Finite representability of operators

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Received 17 April 2002
Submitted by J.B. Conway

Abstract

We introduce operator local supportability as a new type of operator finite representability that generalizes Bellenot finite representability. We prove that local supportability and local representability are mutually independent. New examples of both types of finite representability are given. For instance, for every operator \(T\), we prove that \((T^*U)^*\) is locally supportable in \((T^*)^*U\). We also prove that, given an operator \(T\) with range in \(C[0,1]\), \(T^*\) is locally representable in \(T^*|_{L_1}\).

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1. Introduction

While finite representability is a well stated term for Banach spaces, there are at least six different definitions about what it should mean in the context of operator theory \([4–7,14,18]\). In this paper, we suggest the joining of local theory, operator ideal theory, and operator semigroup theory as the natural frame to analyze and compare all those notions of finite representability. Indeed, the definitions above have been applied to the study of regular, ultrapower-stable ideals. In this sense, the notion of local representability \([18]\), which generalizes Heinrich finite representability \([14]\), is probably the most remarkable. But there are other classes of operators, baptized as semigroups in \([1]\), that are relevant in the study of Banach spaces. Examples of semigroups are the class of semi-Fredholm operators and the class of Tauberian operators. Semigroups exhibit a rich interplay with...
operator ideals; this fact has been used, for instance, in [12] to study weak Calkin algebras.
We consider two classes of ultrapower-stable semigroups: the first, formed by those that are injective and left-stable, and the second, formed by those that are surjective and right-stable. Following the program laid out in [18], we generalize Bellonot finite representability [7] by introducing local supportability, which is perfectly adapted to the first class of semigroups, as we prove in Proposition 4.6. Regarding the semigroups belonging to the second class, we show that local supportability is also the right notion, but it requires delicate work. For, given an operator $T$, we show in Theorems 4.10 and 4.11 a very strong result about finite representability of the conjugate operator $(T_{\Omega})^*$ in the ultrapower $(T^*)_{\Omega}$.
As a consequence, we prove that $T_{\Omega}^*$ is Bellonot finitely representable in $T^*_{\Omega}$. Section 5 is devoted to show that the main types of finite representability involved in this paper, local supportability and local representability, are independent. For, given any operator $T: Y \to \mathcal{C}[0, 1]$, we prove in Theorem 5.3 that $T^*$ is Heinrich finitely representable in $T^*_{|L_1}$ and, in Proposition 5.5, we exhibit an operator $T: \mathcal{C}[0, 1] \to \mathcal{C}[0, 1]$ such that $T^*_{|\mathcal{N}}$ is an isomorphism but $T^*$ is not Tauberian, where $\mathcal{N}$ stands for the subspace of singular measures with respect to the Lebesgue measure on $[0, 1]$.

2. Notation

We denote by $\mathcal{L}(X, Y)$ the class of all bounded operators between the Banach spaces $X$ and $Y$; the kernel and the range of $T \in \mathcal{L}(X, Y)$ are, respectively, denoted by $\ker(T)$ and $\operatorname{ran}(T)$; given a class $\mathcal{A}$ of bounded operators, we denote by $\mathcal{A}(X, Y) := \mathcal{A} \cap \mathcal{L}(X, Y)$ the subset of operators in $\mathcal{A}$ acting between $X$ and $Y$; $B_X$ stands for the unit closed ball of $X$, and $S_X$ denotes the set of all norm one elements of $X$; $K_X$ denotes the natural embedding of $X$ into $X^{**}$; given a closed subspace $Z$ of $X$, $Q_Z$ stands for the quotient operator from $X$ onto $X/Z$; given $f \in X^*$ and $x \in X$, the action of duality will be denoted by $f(x)$ or $(f, x)$; the weak* topology of $X^*$ induced by $X$ is denoted $w^*$; given a subset $A$ of $X$, $\overline{\operatorname{span}}(A)$ stands for the closed subspace of $X$ generated by $A$. Every $T \in \mathcal{L}(X, Y)$ induces an operator $\overline{T} \in \mathcal{L}(X, R(T))$ defined by $\overline{T}(x) = T(x)$. Given $d \geq 1$, we say that $T \in \mathcal{L}(X, Y)$ is a $d$-injection if $T$ is an isomorphism into $Y$ verifying $d^{-1} \leq \|T(x)\| \leq d$ for all $x \in S_X$; if $d = 1$ then $\overline{T}$ is called a metric injection. An operator is said to be an embedding if it is a $d$-injection for some $d$. A metric surjection is an operator $T \in \mathcal{L}(X, Y)$ whose conjugate $T^*$ is a metric injection.

Ultrapowers of Banach spaces play an important role in this paper; proofs and details about the facts listed below can be found in [13]. An ultrafilter $\Omega$ on a set of indices $I$ is said to be $\aleph_0$-incomplete if there is a countable partition of $I$, $\{I_n\}_{n=1}^{\infty}$, such that $I_n \notin \Omega$ for all $n \in \mathbb{N}$. All ultrafilters considered in this paper are $\aleph_0$-incomplete. The ultrapower of a Banach space $X$ following an ultrafilter $\Omega$ on $I$ is the quotient $X_{\Omega} := \ell_\infty(I, X)/N$, where $N$ denotes the closed subspace of all null families following $\Omega$. The element of $X_{\Omega}$ whose representative is $(x_i)_{i \in I}$ will be denoted by $[x_i]$; its norm is $\|[x_i]\| = \lim_{n \to \Omega} \|x_i\|$. $X_{\Omega}$ contains a canonical copy of $X$. Given a family $(A_i)_{i \in I}$ of subsets of $X$, we denote $(A_i)_{\Omega} := \{[x_i] \in X_{\Omega}: x_i \in A_i\}$. Given an ultrafilter $\mathcal{G}$ on $J$, the family $\Omega \times \mathcal{G}$ formed by all the subsets $A \subset I \times J$ verifying $\{|: (i, j) \in A\} \in \Omega \times \mathcal{G}$ is an ultrafilter on $I \times J$. The iteration theorem establishes that $(X_{\Omega})_{\mathcal{G}}$ is canonically isometric to $X_{\Omega \times \mathcal{G}}$. Given
$T \in \mathcal{L}(X, Y)$, the ultrapower of $T$ following $\Delta$ is the operator $T_{\Delta} \in \mathcal{L}(X_{\Delta}, Y_{\Delta})$ that maps $[x_1]$ onto $[Tx_1]$. An operator $T$ is a metric injection if and only if any (and all) of its ultrapowers is also a metric injection [11].

We denote by $\mu$ the usual Lebesgue measure on $[0, 1]$; $\mathcal{M}$ stands for the space of all Radon measures on $[0, 1]$, which is the dual of $\mathcal{C}$, the space of all continuous functions on $[0, 1]$; $L_1$ stands for the space of all $\mu$-integrable functions. We identify $L_1$ with an isometric copy canonically contained in $\mathcal{M}$, so we can write $\mathcal{M} = L_1 \oplus_1 \mathcal{N}$, where $\mathcal{N}$ stands for the subspace of all singular measures with respect to $\mu$.

### 3. Types of operator finite representability

Bellenot [7, Section 3] defines an operator $T \in \mathcal{L}(X, Y)$ to be finitely representable in $S \in \mathcal{L}(W, Z)$ if for every finite dimensional subspace $E$ of $X$ and every $\varepsilon > 0$ there is a $(1 + \varepsilon)$-injection $L \in \mathcal{L}(E, W)$ verifying $\|Tx\| - \|S\phi x\| \leq \varepsilon\|x\|$ for all $x \in E$; equivalently, there are $(1 + \varepsilon)$-injections $U \in \mathcal{L}(E, W)$, $V \in \mathcal{L}(T(E), Z)$ so that $\|SU - VT\| \leq \varepsilon$.

