theorems 7.2 and 8.3, the following are equivalent.

(i) \([0, I] \cap A\) is \(\sigma(A, A')\)-bounded.

(ii) Every \(f \in (A')^+\) is continuous with respect to \(d\) (i.e., if \(d(a_n, a) \to 0\), then \(f(a_n) \to f(a)\)).

**Proof.** (i) \(\Rightarrow\) (ii). By Theorem 8.5, the restriction \(f_b\) of any \(f \in (A')^+\) to \(A_b\) has a positive extension \(f_b \in (Z(A'))^+\). Since \(d\) is translation invariant, we may restrict ourselves to sequences converging to 0, so let \(p(a_n) = d(a_n, 0) \to 0\) as \(n\) tends to infinity. There exists a natural number \(n_0\) such that \(p(a_n) < 1\) for all \(n \geq n_0\). Hence, \(|a_n| \wedge I \leq p(a_n) \cdot I\) and Lemma 8.6 imply that \(|a_n| \leq p(a_n) \cdot I\) for all \(n \geq n_0\). It follows from \(|f(a_n)| \leq f(|a_n|) = f_b(|a_n|) \leq p(a_n) \cdot f_b(I)\) for all \(n \geq n_0\) that \(f(a_n) \to 0\) for all \(f \in (A')^+\).

(ii) \(\Rightarrow\) (i). It is, according to Theorem 8.5, sufficient to show that for every \(0 \leq f \in A'\) the restriction \(f_b\) to \(A_b\) has a positive extension to \(Z(A)\). To this end, take \(f \in (A')^+\). By hypothesis, \(f\) is continuous with respect to \(d\). We assert that \(f_b\) is continuous with respect to the \(I\)-uniform norm \(|\cdot|\) on \(A_b\). For this purpose, take \(a_n \in A_b\) \((n = 1, 2, \ldots)\) and suppose that \(\|a_n\| \to 0\). There exists a natural number \(n_0\) such that \(\|a_n\| < 1\) for all \(n \geq n_0\). Since \(|a_n| \leq \varepsilon e\)
for all \( n \geq n_0 \). It follows that \( p(\alpha_n) \to 0 \). Hence, \( f_b(\alpha_n) \to 0 \) as \( n \) tends to infinity, which proves the claim. We have shown so far that \( f_b \) is a \((\text{norm})\) continuous linear functional on \((A_b, \| \cdot \|)\). By a Hahn–Banach type argument, \( f_b \) has a positive extension to \((Z(A), \| \cdot \|)\) (see, e.g., [18, Proposition II 5.6]).

9. Some Examples

(1) In the \(\mathcal{F}\)-algebra \( A = l_1 \) (with coordinatewise multiplication) there exists an approximate unit \( \{ \alpha_n \}_{n=1}^\infty \) which is not \( \sigma(A, A^\prime) \)-bounded (observe that each approximate unit in \( A \) is countable, as \( A \) is order separable). Take, e.g., \( \alpha_n = (1, \ldots, 1^{(n)}, 0, 0, \ldots) \) \((n = 1, 2, \ldots)\). Evidently, \( \sup_{n} f(\alpha_n) = \infty \) for \( f = (1, \ldots, 1, 1, \ldots) \in A^\prime = l_\infty \). Moreover, \( A \) satisfies all the conditions of Theorem 8.3. Hence, by the same theorem, \((A^\prime)^\prime_n \) does not have a unit element. An easy calculation shows that \((A^\prime)^\prime_n = l_1 \) and that Arens multiplication in \((A^\prime)^\prime_n \) is the coordinatewise multiplication again. It follows that \((A^\prime)^\prime_n \) is semiprime. This follows also from Theorem 7.2, as the above sequence \( \{ \alpha_n \}_{n=1}^\infty \) is a weak approximate unit in \( A \). Observe that contrary to the situation in Theorem 5.4, \((A^\prime)^\prime_n = l_1 \subset l_\infty = \text{Orth}(A)\).

(2) Let \( A = C_c(\mathbb{R}) \), the \(\mathcal{F}\)-algebra of all real continuous functions on \( \mathbb{R} \) with compact support; \( A \) satisfies all the conditions of Theorem 7.2 (and of Theorem 8.3). Introduce the trapezium functions \( \alpha_n \in C_c(\mathbb{R}) \) as follows:

\[
\alpha_n(x) = \begin{cases} 
1 & \text{if } x \in [-n, n], \\
0 & \text{if } x \geq n + 1, x \leq -n - 1, \\
\text{linear} & \text{in-between.}
\end{cases}
\]

It is not hard to verify that \( \{ \alpha_n \}_{n=1}^\infty \) is a weak approximate unit in \( A \). By Theorem 7.2, \((A^\prime)^\prime_n \) is semiprime. Another approach goes as follows. Since \( A \) has the factorization property (see [3, Sect. 4]), we find that \( A = J \). Hence, \( J = \{0\} \). By Corollary 6.3, \((A^\prime)^\prime_n \) is semiprime. However, \( \{ \alpha_n \}_{n=1}^\infty \) is not \( \sigma(A, A^\prime) \)-bounded. By Theorem 8.3, \((A^\prime)^\prime_n \) does not have a unit element.

(3) Take \( A = C([0, 1]) \) and write \( i(x) = x \) for all \( x \in [0, 1] \). Define \( a \ast b = iab \) for all \( a, b \in C([0, 1]) \). Equipped with this multiplication \( \ast \), \( A \) becomes an \(\mathcal{F}\)-algebra. Let \( \delta \) be the point evaluation in 0, i.e.,

\[
\delta(a) = a(0)
\]
for all $a \in A$. If $0 \leq a \in J$, then
\[ a \leq b \ast c = ibc \]
for appropriate $b, c \in A^+$. It follows immediately that $a(0) = \delta(a) = 0$. This holding for all $a \in J^+$, we find that $\delta \in J^0$. By Corollary 6.3, $(A')_n'$ is not semiprime.

(4) Let $X$ be an arbitrary Tychonov space and let $F$ be a proper nonempty closed subset of $X$. The $f$-algebra
\[ A = \{ a \in C(X): a|F = 0 \} \]
(here $a|F$ denotes the restriction of $a$ to $F$) is Archimedean, semiprime, satisfies the Stone condition, but does not have a unit element. Observe that by [17, Theorem 20.3],
\[ A_b \subset Z(A) = C_b(X - F). \]
The $I$-uniform norm of any $a \in A_b$ is equal to
\[ \sup \{ |a(x)|: x \in X - F \} = \sup \{ |a(x)|: x \in X \} = \| a \|_\infty, \]
as $a(x) = 0$ for all $x \in F$. Take $0 \leq f \in A'$. If $f_b$ were not continuous with respect to $\| \cdot \|$, there would exist a sequence $\{a_n\}_{n=1}^\infty$ in $A_b$ such that
\[ \|a_n\|_\infty \leq \frac{1}{n^2}, \quad f(a_n) \geq n \quad (n = 1, 2,...). \]
If we define the uniform convergent sum
\[ a_0(x) = \sum_{n=1}^\infty a_n(x) \]
for all $x \in X$, then we have obviously that $a_0 \in A_b$. On the other hand, $f(a_0) \geq f(a_n) \geq n$ for all $n$, a contradiction. It follows that $f_b$ is $\| \cdot \|$-continuous and has therefore a positive extension to $Z(A)$. By Theorems 8.3 and 8.5, $(A')_n'$ has a unit element (although $A$ has not).

References


