Linear Algebra and its Applications 431 (2009) 495–510

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Dynamics of piecewise linear maps and sets of nonnegative matrices ☆

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ARTICLE INFO

Article history:
Received 17 June 2007
Accepted 11 February 2009
Available online 1 May 2009

Submitted by V. Mehrmann

Keywords:
Nonnegative matrices
Piecewise-linear maps
Generalized eigenvectors
Perron-Frobenius theorem

ABSTRACT

We consider maps $f_{\mathcal{K}}(v) = \min_{A\in\mathcal{K}} Av$ and $g_{\mathcal{K}}(v) = \max_{A\in\mathcal{K}} Av$, where $\mathcal{K}$ is a finite set of nonnegative matrices and by "min" and "max" we mean component-wise minimum and maximum. We transfer known results about properties of $g_{\mathcal{K}}$ to $f_{\mathcal{K}}$. In particular we show existence of nonnegative generalized eigenvectors of $f_{\mathcal{K}}$, give necessary and sufficient conditions for existence of strictly positive eigenvector of $f_{\mathcal{K}}$, study dynamics of $f_{\mathcal{K}}$ on the positive cone. We show the existence and construct matrices A and B, possibly not in $\mathcal{K}$, such that $f^n_{\mathcal{K}}(v) \sim A^n v$ and $g^n_{\mathcal{K}}(v) \sim B^n v$ for any strictly positive vector v.

1. Introduction

The theory of nonnegative matrices has been very well developed since its appearance in papers of Perron and Frobenius. Vast number of applications to dynamic programming, probability theory, numerical analysis, mathematical economics, fractal geometry raise even greater interest to this field. As a result there are many strong generalizations of the Perron–Frobenius theory (see [1,4,6–8,13–15,17]).

The classical Perron–Frobenius theorem shows that a nonnegative matrix has a nonnegative eigenvector associated with its spectral radius, and if the matrix is irreducible then this nonnegative eigenvector can be chosen strictly positive. One important generalization of this result was obtained by Rothblum [17], who studied the structure of the algebraic eigenspaces of nonnegative matrices and described the combinatorics that stands behind the index of the spectral radius and dimensions of the

☆ The author was partially supported by NSF grants DMS-0308985 and DMS-0456185.
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0024-3795/$ - see front matter
doi:10.1016/j.laa.2009.02.038

algebraic eigenspaces. Moreover, it was shown that the algebraic eigenspace of a nonnegative matrix corresponding to its spectral radius is spanned by a set of nonnegative generalized eigenvectors with certain strictly positive entries.

Many generalizations of the Perron–Frobenius theory involve homogeneous monotone functions, especially functions of the form

$$g_{\mathcal{K}}(x) = \max_{A\in\mathcal{K}} Ax,$$

where $\mathcal{K}$ is a set of square nonnegative matrices of fixed dimension and by "max" we mean component-wise maximum. Such functions appear in many well-known problems, such as the theory of controlled Markov chains, Leontief substitution systems, controlled branching processes, parallel computations, transportation networks, etc. The study of maps $g_{\mathcal{K}}$ was initiated by Richard Bellman. Using the Brouwer fixed point theorem he proved existence of a strictly positive eigenvector of the map $g_{\mathcal{K}}$ in the case when each matrix in $\mathcal{K}$ is positive and studied the asymptotic behavior of iterations $g^n_{\mathcal{K}}(v) = g_{\mathcal{K}}(g_{\mathcal{K}}(\ldots g_{\mathcal{K}}(v)\ldots))$ for a nonnegative vector v (see [2] and [3, Chapter XI, Sections 10–11]). These results were generalized to a set of irreducible matrices by Mandl and Seneta [14].