Heinrich [14, Definition 1.1] says an operator $T \in \mathcal{L}(X, Y)$ is finitely representable in $S \in \mathcal{L}(W, Z)$ if for every $\varepsilon > 0$, every finite dimensional subspace $E$ of $X$ and every finite codimensional subspace $F$ of $Y$ there is a finite dimensional subspace $E_1$ of $W$, a finite codimensional subspace $F_1$ of $Z$ and a pair of surjective $(1 + \varepsilon)$-injections $U \in \mathcal{L}(E, E_1)$, $V \in \mathcal{L}(Z/F_1, Y/F)$ such that $\|VQF, SU - QFT\| \leq \varepsilon$.

Our purpose needs more general types of finite representability than those of Bellenot and Heinrich. So we adopt the following definitions.

**Definition 3.1**. Given $d \geq 1$, we say that $T \in \mathcal{L}(X, Y)$ is locally $d$-supportable in $S \in \mathcal{L}(W, Z)$ if for every $\varepsilon > 0$ and every finite dimensional subspace $E$ of $X$ there is a $(d + \varepsilon)$-injection $U \in \mathcal{L}(E, W)$ and an operator $V \in \mathcal{L}(T(E), Z)$ verifying $\|V\| \leq d + \varepsilon$ and $\|SU - VT\| \leq \varepsilon$.

**Definition 3.2** [18, 6.6]. Given $c > 0$, an operator $T \in \mathcal{L}(X, Y)$ is said to be locally $c$-representable in $S \in \mathcal{L}(W, Z)$ if for every $\varepsilon > 0$ and every pair of operators $A \in \mathcal{L}(E, X)$, $B \in \mathcal{L}(Y, F)$ with $E$ and $F$ finite dimensional spaces there is a pair of operators $A_1 \in \mathcal{L}(E, W)$, $B_1 \in \mathcal{L}(Z, F)$ verifying $\|B_1\| \cdot \|A_1\| \leq (c + \varepsilon)\|A\| \cdot \|B\|$ and $BTA = B_1SA_1$.

When we do not need to specify parameters $d$ or $c$ in the above definitions, we will just speak of local supportability or local representability. The next propositions give characterizations for local supportability and local representability in terms of ultrapowers. These characterizations yield the main applications in semigroups and ideals of operators and also show that Bellenot finite representability and Heinrich finite representability are respectively generalized by local supportability and local representability.
Proposition 3.3. Let $T \in L(X, Y)$ and $S \in L(W, Z)$ be a pair of operators. We have:

(a) $T$ is locally $d$-supportable in $S$ if and only if there is an ultrafilter $\mathcal{U}$, a $d$-injection $U \in \mathcal{L}(X, W_{\mathcal{U}})$ and an operator $V \in \mathcal{L}(R(T), Z_{\mathcal{U}})$ so that $S_{\mathcal{U}}U = V T$ and $\|V\| \leq d$;
(b) $T$ is Bellenot finitely representable in $S$ if and only if there is an ultrafilter $\mathcal{U}$ and metric injections $U \in \mathcal{L}(X, W_{\mathcal{U}})$, $V \in \mathcal{L}(R(T), Z_{\mathcal{U}})$ such that $S_{\mathcal{U}}U = V T$.

Proof. For part (a), assume that $T$ is locally $d$-supportable in $S$. Let $\mathcal{F}$ be the collection of all finite dimensional subspaces of $X$ and consider the order filter on $\mathcal{F}$, which consists of all sets $\{E \in \mathcal{F} : E \supset F\}$ for every $F \in \mathcal{F}$. Let $\mathcal{U}$ be an ultrafilter on $\mathcal{F}$ containing the order filter. For each $E \in \mathcal{F}$, we write $\varepsilon_E := (\dim E)^{-1}$, and choose a $(d + \varepsilon_E)$-injection $U_E \in \mathcal{L}(E, W)$ and an operator $V_E \in \mathcal{L}(T(E), Z)$ so that $\|V_E\| \leq d + \varepsilon_E$ and $\|S_{\mathcal{U}}U - V_E T\|_{E} \leq \varepsilon_E$. We define an operator $U \in \mathcal{L}(X, W_{\mathcal{U}})$ by $U(x) = [x_E]$, where $x_E := U_E(x)$ if $x \in E$ and $x_E = 0$ otherwise. Thus, for $x \in S_X$ we have

$$\lim_{E \to \mathcal{U}} (d + \varepsilon_E)^{-1} \leq \|U(x)\| \leq \lim_{E \to \mathcal{U}} (d + \varepsilon_E),$$

and as $\lim_{E \to \mathcal{U}} \varepsilon_E = 0$, we obtain that $U$ is a $d$-injection. Analogously, for each $y \in T(X)$ and every $E \in \mathcal{F}$, we write $y_E := V_E(y)$ if $y \in T(E)$ and $y_E = 0$ otherwise. Hence $V(y) = [y_E]$ defines an operator from $R(T)$ into $Z_{\mathcal{U}}$ such that $\|V\| \leq d$. The identity $S_{\mathcal{U}}U - V T = 0$ follows from the fact that $\|S_{\mathcal{U}}U - V T\| \leq \varepsilon_E$.

For the converse of (a), let us assume there is an ultrafilter $\mathcal{U}$ on a set $I$, a $d$-injection $U \in \mathcal{L}(X, W_{\mathcal{U}})$ and an operator $V \in \mathcal{L}(R(T), Z_{\mathcal{U}})$ verifying $\|V\| \leq d$ and $S_{\mathcal{U}}U - V T = 0$. Fix a finite dimensional subspace $E$ of $X$, $\varepsilon > 0$, and a basis $\{x_j\}_{j=1}^n$ of $E$. For each $j \in [1, \ldots, n]$, let $U e_j = [x_j]$, and for every $i$, let $y_i^j := S x_i$.

By Proposition 6.1 in [14], there is $J \in \mathcal{U}$ such that, for every $i \in J$, the operators $U_i \in \mathcal{L}(E, X)$, $V_i \in \mathcal{L}(E, Y)$ defined by $U_i (e_j) := x_j$, $V_i (T e_j) := y_i^j$ verify that $\|V_i\| \leq d + \varepsilon$ and $U_i$ is a $(d + \varepsilon)$-injection. Moreover, the identity $S_{\mathcal{U}}U - V T|_E = 0$ implies $\lim_{i \to \mathcal{U}} \|SU_i - V_i T|_E\| \leq \varepsilon$. Thus $T$ is locally $d$-supportable in $S$.

Proof of part (b) is similar to that of part (a). \qed

Proposition 3.4. Given $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(W, Z)$, we have:

(a) [18, 6.6] $T$ is locally $c$-representable in $S$ if and only if there is an ultrafilter $\mathcal{U}$ and a pair of operators $U \in \mathcal{L}(X, W_{\mathcal{U}})$, $V \in \mathcal{L}(Z_{\mathcal{U}}, Y^{**})$ such that $V S_{\mathcal{U}}U = K_T$ and $\|U\| \cdot \|V\| \leq c$;
(b) [14, Theorem 1.2] $T$ is Heinrich finitely representable in $S$ if and only if there is an ultrafilter $\mathcal{U}$, a metric injection $U \in \mathcal{L}(X, W_{\mathcal{U}})$ and a metric surjection $V \in \mathcal{L}(Z_{\mathcal{U}}, Y^{**})$ such that $V S_{\mathcal{U}}U = K_T$.

When the final space of the operator $T$ is a conjugate space, the statement of Proposition 3.4 admits the following reformulation that will ease further reasonings.

