JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 87, 454–459 (1982)

Monotone Iterative Technique for Differential Equations in a Banach Space*

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I. INTRODUCTION

Let E be a real Banach space with norm $\|\cdot\|$. Consider the initial value problem

$$u' = f(t, u), \qquad u(0) = u_0, \tag{1.1}$$

where $u, f \in E$. Generally speaking, the methods of proving existence of solutions of (1.1) consist of three steps, namely,

(i) constructing a sequence of approximate solutions of some kind for (1.1);

(ii) showing the convergence of the constructed sequence;

(iii) proving that the limit function is a solution.

If f is continuous, steps (i) and (iii) are standard and straightforward. It is step (ii) that deserves attention. This in turn leads to three possibilities, namely, to show that the sequence of approximate solutions is (a) a Cauchy sequence, (b) relatively compact so that one can appeal to Ascoli's theorem, and (c) a monotone sequence in a cone. The first two possibilities are well known and are discussed in [2, 3]. This paper is devoted to the investigation of (c) which leads to the development of a monotone iterative technique in an arbitrary cone.

II. PRELIMINARIES

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* denote the set of continuous linear functionals on E. A proper subset K of E is said to be a

* Research partially supported by U. S. Army Research Grant DAAG29–80–C–0060.

0022-247X/82/060454–06$02.00/0

cone if $\lambda K \subset K$ for $\lambda \geqslant 0$, $K + K \subset K$, $K = \bar{K}$ and $K \cap (-K) = \{0\}$. Here $\bar{K}$ denotes the closure of K. The cone K induces a partial ordering on E defined by $u \leqslant v$ iff $v - u \in K$. Given a cone K, we let $K^* = \{\phi \in E^*. \phi(u) \geqslant 0$ for all $u \in K\}$. A cone K is said to be normal if there exists a real number $N > 0$ such that $0 \leqslant u \leqslant v$ implies $\|\cdot\| \leqslant N \|v\|$, where N is independent of u and v. We shall assume in this paper that K is a normal cone.

Let α denote the Kuratowski's measure of noncompactness the properties of which may be found in [2, 3].

For any $v_0, w_0 \in C[I, E]$ such that $v_0(t) \leqslant w_0(t)$ on I where $I = [0, T]$, we define the conical segment $[v_0, w_0] = \{u \in E: v_0(t) \leqslant u \leqslant w_0(t), t \in I\}$.

Let us consider the IVP

$$u' = f(t, u), \qquad u(0) = u_0, \tag{2.1}$$

where $f \in C[I \times E, E]$. Let us list the following assumptions for convenience.

(A1) For any bounded set B in E

$$\alpha(f(I \times B)) \leqslant L\alpha(B), \qquad \text{for some} \quad L > 0;$$

(A2) $v_0, w_0 \in C^1[I, E]$ with $v_0(t) \leqslant w_0(t)$ on I such that

$$v_0' \leqslant f(t, v_0),$$
$$w_0' \geqslant f(t, w_0) \qquad \text{on } I;$$

(A3) $f(t, u) - f(t, v) \geqslant -M(u - v)$ whenever $u \geqslant v$ and $u, v \in [v_0, w_0]$ for some $M > 0$.

A function f is said to be quasimonotone nondecreasing relative to K if $v \leqslant u$ and $\phi(v - u) = 0$, $\phi \in K^*$ implies

$$\phi(f(t, v)) \leqslant \phi(f(t, u)).$$

Clearly when (A3) holds, f is quasimonotone.

III. Monotone Iterative Technique

In order to develop the monotone iterative technique, we need to consider the linear IVP

$$u' = F(t, u), \qquad u(0) = u_0, \tag{3.1}$$

where $F(t, u) = f(t, \eta(t)) - M(u - \eta(t))$ and $\eta \in C[I, E]$ such that $v_0(t) \leqslant \eta(t) \leqslant w_0(t)$ on I. The following lemma shows that problem (3.1) has a unique solution on I.

LEMMA 3.1. *Let assumption* (A1) *hold. Then the IVP* (3.1) *possesses a unique solution on I.*

Proof. For any bounded set B in E, we have, by (A1),

$$\alpha(F(I \times B)) \leqslant (L+M)\,\alpha(\eta(I)) + M\alpha(B) = M\alpha(B),$$

since a continuous function maps a compact set into a compact set. Hence for any u_0 such that $v_0(0) \leqslant u_0 \leqslant w_0(0)$ problem (3.1) has a solution on an interval $[0, a)$, $a < T$. See [2, 3]. Furthermore,

$$\|F(t, u)\| \leqslant M\,\|u\| + C,$$

where $C = \max_I \|f(t, \eta(t))\| + M \max_I \|\eta(t)\|$.

Setting $g(t, r) = Mr + C$, we see that $g(t, r)$ is monotone nondecreasing in r for each t, and the solutions of the scalar differential equation $r' = g(t, r)$, $r(0) - r_0 \geqslant 0$ are bounded on I. Hence the solutions of IVP (3.1) exist on I. See [3, Theorem 4.1.1]. Since the uniqueness of solutions of (3.1) follows trivially from the linearity of F, proof of lemma is complete.