The most important results for our investigation were obtained by Zijm in [25]. He showed that there is a simultaneous block-triangular decomposition of the set of matrices $\mathcal{K}$, which was used to give the necessary and sufficient conditions for the existence of a strictly positive eigenvector of $g_{\mathcal{K}}$ and extend the above mentioned result of Rothblum on nonnegative generalized eigenvectors to $g_{\mathcal{K}}$. Related results can be found in [24, Chapter 35]. Independently, Sladký [21,22] obtained the same block-triangular decomposition and used it to get bounds on the asymptotic behavior of iterations $g^n_{\mathcal{K}}(v)$ for a nonnegative vector v. Stronger results about asymptotic behavior of iterations $g^n_{\mathcal{K}}(v)$ were obtained in [22,23,26] for the case when some special matrices in $\mathcal{K}$ are aperiodic.

We consider maps of similar form, but with "minimum" instead of "maximum":

$$f_{\mathcal{K}}(x) = \min_{A\in\mathcal{K}} Ax.$$

These maps appear in [12] in connection with the construction of "self-similar" metrics on self-similar sets and finding their Hausdorff dimensions. Also such maps appear in the study of growth of Schreier graphs of groups generated by finite automata [5,11]. These problems demand us to study spectral properties of maps $f_{\mathcal{K}}$ and describe asymptotic behavior of its iterations.

Considering maps $f_{\mathcal{K}}$ and $g_{\mathcal{K}}$ we can always suppose that the set $\mathcal{K}$ satisfies the product property, i.e. $\mathcal{K}$ is constructed by all possible interchanges of corresponding rows selected from a finite set of square nonnegative matrices (see the precise definition and explanation in Section 3). Under this assumption, for every vector v there exist $A = A_v \in \mathcal{K}$ and $B = B_v \in \mathcal{K}$ such that $f_{\mathcal{K}}(v) = Av$ and $g_{\mathcal{K}}(v) = Bv$. In the theory of Markov decision processes this property is usually called the optimal choice property (see [9,3]).

The asymptotic behavior of iterations $f^n(v)$ is studied with respect to the following equivalence. Let $a_n, b_n, n \geqslant 1$, be sequences of nonnegative numbers or vectors of the same dimension. We say that $a_n \preceq b_n$ if there exists constant $q > 0$ such that $a_n \leqslant q \cdot b_n$ for all n large enough. If $a_n \preceq b_n$ and $b_n \preceq a_n$ then we say that $a_n \sim b_n$ and that a_n and b_n have the same *growth*. Then $h^n(v) \sim h^n(u)$ for any homogenous nondecreasing function $h : \mathbb{R}^N_+ \to \mathbb{R}^N_+$ and any strictly positive vectors v, u. Hence we can and we will change one strictly positive vector to another one considering asymptotic behavior if it is necessary.

Considering maps $f_{\mathcal{K}}$ we follow as close as possible to the ideas of Zijm and use his paper [25] as a model. Notice that we cannot use Zijm's results for $-f_{\mathcal{K}}$, which can be expressed using maximum, because matrices should be nonnegative and dynamics is considered on the nonnegative cone. The problem in transferring the results obtained for $g_{\mathcal{K}}$ to $f_{\mathcal{K}}$ lies in the convexity property which $f_{\mathcal{K}}$ lacks. In particular, there is no simultaneous block-triangular decomposition, which was extremely important in [25,21,22]. To overcome this difficulty we show that if the set $\mathcal{K}$ satisfies the product property then there exist matrices B and C in $\mathcal{K}$ which give the lowest and the greatest asymptotic behavior over all matrices in $\mathcal{K}$, i.e. $B^n v \preceq A^n v \preceq C^n v$ for all $A \in \mathcal{K}$ and every strictly positive vector v. These matrices we call respectively $\preceq$-*minimal* and $\preceq$-*maximal* for the set $\mathcal{K}$. Using these notions we study spectral properties of $f_{\mathcal{K}}$. In particular, we prove that $f_{\mathcal{K}}$ possesses a strictly positive eigenvector if and only if some (every) $\preceq$-minimal matrix possesses a strictly positive eigenvector. The main result shows existence of nonnegative generalized eigenvectors of $f_{\mathcal{K}}$. Finally as a corollary we describe the asymptotic behavior of each component of $f^n_{\mathcal{K}}(v)$ by showing that $f^n_{\mathcal{K}}(v) \sim A^n v$ for some (every) $\preceq$-minimal matrix A. We prove some other propositions similar to the results of [25], sometimes assuming additional conditions.