Proposition 3.5. Given a pair of operators $T \in \mathcal{L}(X, Y^{*})$ and $S \in \mathcal{L}(W, Z)$, the following statements hold:
(a) $T$ is locally $c$-representable in $S$ if and only if there is an ultrafilter $\mathcal{U}$ and operators $A \in \mathcal{L}(X, W_{\mathcal{U}})$, $B \in \mathcal{L}(Z_{\mathcal{U}}, Y^*)$ such that $T = BS_{\mathcal{U}}A$ and $\|B\| \cdot \|A\| \leq c$;

(b) $T$ is finitely representable in $S$ in the sense of Heinrich if and only if there is an ultrafilter $\mathcal{U}$, a metric injection $U \in \mathcal{L}(X, W_{\mathcal{U}})$ and a metric surjection $V \in \mathcal{L}(Z_{\mathcal{U}}, Y^*)$ verifying $T = VS_{\mathcal{U}}U$.

**Proof.** (a) Assume $T$ is locally $c$-representable in $S$. By Proposition 3.4 there exists an ultrafilter $\mathcal{U}$ and operators $A \in \mathcal{L}(X, W_{\mathcal{U}})$, $C \in \mathcal{L}(Z_{\mathcal{U}}, Y^{3(*)})$ such that $KY \cdot T = CS_{\mathcal{U}}A$ and $\|C\| \cdot \|A\| \leq c$. Let $P \in \mathcal{L}(Y^{3(*)}, Y^{3(*)})$ be the norm one projection onto $Y^*$ associated to the decomposition $Y^{3(*)} = Y^* \oplus Y^\perp$ and define $B := PC$. Clearly, $T = BS_{\mathcal{U}}A$ and $\|B\| \cdot \|A\| \leq c$.

For the converse, assume there is an ultrafilter $\mathcal{U}$ and operators $A \in \mathcal{L}(X, W_{\mathcal{U}})$, $B \in \mathcal{L}(Z_{\mathcal{U}}, Y^*)$ verifying $T = BS_{\mathcal{U}}A$ and $\|B\| \cdot \|A\| \leq c$. Let $C := KY \cdot T = CS_{\mathcal{U}}A$ and $\|C\| \cdot \|A\| \leq c$, so Proposition 3.4 shows $T$ is locally $c$-representable in $S$.

(b) The direct implication of part (b) is similar to that of part (a). For the converse, assume there is an ultrafilter $\mathcal{U}$, a metric injection $U \in \mathcal{L}(X, W_{\mathcal{U}})$ and a metric surjection $V \in \mathcal{L}(Z_{\mathcal{U}}, Y^*)$ verifying $T = VS_{\mathcal{U}}U$. Since $T$ is Heinrich finitely representable in itself, by Proposition 3.4 there is an ultrafilter $\mathcal{M}$, a metric injection $U_1 \in \mathcal{L}(X, X_{\mathcal{M}})$ and a metric surjection $V_1 \in \mathcal{L}(Z_{\mathcal{M}}, Y^{3(*)})$ verifying $KY \cdot T = V_1T WS_{\mathcal{M}} U_1$. Thus $A := U_{\mathcal{M}}U_1 \in \mathcal{L}(X, W_{\mathcal{M}})$ is a metric injection, $B := V_1 V_{\mathcal{M}} \in \mathcal{L}(Z_{\mathcal{M}} \times X_{\mathcal{M}}, Y^{3(*)})$ is a metric surjection, and $KY \cdot T = BS_{\mathcal{M}}X_{\mathcal{M}} A$. By Proposition 3.4, $T$ is Heinrich finitely representable in $S$. \end{proof}

Henceforth, it will be very convenient to adopt the following notations: $T <_{h} S$ means that the operator $T$ is locally supportable in $S$, and $T <_{lr} S$ means that $T$ is locally representable in $S$.

### 4. Semigroups and ideals of operators

Pietsch [18] has proved that every regular, ultrapower-stable ideal of operators $\mathcal{A}$ is stable under local representability; namely, if $T <_{lr} S$ and $S \in \mathcal{A}$ then $S \in \mathcal{A}$. But besides ideals, there are other classes of operators, called *semigroups* in [1], which are also remarkable in Banach space theory. Examples of semigroups are the class of all upper semi-Fredholm operators and the class of all Tauberian operators. We follow the program laid out in [18] to show that certain types of ultrapower-stable semigroups are stable under local supportability. Applications of this fact are given in [15].

We recall that a class of operators $\mathcal{A}$ is said to be ultrapower-stable if, for every $T \in \mathcal{A}$, all ultrapowers $T_{\mathcal{U}}$ belong to $\mathcal{A}$. If $\mathcal{A}$ is endowed with a preorder $\leq$, $\mathcal{A}$ is said to be $\leq$-stable if $T \in \mathcal{A}$ and $S \leq T$ imply $S \in \mathcal{A}$.

**Proposition 4.1. Local supportability and local representability are preorders.**

**Proof.** The proof for local representability can be found in [18]. Assume now that the operators $T_1 \in \mathcal{L}(X_1, Y_1)$, $T_2 \in \mathcal{L}(X_2, Y_2)$, $T_3 \in \mathcal{L}(X_3, Y_3)$ verify $T_1 <_{lr} T_2$ and $T_2 <_{lr} T_3$. Then...
By Proposition 3.3, there are ultrafilters \( \mathcal{U} \) and \( \mathcal{V} \), embeddings \( U_1 \in L(X_1, (X_2)_\mathcal{U}) \), \( U_2 \in L(X_2, (X_3)_\mathcal{V}) \) and operators \( V_1 \in L(R(T_1), (Y_2)_\mathcal{U}) \), \( V_2 \in L(R(T_2), (Y_3)_\mathcal{V}) \) so that \( V_1 \mathcal{T}_1 = (T_2)_\mathcal{U} U_1 \) and \( V_2 \mathcal{T}_2 = (T_3)_\mathcal{V} U_2 \). Thus, by the iteration theorem for ultrapowers, we can write \( (V_2)_\mathcal{U} V_1 \mathcal{T}_1 = (T_3)_\mathcal{V} (U_2)_\mathcal{U} U_1 \). But \( (U_2)_\mathcal{U} \) is an embedding, so Proposition 3.3 yields \( T_1 \prec_\mathcal{U} T_3 \).  

For the sake of completeness, we recall the following definitions.

Definition 4.2 [1]. A class of operators \( S \) is said to be an operator semigroup if the following conditions hold:

(i) \( S \) contains all bijective operators;
(ii) If \( T \in S(X, Y) \) and \( S \in S(Y, Z) \) then \( ST \in S(X, Z) \);
(iii) \( T \in S(X, Y) \) and \( S \in S(U, V) \) if and only if \( T \times S \in S(X \times U, Y \times V) \).

We note that if \( S \) is a semigroup of operators, then the class \( S^d := \{ T : T^* \in S \} \) is also a semigroup.

Definition 4.3 [17]. A class of operators \( A \) is said to be an operator ideal if the following conditions hold:

(i) \( A \) contains all finite dimensional range operators;
(ii) For every \( X \) and \( Y \), \( A(X, Y) \) is a subspace of \( L(X, Y) \);
(iii) If \( S \in L(W, X) \), \( T \in A(X, Y) \) and \( U \in L(Y, Z) \) then \( UT S \in A(W, Z) \).

Linkages between semigroups and ideals are given in [1]. We recall the following: given an operator ideal \( A \), the classes \( A^+ := \{ T : TS \in A \Rightarrow S \in A \} \) and \( A^- := \{ T : ST \in A \Rightarrow S \in A \} \) are operator semigroups.