For each $\eta \in C[I, E]$ such that $v_0(t) \leqslant \eta(t) \leqslant w_0(t)$ on I, we define the mapping A by $A\eta = u$, where u is the unique solution of (3.1) corresponding to η. The following result concerning mapping A holds.

LEMMA 3.2. *Suppose that assumptions* (A1), (A2) and (A3) *hold; then*

(i) $v_0 \leqslant Av_0$ *and* $w_0 \geqslant Aw_0$;

(ii) A *is monotone on* $[v_0, w_0]$, *that is, if* $\eta_1, \eta_2 \in [v_0, w_0]$ *with* $\eta_1 \leqslant \eta_2$ *then* $A\eta_1 \leqslant A\eta_2$.

Proof. (i) Suppose that $Av_0 = v_1$. Set $p(t) = \phi[v_1(t) - v_0(t)]$ so that $p(0) \geqslant 0$, where $\phi \in K^*$. Then

$$p' \geqslant \phi[f(t, v_0) - M(v_1 - v_0) - f(t, v_0)] = -Mp$$

in view of (A2). As a result, we have $p(t) \geqslant p(0)\, e^{-Mt} \geqslant 0$ on I. Since $\phi \in K^*$ is arbitrary, this implies $v_1 \geqslant v_0$ on I proving $v_0 \leqslant Av_0$. Similarly we can show that $w_0 \geqslant Aw_0$.

To prove (ii), let $\eta_1, \eta_2 \in C[I, E]$ such that $\eta_1 \leqslant \eta_2$ on I and suppose that $A\eta_1 = u_1$, $A\eta_2 = u_2$. We set $p(t) = \phi[u_2(t) - u_1(t)]$ so that $p(0) \geqslant 0$, where, as before, $\phi \in K^*$. It then follows by using (A3) that

$$\begin{aligned} p' &= \phi[f(t, \eta_2) - M(u_2 - \eta_2) - f(t, \eta_1) + M(u_1 - \eta_1)] \\ &\geqslant \phi[-M(\eta_2 - \eta_1) - M(u_2 - \eta_2) + M(u_1 - \eta_1)] = -Mp. \end{aligned}$$

Consequently $p(t) \geqslant p(0)\, e^{-Mt} \geqslant 0$ on I and this proves $A\eta_1 \leqslant A\eta_2$. The proof of the lemma is complete.

In view of Lemma 3.2, we can define the sequences $\{v_n\}$, $\{w_n\}$ as follows:

$$v_n = Av_{n-1} \qquad \text{and} \qquad w_n = Aw_{n-1}.$$

It is easy to see tat $\{v_n\}$, $\{w_n\}$ are monotone sequences such that $v_n \leqslant w_n$ and $v_n, w_n \in [v_0, w_0]$. We shall now show that there exist subsequences of $\{v_n\}$, $\{w_n\}$ which converge uniformly on I.

LEMMA 3.3. *Under the assumptions of Lemma* 3.2, *the sequences* $\{v_n\}$, $\{w_n\}$ *are uniformly bounded, equicontinuous and relatively compact on* I.

Proof. Since the cone K is assumed to be normal, it follows from $v_n, w_n \in [v_0, w_0]$ that $\{v_n\}$, $\{w_n\}$ are uniformly bounded. This implies the equicontinuity of the sequences by using standard estimates and the fact that f maps bounded sets into bounded sets which is a consequence of (A1). Now we set $B(t) = \{v_n(t)\}_{n=0}^{\infty}$ so that $B'(t) = \{v_n'(t)\}_{n=0}^{\infty}$ and $m(t) = \alpha(B(t))$. Using the standard arguments as in [2, 3], we get

$$D^- m(t) \leqslant \alpha\left(\left\{\frac{v_n(t) - v_n(t-h)}{h}\right\}_{n=0}^{\infty}\right) \leqslant \alpha(\overline{\text{conv}}(\{v_n'(t)\}_{n=0}^{\infty})).$$

Thus we have

$$D^- m(t) \leqslant \lim_{h \to 0^+} \alpha\left(\bigcup_{J_h} B'(x)\right), \qquad \text{where } J_h = [t-h, t] \subset I.$$

Evidently

$$\begin{aligned}\alpha\left(\bigcup_{J_h} B'(s)\right) &\leqslant \alpha\left(\bigcup_{J_h} \{f(t, v_{n-1}(t))\}_{n=1}^{\infty}\right) + 2M\alpha\left(\bigcup_{J_h} \{v_n(t)\}_{n=0}^{\infty}\right)\\ &\leqslant \alpha\left(\left\{f\left(I, \bigcup_{J_h} B(s)\right)\right\}\right) + 2M\alpha\left(\bigcup_{J_h} B(s)\right)\\ &\leqslant (L + 2M)\,\alpha\left(\bigcup_{J_h} B(s)\right).\end{aligned}$$

The equicontinuity of $v_n(t)$ now yields

$$D^- m(t) \leqslant (L + 2M)\, m(t), \qquad t \in I.$$

Since $m(0) = \alpha(\{v_n(0)\}_{n=0}^{\infty}) = \alpha(u_0, v_0(0)) = 0$, it is immediate that $m(t) \equiv 0$ on I, which implies the relative compactness of the sequence $\{v_n(t)\}$ for each $t \in I$. Similarly $\{w_n(t)\}$ is relatively compact for each $t \in I$. The proof of the lemma is complete.