I would like to thank Stéphane Gaubert for bringing my attention to Zijm's articles and Volodymyr Nekrashevych for helpful suggestions.

2. Nonnegative matrices: definitions, notations, results

We recall in this section some (usually well-known) definitions and results that we need about nonnegative matrices (for the references see [4, Chapter 2], [20], [1, Chapter 1], [6, Chapter XIII]).

A matrix $A = (a_{ij})$ is called *nonnegative* (*positive*) if $a_{ij} \geqslant 0$ ($a_{ij} > 0$) for all indices i, j. Denote by A_i the ith row of the matrix A and by v_i the ith component of a vector v. A vector v is called *strictly positive* if $v_i > 0$ for all i. Unless otherwise stated, all matrices will be square of a fixed dimension N. Following [9,16,25] the set $\{1, 2, \ldots, N\}$ is called the *state space* and denoted by S. If $S_1, S_2 \subset S$ then we denote by $A|_{(S_1,S_2)}$ the restriction of the square matrix A to $S_1 \times S_2$ and by $v|_{S_1}$ the restriction of the vector v to S_1.

The spectral radius of a matrix A is denoted by $spr(A)$.

We say that state i *has access to state* j if there exists a nonnegative integer n such that the ijth entry of A^n is positive. Matrix A is called *irreducible* if any two states have access to each other. In the other case A is called *reducible*.

The following theorem states some important properties of square nonnegative matrices.

Theorem 1 (4, Chapter 2). *Let A be a nonnegative matrix with spectral radius λ. Then*

(*a*) *λ is an eigenvalue of A.*
(*b*) *There exists a nonnegative eigenvector v associated with λ.*
(*c*) *If $Au \geqslant \sigma u$ for $u \gneq 0$ then $\lambda \geqslant \sigma$.*
If moreover A is irreducible then
(*d*) *There exists a strictly positive eigenvector v associated with λ and any nonnegative eigenvector of A is a scalar multiple of v.*
(*e*) *If $Au \geqslant \lambda u$ or $Au \leqslant \lambda u$ for $u \geqslant 0$ then $Au = \lambda u$.*
(*f*) *$(\sigma I - A)^{-1} > 0$ for any $\sigma > \lambda$.*
(*g*) *$spr(A|_{(C,C)}) < \lambda$ for any $C \subsetneq S$. If A is reducible then $spr(A|_{(C,C)}) \leqslant \lambda$ for any $C \subsetneq S$ and $spr(A|_{(C,C)}) = \lambda$ for some $C \subsetneq S$.*
(*h*) *If $Au > \sigma u$ ($Au < \sigma u$) for $u \geqslant 0$ then $\lambda > \sigma$ (respectively, $\lambda < \sigma$).*

Iterations of a matrix heavily depend on its block-triangular structure. We will describe this following [4,20].

A *class* of a nonnegative matrix A is a subset C of the state space S such that $A|_{(C,C)}$ is irreducible and such that C cannot be enlarged without destroying the irreducibility. A class C is called *basic* if $spr(A|_{(C,C)}) = spr(A)$, otherwise *nonbasic* (when $spr(A|_{(C,C)}) < spr(A)$). It follows that for any matrix A we have a partition of the state space S into classes, say $C_1, C_2, \ldots, C_n$. Then, after possibly permuting the states and renumbering the classes, A can be written in the form, sometimes called *the Frobenius Normal Form*,

$$A = \begin{pmatrix} A_{(1,1)} & A_{(1,2)} & \cdots & A_{(1,n)} \\ 0 & A_{(2,2)} & \cdots & A_{(2,n)} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & A_{(n,n)} \end{pmatrix}$$

where $A_{(i,j)}$ denotes $A|_{(C_i,C_j)}$. Hence classes can be partially ordered by accessibility relation. We say that a class C has *access to (from)* a class C' if there is an access to (from) some (or equivalently any) state in C to some (or equivalently any) state in C'. A class is called *final* if it has no access to any other class.