Definition 4.4 [1]. Let \( S \) be an operator semigroup:

(i) \( S \) is said to be left-stable if \( ST \in S \) implies \( T \in S \);
(ii) \( S \) is said to be right-stable if \( ST \in S \) implies \( S \in S \).

Definition 4.5 [1]. Let \( S \) be an operator semigroup:

(i) \( S \) is said to be injective if every upper semi-Fredholm operator belongs to \( S \);
(ii) \( S \) is said to be surjective if every lower semi-Fredholm operator belongs to \( S \).

Given an ideal \( A \), the semigroup \( A^+ \) (respectively, \( A^- \)) is injective (surjective) if and only if \( A \) is an injective (surjective) ideal [1, Proposition 2.12].

The interesting semigroups are those whose elements preserve some isomorphic property. Therefore, we agree in calling trivial a semigroup that contains the null operator 0_X.
for every Banach space $X$. Note that if $\mathcal{A}$ is an operator ideal and $\mathcal{A}_+$ (or $\mathcal{A}_-$) is trivial then

$$\mathcal{A}(X, Y) = \mathcal{A}_+(X, Y) = \mathcal{A}_-(X, Y) = \mathcal{L}(X, Y) \quad \text{for all } X \text{ and } Y.$$ 

It is immediate that a semigroup holding any combination of conditions (i) and (ii) of those stated in Definitions 4.4 and 4.5 is trivial. So we are only concerned about semigroups which are either left-stable and injective or right-stable and surjective.

It is also immediate after Proposition 3.4 that each ultrapower-stable, regular ideal is $\prec_{lh}$-stable. The necessary result for injective, left-stable, ultrapower-stable semigroups is the following.

**Proposition 4.6.** Let $S$ be an injective, left-stable, ultrapower-stable semigroup. Then $S$ is $\prec_{lh}$-stable.

**Proof.** Assume $S \in \mathcal{S}$ and $T \prec_{lh} S$. By Proposition 3.3 there is an ultrafilter $\mathcal{U}$, an isomorphism $U$ and an operator $V$ such that $V^{\mathcal{U}} = S_{\mathcal{U}} U$. Since $S$ is ultrapower-stable, we have $S_{\mathcal{U}} \in \mathcal{S}$. Moreover, $S$ is injective so $U \in \mathcal{S}$; therefore $S_{\mathcal{U}} U \in \mathcal{S}$. Left-stability yields $T = S \in \mathcal{S}$, and again, the injectivity of $S$ leads to $T \in \mathcal{S}$. \(\Box\)

If $\mathcal{R}$ is a surjective, right-stable semigroup, it is immediate that $\mathcal{R}^d$ is injective and left-stable. In Proposition 4.12 we prove that if moreover $\mathcal{R}$ is ultrapower-stable then so is $\mathcal{R}^d$. Hence we conclude in Proposition 4.13 that local supportability is also the right notion of finite representability for $\mathcal{R}$. The way to Proposition 4.12 needs some preliminary results.

**Lemma 4.7.** Let $E \subset X$ be a finite dimensional subspace with $\dim E = n$ and let $0 < \varepsilon < 1/n$. Then every $\varepsilon$-net in $S_E$ contains a basis whose coordinate functionals are norm bounded by $(1 - n\varepsilon)^{-1}$.

**Proof.** Let $E$ be an $\varepsilon$-net in $S_E$. By Auerbach’s lemma, there is a biorthogonal system $(u_i, h_i)_{i=1}^n \subset E \times X^*$. For every $u_i$, we choose $e_i \in \mathcal{E}$ so that $\|u_i - e_i\| \leq \varepsilon$. We define the operator $L \in \mathcal{L}(E, E)$ by $L(e) := \sum_{i=1}^n h_i(e) \cdot e_i$. Note that $L(u_i) = e_i$ and $\|I_E - L\| \leq n\varepsilon$, so $L$ is an isomorphism, hence $\{e_i\}_{i=1}^n$ is a basis of $E$. Moreover, given $x = \sum_{i=1}^n \lambda_i e_i \in S_E$, and writing $u := L^{-1}(e) = \sum_{i=1}^n \lambda_i u_i$, we get $\|x - u\| \leq n\varepsilon \|u\|$. Hence, for every $i$, $|\lambda_i| = |h_i(u)| \leq \|u\| \leq (1 - n\varepsilon)^{-1}$. Thus, the coordinate functionals associated to $\{e_i\}_{i=1}^n$ are norm bounded by $(1 - n\varepsilon)^{-1}$. \(\Box\)

**Lemma 4.8.** Let $E \subset X^*$ be a finite dimensional subspace with $\dim E = n$, $\{e_i\}_{i=1}^n$ an $\varepsilon$-net in $S_E$ with $0 < \varepsilon < (2n)^{-1}$, and $\mathcal{V}$ a weak* neighborhood of $0 \in X^*$. If $\{L_\alpha\}_{\alpha \in \Lambda}$ is a net of operators from $E$ into $X^*$ such that $\|L_\alpha(e_i)\| \leq 1$ for all $\alpha$ and $w^*\text{-lim}_{\alpha} L_\alpha(e_i) = e_i$ for all $1 \leq i \leq p$, then there is $\alpha \in \Lambda$ such that $L_\alpha$ is a $(1 - 2n\varepsilon)^{-1}$-injection and $L_\alpha(e) \in \varepsilon + \mathcal{V}$ for all $e \in S_E$.

**Proof.** By Lemma 4.7, we can assume that $\{e_i\}_{i=1}^n$ is a basis of $E$ whose coordinate functionals are norm bounded by $(1 - n\varepsilon)^{-1}$. Consequently, $\|L_\alpha\| \leq (1 - n\varepsilon)^{-1}n$ for all $\alpha$. Since the equalities $w^*\text{-lim}_{\alpha} L_\alpha e_i = e_i$ hold and $(1 - n\varepsilon)^{-1} < 2$, we can select $\beta \in \Lambda$ such that
verifying $\|L_\beta e_i\| \geq 1 - n\varepsilon(2 - (1 - n\varepsilon)^{-1})$ for all $1 \leq i \leq p$, so $L_\beta$ is a $(1 - 2n\varepsilon)^{-1}$-injection. Indeed, given $e \in S_F$, by choosing $e_i$ so that $\|e - e_i\| \leq \varepsilon$, we obtain
\[
\|L_\beta(e)\| \leq \|L_\beta(e_i)\| + \|L_\beta(e - e_i)\| \leq 1 + n\varepsilon(1 - n\varepsilon)^{-1} < (1 - 2n\varepsilon)^{-1}
\]
and
\[
\|L_\beta(e)\| \geq \|L_\beta(e_i)\| - \|L_\beta(e_i)\| \geq 1 - n\varepsilon(2 - (1 - n\varepsilon)^{-1}) - n\varepsilon(1 - n\varepsilon)^{-1} = 1 - 2n\varepsilon,
\]
which proves that $L_\beta$ is a $(1 - 2n\varepsilon)^{-1}$-injection.

In order to finish, let $\mathcal{U}$ be an absolutely convex weak* neighborhood of $0$ such that $n(1 - n\varepsilon)\mathcal{U} \subset V$. By choosing $\beta$ with the additional conditions $L_\beta(e_i) \in e_i + \mathcal{U}$ for all $i$, it follows that for every $e = \sum_{i=1}^n \lambda_i e_i \in S_F$ we have $L_\beta(e) - e \in \sum_{i=1}^n \lambda_i \mathcal{U} \subset V$. □

**Proposition 4.9.** For every operator $T \in \mathcal{L}(X, Y)$ and every ultrafilter $\mathcal{U}$, the set $B = \{ h \in B_{Y^*_\mathcal{U}} : \|T_{\mathcal{U}}^*(h)\| \leq 1 \}$ is the weak* closure in $Y_{\mathcal{U}}^*$ of $A = \{ h \in B_{Y^*_\mathcal{U}} : \|T_{\mathcal{U}}^*(h)\| \leq 1 \}$.