We now apply Ascoli's theorem to the sequences $\{v_n\}$, $\{w_n\}$ to obtain subsequences $\{v_{n_k}\}$, $\{w_{n_k}\}$ which converge uniformly on I. Since the sequences $\{v_n\}$, $\{w_n\}$ are monotone, this then shows that the full sequences converge uniformly and monotonically to continuous functions, that is, $\lim_{n\to\infty} v_n(t) = \rho(t)$ and $\lim_{n\to\infty} w_n(t) = r(t)$ on I. It then follows easily from (3.1) that $\rho(t)$ and $r(t)$ are solutions of IVP (2.1) on I.

Finally we show that $\rho(t)$, $r(t)$ are minimal and maximal solutions of (2.1). To this end, let $u(t)$ be any solution of (2.1) on I such that $u \in [v_0, w_0]$. Assume that $v_n \leqslant u \leqslant w_n$ on I. Set $p(t) = \phi[u(t) - v_{n+1}(t)]$, so that $p(0) = 0$, where, as before, $\phi \in K^*$. Then by (A3) and the assumption $v_n \leqslant u$, we have

$$p' = \phi[f(t, u) - f(t, v_n) + M(v_{n+1} - v_n)]$$
$$\geqslant \phi[-M(u - v_n) + M(v_{n+1} - v_n)] = -Mp.$$

This implies $v_{n+1} \leqslant u$ on I. Similarly, we can show $u \leqslant w_n + 1$ on I. Since $u \in [v_0, w_0]$, we have, by induction, $v_n \leqslant u \leqslant w_n$ on I for all n. Thus we obtain, taking the limit as $n \to \infty$, $\rho(t) \leqslant u(t) \leqslant r(t)$ on I, proving $\rho(t)$, $r(t)$ are minimal and maximal solutions of (2.1) on I. We have therefore proved the following main result.

THEOREM 3.1. *Let the cone K be normal and assumptions* (A1), (A2) and (A3) *hold. Then there exist monotone sequences* $\{v_n\}, \{w_n\}$ *which converge uniformly and monotonically to the minimal and maximal solutions* $\rho(t)$, $r(t)$, *respectively, of the IVP* (2.1) *on* $[v_0, w_0]$. *That is, if u is any solution of* (2.1) *in* $[v_0, w_0]$, *then*

$$v_0 \leqslant v_1 \leqslant \cdots \leqslant v_n \leqslant \rho \leqslant u \leqslant r \leqslant w_n \leqslant \cdots \leqslant w_1 \leqslant w_0 \qquad \text{on } I.$$

COROLLARY 3.1. *If the solutions of IVP* (2.1) *are unique, then the assumptions of Theorem* 3.1 *imply that* $\rho(t) = u(t) = r(t)$ *on I.*

Remark 1. If f is quasimonotone relative to K where K is a solid cone, the existence of extremal solutions is given in [3]. On the other hand, when K is not assumed to be solid and function f maps $I \times K$ into E, existence of extremal solutions is also known. See [1].

Remark 2. Suppose that $E = \mathbb{R}^n$ and $K = \mathbb{R}^n_+$, the standard cone. Let $f(t, u)$ be quasimonotone nondecreasing in u for each $t \in I$, that is, $v \leqslant u$ and $u_i = v_i$ implies $f_i(t, v) \leqslant f_i(t, u)$. Suppose further that for each i, $i = 1, 2, \ldots, n$,

$$f_i(t, v_1, \ldots, u_i, \ldots, v_n) - f_i(t, v_1, \ldots, v_i, \ldots, v_n) \geqslant -M(u_i - v_i),$$

where $v_{0i}(t) \leqslant v_i \leqslant u_i \leqslant w_{0i}(t)$. Then the conclusion of Theorem 3.1 is valid, provided (A2) holds, in addition [4]. In this case, the forgoing two conditions on f imply

$$f(t, u) - f(t, v) \geqslant -M(u - v) \qquad \text{where} \quad v_0 \leqslant v \leqslant u \leqslant w_0,$$

which corresponds to (A3). Thus when we assume the last condition, the quasimonotonicity of f is subsumed in it. Of course, one does not need (A1) when $E = \mathbb{R}^n$. In [4], f is allowed to satisfy a mixed quasimonotone condition and the results obtained there are in that general set up. If $q_i = 0$ for each i, the results of [4] reduce to the case considered in this remark as was noted there.

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