**Proof.** Let $I$ be the set of indices on which $\mathcal{U}$ is taken. Take $f \notin A^{w^*}$, and prove that $f \notin B$. By Hahn–Banach theorem there is $y_0 = [y_i] \in Y_{\mathcal{U}}$ and a pair of real numbers $a, b$ such that $b(y_0) \leq a < b < f(y_0)$ for all $h \in A$. For every $i \in I$, let $V_i := \{ f \in B_{Y^*_\mathcal{U}} : b < f(y_i) \}$. Since $Y_{\mathcal{U}}^*$ is weak* dense in $Y_{\mathcal{U}}^*$, it follows that $[i \in I : V_i \neq \emptyset] \in \mathcal{U}$. Let $W := (V_i)_{\mathcal{U}}$ and note that $f \notin W^{w^*}$ and $A \cap W = \emptyset$. Thus $T_{\mathcal{U}}^*(f) \in T_{\mathcal{U}}^*(W)^{w^*}$ and $\|T_{\mathcal{U}}^*(w)\| > 1$ for all $w \in W$. Therefore there exist $\theta > 1$ and $J \in \mathcal{U}$ such that
\[
\|T_{\mathcal{U}}^*(v)\| \geq \theta \quad \forall i \in J \text{ and all } v \in V_i;
\]
otherwise, for every $n \in N$ and for every $J \in \mathcal{U}$, we would have
\[
J_n := \{ j \in J : \text{there is } v_i \in V_j \text{ such that } \|T_{\mathcal{U}}^*(v_i)\| < 1 + n^{-1} \} \in \mathcal{U}. \tag{2}
\]
Since $\mathcal{U}$ is $\mathbb{N}$-incomplete, we would take subsets $G_n \subset J_n$ so that $G_n \supset G_{n+1}$ for all $n$ and $\bigcap_{n=1}^\infty G_n = \emptyset$. For every $i \in G_1$, let $n_i$ be the unique positive integer such that $i \in G_{n_i} \setminus G_{n_i+1}$; by formula (2), there would exist $v_i \in V_i$ such that $\|T_{\mathcal{U}}^*(v_i)\| < 1 + n_i^{-1}$. By defining $v_i := 0$ when $i \notin J \setminus G_1$, we would get $\|T_{\mathcal{U}}^*(v_i)\| = \lim_{i \to \mathcal{U}} \|T_{\mathcal{U}}^*(v_i)\| \leq 1$, so $[v_i] \notin A \cap W$, in contradiction with $A \cap W \neq \emptyset$. Therefore formula (1) holds.

Now we choose $\theta > n > 1$. Since each $V_i^{w^*}$ is weak* compact, by formula (1) there exists $x_i \in B_X$ such that $(T^*(v))(x_i) > \eta$ for all $v \in V_i$. Hence, for $x := [x_i]$, we get $(T_{\mathcal{U}}^*(w))(x) > \eta$ for all $w \in W$. Moreover, $T_{\mathcal{U}}^*(f) \in T_{\mathcal{U}}^*(W)^{w^*}$, so
\[
\|T_{\mathcal{U}}^*(f)\| \geq (T_{\mathcal{U}}^*(f))(x) \geq \eta,
\]
hence $f \notin B$. □

**Theorem 4.10.** Let $T \in \mathcal{L}(X, Y)$, $\mathcal{U}$ an ultrafilter on $I$, and finite dimensional subspaces $F \subset Y_{\mathcal{U}}^*$, $G \subset X_{\mathcal{U}}^*$ such that $T_{\mathcal{U}}^*(F) \cap G = \emptyset$. Then, given a weak* neighborhood $\mathcal{U}$ of $0 \in Y_{\mathcal{U}}^*$, a weak* neighborhood $\mathcal{V}$ of $0 \in X_{\mathcal{U}}^*$ and $\varepsilon > 0$, there is a pair of $(1 + \varepsilon)$-injections $U \in \mathcal{L}(F, X_{\mathcal{U}}^*)$ and $V \in \mathcal{L}(T_{\mathcal{U}}^*(F) \oplus G, X_{\mathcal{U}}^*)$ verifying
\[
(a) \quad \|T_{\mathcal{U}}^* U - V T_{\mathcal{U}}^* F\| \leq \varepsilon.
\]
(b) \( U(f) \in f + \mathcal{U} \) for all \( f \in S_F \), and
(c) \( V(h) \in h + \mathcal{V} \) for all \( h \in S_H \), where \( H := T_{\mathcal{U}}^*(F) \oplus G \).

**Proof.** Let \( \{f_i\}_{i=1}^l \) be an orthonormal basis of the kernel \( N(T_{\mathcal{U}}^*(F)) \), which is completed up to a normalized basis of \( F \), \( \{f_i\}_{i=1}^l \). Take a normalized basis of \( G \), \( \{h_i\}_{i=l+1}^m \), and write \( h_i := T_{\mathcal{U}}^*(f_i) \) for \( i = k + 1, \ldots, l \). Let

\[
0 < \delta < \min \left\{ \epsilon, \frac{1 - (1 + \epsilon)^{-1}}{2m} \right\}.
\]

Note that \( \delta < (2m)^{-1} \). Take \( \delta \)-nets \( \{e_i\}_{i=1}^n \) in \( S_F \) and \( \{c_i\}_{i=1}^n \) in \( S_H \), and write \( e_j = \sum_{i=1}^{j} \lambda_{ij} f_i \) and \( c_j = \sum_{i=k+1}^{m} \mu_{ij} h_i \) for all \( j \).

Notice that for any Banach space \( Z \) and any \( p \in \mathbb{N} \), there is a natural isometry \( K : \ell_p^0(Z) \to \ell_p^0(Z) \) that maps \( \{(c_j)_{j=1}^m\} \) onto \( \{(c_j)_{j=1}^m\} \); moreover, if we identify the dual of \( \ell_1^0(Z) \) with \( \ell_\infty^0(Z) \), then \( K^* \) maps the subspace \( \ell_\infty^0(Z) \) onto \( \ell_\infty^0(Z^*) \). Consider the natural isometries \( B : (\ell_1^0(Y) \oplus_1 \ell_1^0(X) \oplus_1 \ell_1^0(X)) \to \ell_1^0(Y) \oplus_1 \ell_1^0(X) \oplus_1 \ell_1^0(X) \) and \( C : (\ell_1^0(Y) \oplus_1 \ell_1^0(X) \oplus_1 \ell_1^0(X)) \to \ell_1^0(Y) \oplus_1 \ell_1^0(X) \oplus_1 \ell_1^0(X) \). Take the operator \( A : \ell_1^0(Y) \oplus_1 \ell_1^0(X) \oplus_1 \ell_1^0(X) \to \ell_1^0(Y) \oplus_1 \ell_1^0(X) \) that maps \((a_j)_{j=1}^n, (b_j)_{j=1}^m, (c_k)_{k=1}^l\) onto

\[
\left( \left( \sum_{j=1}^{l} j \lambda_{j} a_{i} \right)_{i=1}^{n} + \left( \sum_{j=1}^{m} j \mu_{j} b_{i} \right)_{i=1}^{l+1} \right).
\]

Let \( L := C A_{\mathcal{U}} B^{-1} \), so \( L^* : \ell_\infty^0(Y^*) \oplus_\infty \ell_\infty^0(X^*) \to \ell_\infty^0(Y^*) \oplus_\infty \ell_\infty^0(X^*) \oplus_\infty \ell_\infty^0(X^*) \) maps \((w_j)_{j=1}^n, (w_j)_{j=m+1}^l\) onto

\[
\left( \left( \sum_{j=1}^{l} j \lambda_{j} v_{i} \right)_{j=1}^{n} + \left( \sum_{j=1}^{m} j \mu_{j} w_{i} \right)_{j=m+1}^{l} \right).
\]

Moreover, \( L^* \) maps \( \ell_\infty^0(Y^*) \oplus_\infty \ell_\infty^0(X^*) \) into \( \ell_\infty^0(Y^*) \oplus_\infty \ell_\infty^0(X^*) \oplus_\infty \ell_\infty^0(X^*) \). Therefore, as \( \|L^*((f)_{j=1}^l, (h)_{j=m+1}^l)\| \leq 1 \), Proposition 4.9 provides us with a net \((f_{j=1}^l, (h)_{j=m+1}^l)\) in the ball of \( \ell_\infty^0(Y^*) \oplus_\infty \ell_\infty^0(X^*) \) which is weak* converging to \((f)_{j=1}^l, (h)_{j=m+1}^l\) and \( \|L^*((f)_{j=1}^l, (h)_{j=m+1}^l)\| \leq 1 \) for all \( \alpha \). For each \( \alpha \), we define operators \( U_{\alpha} \in L(F, Y_{\alpha}^*) \) and \( V_{\alpha} \in L(H, X_{\alpha}^*) \) by

\[
U_{\alpha}(f_i) := f_{i}^{\alpha} \quad \text{for all } i \in \{1, \ldots, l\},
\]

\[
V_{\alpha}(h_i) := \left\{ \begin{array}{ll}
T_{\mathcal{U}}^*(f_{i}^{\alpha}), & \text{if } i \in [k+1, \ldots, l], \\
0, & \text{if } i \in [l+1, \ldots, m].
\end{array} \right.
\]

Therefore, we obtain

\[
\|T_{\mathcal{U}}(U_{\alpha} f_{i})\| \leq k^{-1} \delta \quad \text{for all } i \in \{1, \ldots, k\}.
\]

\[\text{(3)}\]

\[\text{(4)}\]

\[\text{(5)}\]
The election of δ, Lemma 4.8, and formulas (3) and (4) allow us to choose an index β such that $U_β$ and $V_β$ are $(1 + \varepsilon)$-injections, and such that statements (b) and (c) are satisfied.

For statement (a), given $v = \sum_{i=1}^j v_i f_i \in S_F$, we have

$$\|T U_a - V_a T_{U^*}\| v = \left\| \sum_{i=1}^k v_i T U_a (f_i) \right\| \leq k^{-1} \delta \sum_{i=1}^k |v_i| \leq \delta$$

because $|f_i|_{1=1}^k$ is an orthonormal basis. \(\square\)

The following theorem is a useful translation of Theorem 4.10 to ultraproduct language.

**Theorem 4.11.** For every operator $T \in \mathcal{L}(X, Y)$ and every ultrafilter $\mathcal{U}$ there are an ultrafilter $\mathcal{W}$, metric injections $U \in \mathcal{L}(Y_{\mathcal{U}}^*, (Y_{\mathcal{U}}^*)_{\mathcal{W}})$ and $V \in \mathcal{L}(X_{\mathcal{U}}^*, (X_{\mathcal{U}}^*)_{\mathcal{W}})$, and metric surjections $P \in \mathcal{L}((Y_{\mathcal{U}}^*)_{\mathcal{W}}, Y_{\mathcal{U}}^*)$ and $Q \in \mathcal{L}((X_{\mathcal{U}}^*)_{\mathcal{W}}, X_{\mathcal{U}}^*)$ verifying

(a) $(T_{\mathcal{U}}^*)_{\mathcal{W}} \circ U = V \circ T_{\mathcal{U}}^*$,
(b) $T_{\mathcal{U}}^* \circ P = Q \circ (T_{\mathcal{U}}^*)_{\mathcal{W}}$,
(c) $T_{\mathcal{U}}^* = Q \circ (T_{\mathcal{U}}^*)_{\mathcal{W}} \circ U$.

**Proof.** Let $J$ be the set of all tuples $j \equiv (F_j, E_j, \varepsilon_j, U_j, V_j)$, where $F_j$ and $E_j$ are finite dimensional subspaces of $Y_{\mathcal{U}}^*$ and $X_{\mathcal{U}}^*$, respectively, $\varepsilon_j \in (0, 1)$, $U_j$ is a weak* neighborhood of 0 in $Y_{\mathcal{U}}^*$, and $V_j$ is a weak* neighborhood of 0 in $X_{\mathcal{U}}^*$. We define an order $\leq$ in $J$ by $i \leq j$ if $F_i \subseteq F_j$, $E_i \subseteq E_j$, $\varepsilon_i \geq \varepsilon_j$, $U_i \supseteq U_j$, and $V_i \supseteq V_j$. Let $\mathcal{W}$ be an ultrafilter refining the order filter on $J$.

For every index $j \in J$, Theorem 4.10 gives a pair of $(1 + \varepsilon_j)$-injections $U_j \in \mathcal{L}(F_j, Y_{\mathcal{U}}^*)$ and $V_j \in \mathcal{L}(T_{\mathcal{U}}^*(F_j) + E_j, X_{\mathcal{U}}^*)$ verifying

$$\|T U_j - V_j T_{U^*}\| F_j \leq \varepsilon_j,$$

$U_j(v) \in v + U_j$ for all $v \in S_{F_j}$,

$V_j(w) \in w + V_j$ for all $w \in S_{T_{\mathcal{U}}^*(F_j) + E_j}$.

The operators $U$, $V$, $P$ and $Q$ are defined as follows:

$U(v) = [f_j]$, where $f_j := U_j(v)$ if $v \in F_j$ and $f_j := 0$ otherwise,

$V(w) = [g_j]$, where $g_j := V_j(w)$ if $w \in T_{\mathcal{U}}^*(F_j) + E_j$ and $g_j := 0$ otherwise,

$P([v_j]) = w^* - \lim_{j \rightarrow \mathcal{W}} v_j \in Y_{\mathcal{U}}^*$ for all $[v_j] \in (Y_{\mathcal{U}}^*)_{\mathcal{W}},$

$Q([w_j]) = w^* - \lim_{j \rightarrow \mathcal{W}} w_j \in X_{\mathcal{U}}^*$ for all $[w_j] \in (X_{\mathcal{U}}^*)_{\mathcal{W}}$.

Typical ultrapower arguments as those given in Proposition 3.3 show that $U$ and $V$ are metric injections.
In order to prove that $P$ is a metric surjection, take any $v \in S_{Y^*_\mathcal{U}}$. By the principle of local reflexivity for ultrapowers, we can choose a family $\{v_j\}_{j \in J}$ in $Y^*_\mathcal{U}$ such that $w^*-\lim_{j \to \mathcal{U}} v_j = v$ and $\lim_{j \to \mathcal{U}} \|v_j\| = 1$. Hence, since $\|P\| \leq 1$, we have $P(B_{(Y^*_\mathcal{U})\mathcal{U}}) = B_{Y^*_\mathcal{U}}$; that means that $P$ is a metric surjection. The same argument applies for $Q$.

To prove (a), take $v \in S_{Y^*_\mathcal{U}}$ and $\varepsilon > 0$. Let $j_0 \in J$ such that $v \in F_{j_0}$ and $\varepsilon_{j_0} < \varepsilon$. Then

$$\{ j \in J : \|(T^*_\mathcal{U} U_j - V_j T^*_\mathcal{U}) v\| < \varepsilon \} \supset \{ j \in J : j_0 \leq j \} \in \mathcal{U},$$

so $((T^*_\mathcal{U})\mathcal{U} U - VT^*_\mathcal{U}) v = 0$. For statement (b), take $[v_j] \in (Y^*_\mathcal{U})\mathcal{U}$. Then

$$T^*_\mathcal{U} P([v_j]) = T^*_\mathcal{U} (w^*-\lim_{j \to \mathcal{U}} v_j) = w^*-\lim_{j \to \mathcal{U}} T^*_\mathcal{U} (v_j) = Q(T^*_\mathcal{U})\mathcal{U} ([v_j]) .$$

The proof of statement (c) is similar to that of (a) and (b).

\[ \square \]

**Proposition 4.12.** Given an ultrapower-stable semigroup $S$, the following statements hold:

(a) If $S$ is surjective and right-stable then $S^d$ is ultrapower-stable;

(b) If $S$ is injective and left-stable then $S^d$ is ultrapower-stable.

**Proof.** (a) Take $T \in S^d$, that is, $T^* \in S$. Given any ultrafilter $\mathcal{U}$, we have $T^*_\mathcal{U} \in S$. By Theorem 4.11(b), there exists a pair of metric surjections $P$ and $Q$ and an ultrafilter $\mathcal{U}$ such that $T^*_\mathcal{U} \circ P = Q \circ (T^*_\mathcal{U})\mathcal{U}$. Since $S$ is ultrapower-stable and surjective, we have $Q \circ (T^*_\mathcal{U})\mathcal{U} \in S$. But $S$ is also right-stable, so $T^*_\mathcal{U} \in S$ and $T^*_\mathcal{U} \in S^d$.

The proof of part (b) follows a similar argument to that of part (a), but here we need statement (a) of Theorem 4.11.

\[ \square \]

**Proposition 4.13.** Let $S$ be a surjective, right-stable, ultrapower-stable semigroup. Then $S^d$ is $\prec_{\mathcal{U}}$-stable.

**Proof.** It follows from Propositions 4.6 and 4.12.

\[ \square \]

5. Independence between local supportability and local representability

Although local supportability and local representability are closely related in situations like that of Theorem 4.11, we prove in this section that both notions are in fact mutually independent. The proof follows the next argument: given a $\prec_{\mathcal{U}}$-stable semigroup $S$, if $S \in \mathcal{A}$ and $T \in S$ then $T$ is not locally supportable in $S$. Analogously, if $T \in S$ is $\prec_{\mathcal{U}}$-stable, then $S \in \mathcal{A}$ and $T \notin \mathcal{A}$ then $T$ is not locally representable in $T$.

We introduce some notation. Given $k \in \mathbb{N}$, we write $I_k = [2^{-k}(i - 1), 2^{-k}i)$ for $1 \leq i \leq 2^k - 1$, and $I_k^* = [2^{-k}(2^k - 1), 1]; \chi_k^*$ is the characteristic function associated to $I_k^*$. Given a function $f : [0, 1] \to \mathbb{R}$, and a positive integer $1 \leq i \leq 2^k$, we write $m_i^k(f) := \inf f(I_i^*)$, $M_i^k(f) := \sup f(I_i^*)$, and $\rho_k(f) := \max\{|M_i^k(f) - m_i^k(f)| : 1 \leq i \leq 2^k\}$. 
We consider a system of positive, norm one measures \( \{ \nu^k \}_{i=1}^{\infty} \) in \( \mathcal{M} \) such that every \( \nu^k_i \) is concentrated on \( I^k_i \). Let \( G_k \in L(\mathcal{M}, \mathcal{M}) \) be the norm one projection defined by
\[
G_k(\lambda) = \sum_{i=1}^{2^k} \lambda(I^k_i) \nu^k_i.
\] (6)

**Lemma 5.1.** Given \( k \in \mathbb{N} \), \( f \in C \), and \( \lambda \in \mathcal{M} \), we have \( |\lambda(f) - G_k(\lambda)(f)| \leq \| \lambda \| \rho_k(f) \).

**Proof.** It is sufficient to show the result for a positive measure \( \lambda \). We define the functions \( m_f(x) := \sum_{i=1}^{2^k} m^k_i(f) \chi^k_i(x) \), \( M_f(x) := \sum_{i=1}^{2^k} M^k_i(f) \chi^k_i(x) \). Note that
\[
\int_0^1 m_f \, d\lambda = \int_0^1 M_f \, d\lambda = \int_0^1 f \, dG_k(\lambda).
\]
and
\[
\int_0^1 m_f \, dG_k(\lambda) \leq \int_0^1 f \, dG_k(\lambda) \leq \int_0^1 M_f \, dG_k(\lambda),
\]
therefore, we get
\[
\left| \int_0^1 f \, d\lambda - \int_0^1 f \, dG_k(\lambda) \right| \leq \int_0^1 (M_f - m_f) \, d\lambda \leq \sum_{i=1}^{2^k} \rho_k(f) \lambda(I^k_i) = \| \lambda \| \rho_k(f).
\]
\[ \square \]

**Proposition 5.2.** Let \( \mathcal{U} \) be an ultrafilter on \( \mathbb{N} \) and define \( G \in L(\mathcal{M}, L_1(\mu)_\mathcal{U}) \) by \( G(\lambda) := (G_n(\lambda), f) \) (\( G_n \) defined as in formula (6)). Then the next statements hold:

(i) \( \lim_n G_n(\lambda, f) = (\lambda, f) \) for all \( \lambda \in \mathcal{M} \) and all \( f \in C \);

(ii) \( \lim_n \| G_n(\lambda) \| = \| \lambda \| \) for all \( \lambda \in \mathcal{M} \), so \( G \) is a metric injection.

**Proof.** (i) Let \( \lambda \in \mathcal{M} \) and \( f \in C \). By uniform continuity of \( f \), there is a positive integer \( n_0 \) verifying \( \rho_n(f) < \varepsilon \). So, by Lemma 5.1, we have \( |(\lambda - G_n(\lambda), f)| < \varepsilon \) for all \( n \geq n_0 \).

(ii) Let \( \lambda \in \mathcal{M} \) and \( \varepsilon > 0 \). Choose \( f \in B_C \) so that \( (\lambda, f) > \| \lambda \| - 2^{-1} \varepsilon \). By statement (i), there is \( n_0 \) such that \( |(G_n \lambda, f)| > |(\lambda, f)| - 2^{-1} \varepsilon \) for all \( n \geq n_0 \), so
\[
\| \lambda \| - \varepsilon < |(G_n \lambda, f)| \leq \| G_n(\lambda) \| \leq \| \lambda \|. \quad \square
\]

The following theorem is a new general example of Heinrich finite representability concerning the space \( L_1 \).

**Theorem 5.3.** Let \( T \in L(Y, C) \) and \( \{ \nu^k \}_{i=1}^{\infty} \subset \mathcal{M} \) a system of positive, norm one measures such that every \( \nu^k_i \) is concentrated in \( I^k_i \). Let \( Z \) be the closed subspace of \( \mathcal{M} \)
generated by \( \{ \chi^k_{ij} \}_{k=1}^{\infty} \). Consider the metric injection \( G \) given in Proposition 5.2, and the metric surjection \( P \in \mathcal{L}(Y^s, Y^*) \) defined by \( P(\chi^k_{ij}) := w^s - \lim_{n \to \infty} \chi^k_{ij} \). Then \( T^* = P(T^*|Z) \) of \( G \). Hence, \( T^* \) is Heinrich finitely representable in \( T^*|Z \).

**Proof.** Given \( \lambda \in \mathcal{M} \), Proposition 5.2 shows

\[
P(T^*|Z) G(\lambda) = \lim_{n \to \infty} T^* G_n(\lambda) = T^*(\lambda),
\]

so Proposition 3.5 proves that \( T^* \) is Heinrich finitely representable in \( T^*|Z \). \( \square \)

**Corollary 5.4.** For every \( T \in \mathcal{L}(Y, C) \), the conjugate \( T^* \) is Heinrich finitely representable in \( T^*|L_1 \) and in \( T^*|\mathcal{N} \).

**Proof.** For every dyadic interval \( I^k_i \), we define \( \mu^k_i(A) := 2^k \mu(A \cap I^k_i) \) for every Borelian subset \( A \) of \([0, 1]\), and we denote by \( \delta^k_i \) the Dirac delta associated to the middle point of \( I^k_i \). Let \( Z := \bigcap_{i=1}^{\infty} \delta^k_i \), and note that \( L_1 = \bigcap_{i=1}^{\infty} \mu^k_i \). By Theorem 5.3, \( T^* \) is Heinrich finitely representable in \( T^*|L_1 \) and in \( T^*|\mathcal{N} \). \( \square \)

Next proposition involves the classes of Tauberian operators and super-Tauberian operators. We note that the class of super-Tauberian operators has been identified with the semigroup \( \mathcal{W}^{\text{sup}}_+ \), where \( \mathcal{W}^{\text{sup}}_+ \) stands for the ideal of super-weakly compact operators [9, Theorem 18]. It is immediate that \( \mathcal{W}^{\text{sup}}_+ \) is injective and left-stable ([9, Proposition 2] or [9, Proposition 7]). Moreover, \( \mathcal{W}^{\text{sup}}_+ \) is ultrapower-stable; indeed, given an ultrafilter \( \mathcal{U} \) and a super-Tauberian operator \( T \), it follows from [9, Theorem 9(b)] that \( T_{\mathcal{U}^*} \) is Tauberian. As the iteration theorem identifies \( (T_{\mathcal{U}})_{\mathcal{U}} \) with \( T_{\mathcal{U}^*} \), it follows that \( \mathcal{W}^{\text{sup}}_+ \) is super-Tauberian [9, Theorem 9(a)].

**Proposition 5.5.** There is a non-Tauberian operator \( T^* \in \mathcal{L}(\mathcal{M}, \mathcal{M}) \) such that \( T^*|\mathcal{N} \) is an isomorphism. Hence, Heinrich finite representability does not imply local supportability.

**Proof.** For every \( n \in \mathbb{N} \), we denote \( J_n := [1/2^n, 2/2^n] \), \( J_n^+ := (2/2^n - 1, 3/2^n - 1) \), and \( J_n^- := (3/2^n - 1, 4/2^n - 1) \), and define the functions \( f_n(t) := \sin(2^n + 1) t \chi_{J_n}(t) \in \mathcal{C} \), \( h_n := 2^n(\chi_{J_n} - \chi_{J_n^+}) \in L_1 \). Note that \( \|h_n\|_1 = 1 \) and \( (h_n, f_n) = 2\pi^{-1} h_{nm} \).

Since \( \lim_{n \to \infty} (h_n, f) = 0 \) for all \( f \in \mathcal{C} \), given any null sequence \( \alpha \equiv (\alpha_n)_{n \in \mathbb{N}} \in (0, 1) \) we can define \( P_\alpha \in \mathcal{L}(C, C) \) by \( P_\alpha(f) := \sum_{n=1}^{\infty} (1 - \alpha_n)(h_n, f) f_n \) and \( T := \mathcal{I}_C - 2\pi^{-1} P_\alpha \), so

\[
T^*(\lambda) = \lambda - 2\pi^{-1} \sum_{n=1}^{\infty} (1 - \alpha_n)(\lambda_n, f_n) h_n.
\]

Note that \( P_\alpha^*(\lambda) \in L_1 \) for all \( \lambda \in \mathcal{M} \), so the decomposition \( \mathcal{M} = L_1 \oplus \mathcal{N} \) yields \( \|T^*v\| \geq \|v\| \) for all \( v \in \mathcal{N} \), hence \( T^*|\mathcal{N} \) is an isomorphism. However, \( T^*(h_n) = \alpha_n h_n \), so \( \lim_{n} T^*(h_n) = 0 \). Moreover, \( (h_n) \) is a normalized disjoint sequence, hence \( T^*|L_1 \) is not Tauberian, so \( T^*|L_1 \) is super-Tauberian either [10, Theorem 2]. As \( \mathcal{W}^{\text{sup}}_+ \) is an injective semigroup, it follows that \( T^* \not\in \mathcal{W}^{\text{sup}}_+ \). Moreover, as we pointed out before, the semigroup \( \mathcal{W}^{\text{sup}}_+ \) of the super-Tauberian operators is ultrapower-stable and left-stable, so Proposition 4.6 shows that \( T^* \) is not locally supportable in \( T^*|\mathcal{N} \), and by Corollary 5.4, we conclude that Heinrich finite representability does not imply local supportability. \( \square \)
Proposition 5.6. Let $\mathcal{T} \in \mathcal{L}(\ell_2, \ell_\infty)$ be a metric injection. Then the identity $I_{\ell_2}$ is Bellenot finitely representable in $\mathcal{T}$, but $I_{\ell_2}$ is not locally $c$-representable in $\mathcal{T}$, for any $c \geq 0$. Hence, Bellenot finite representability does not imply local representability.

Proof. It is immediate that $I_{\ell_2}$ is Bellenot finitely representable in $\mathcal{T}$. Let us assume that $I_{\ell_2}$ is locally $c$-representable in $\mathcal{T}$. Then, by Proposition 3.4, there is an ultrafilter $\mathcal{U}$ and operators $A \in \mathcal{L}(\ell_2, (\ell_2)_\mathcal{U})$, $B \in \mathcal{L}((\ell_\infty)_\mathcal{U}, \ell_2)$ so that $I_{\ell_2} = BY_{\mathcal{U}}A$. Since $(\ell_\infty)_\mathcal{U}$ is isometric to a space of continuous functions on some compact set [13], it follows that $(\ell_\infty)_\mathcal{U}$ has the Dunford–Pettis property. As $\ell_2$ is reflexive, $(\ell_\infty)_\mathcal{U}$ and $\ell_2$ are essentially incomparable, thus $B$ is an inessential operator [8, Theorem 1]. But the class of all inessential operators is an ideal, and moreover, is the perturbation class for Fredholm operators [16], so $0 = I_{\ell_2} - BY_{\mathcal{U}}A$ is a Fredholm operator, a contradiction. $\square$

6. Final remarks

Little room has been left for Beauzamy’s first definition [4], second definition [5, p. 221] (involving only two operators) and third definition [6, p. 241] of finite representability. The following remarks, complemented with some results in [2,5,14], provide us with a complete picture about the subject:

(a) Let $T$ be an injective, non-closed range operator. Thus, no non-trivial ultrapower $T_\mathcal{U}$ is injective, so $T_\mathcal{U}$ is not finitely representable in $T$ following Beauzamy second definition nor third definition. This is a serious handicap in studying ultrapower-stable classes of operators.

(b) Beauzamy second definition implies Beauzamy first definition and Bellenot finite representability. It follows, respectively, from [5] and our remark (a) that both converses fail.

(c) Our example in Proposition 5.6 also shows that Beauzamy second definition and third definition do not imply local representability.

(d) Heinrich finite representability does not imply Beauzamy first definition. Indeed, since $\ell_1$ is finitely representable in $c_0$, we have that $\ell_1^*$ is finitely representable by quotients in $c_0^*$ [3, Proposition 3.5], hence $A:0 \to \ell_1^*$ is Heinrich finitely representable in $B:0 \to c_0^*$. However, $A$ is not finitely representable in $B$ in the sense of Beauzamy first definition.

References