The *spectral radius of a class C* is the spectral radius of $A|_{(C,C)}$.

The next proposition describes when a matrix A has a strictly positive eigenvector and, what is more important for the subject of this paper, when $(A^n v)_i \sim (A^n v)_j$ for all indices i, j and any strictly positive vector v.

Proposition 2. *Let A be a nonnegative matrix with spectral radius λ. Then the following conditions are equivalent*:

(*a*) *The matrix A has a strictly positive eigenvector*.
(*b*) *The basic classes of A are precisely its final classes*.
(*c*) $(A^n v)_i \sim \lambda^n$ *for all i and for some* (*every*) *vector* $v > 0$.
(*d*) $(A^n v)_i \sim (A^n v)_j$ *for all i, j and for some* (*every*) *vector* $v > 0$.

Proof. The proof of equivalence (a) and (b) can be found in [4, Theorem 3.10]. The proof of the rest will follow directly from Corollary 5. $\square$

Also notice that if a nonnegative matrix possesses a strictly positive eigenvector then it is associated with the spectral radius of this matrix.

Already the last proposition indicates importance of the position of basic and nonbasic classes of a square nonnegative matrix A for existence of a strictly positive eigenvector and behavior of its iterations. These positions can be defined precisely by introducing the concept of a chain. A *chain of classes* of A is an ordered collection of classes $\{C_1, C_2, \ldots, C_n\}$ such that C_i has access to $C_{i+1}, i = 1, \ldots, n-1$. The *length of a chain* is the number of basic classes it contains. The *depth of a class C* of A is the length of the longest chain that starts with C. The *degree* $\nu(A)$ of A is the length of its longest chain. Let S_i be the union of all classes of depth i. The partition $\{S_0, S_1, \ldots, S_\nu\}$ of the state space S is called the *principal partition of S with respect to A*. Principal partitions play a fundamental role in this paper. The next result is then straight forward.

Proposition 3. *Let* $\{S_0, S_1, \ldots, S_\nu\}$ *be the principal partition of S with respect to A. Then, after possibly permuting the states, A can be written in the form*

$$A = \begin{pmatrix} A_{(\nu,\nu)} & A_{(\nu,\nu-1)} & \cdots & A_{(\nu,0)} \\ 0 & A_{(\nu-1,\nu-1)} & \cdots & A_{(\nu-1,0)} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & A_{(0,0)} \end{pmatrix},$$

where $A_{(i,j)}$ *denotes* $A|_{(S_i,S_j)}$. *We have that* $spr(A_{(0,0)}) < spr(A)$ (*if* S_0 *is not empty*); $spr(A_{(i,i)}) = spr(A)$ *and the final classes and basic classes of* $A_{(i,i)}$ *coincide for* $i = 1, \ldots, \nu$. *Each state in* S_{i+1} *has access to some state in* S_i *for* $i \geqslant 1$ (*here* $A_{(i,i-1)}$ *is non-zero*).

Notice that it follows from Proposition 2 that $A|_{(S_i,S_i)}$ possesses a strictly positive eigenvector for every $i = 1, \ldots, \nu$.

We need the following useful lemma.

Lemma 1 (25, Lemma 2.5). *Let A be a nonnegative matrix with spectral radius* λ.

(*a*) *If* $Av \geqslant \sigma v$ *for some real number* σ *and a real vector v with at least one positive component, then* $\lambda \geqslant \sigma$.
(*b*) *If* $Av \geqslant \lambda v$ *with* $v > 0$ *then every final class of A is basic and* $(Av)_i = \lambda v_i$ *for every i in a final class of A*.

Matrices which possess strictly positive eigenvectors have the following additional properties.

Lemma 2. *Let A be a nonnegative matrix with spectral radius λ which has a strictly positive eigenvector. Let $S_1 \subset S$ be the union of all final classes of A. If $Au = \lambda u$ with $u|_{S_1} > 0$ then $u > 0$.*

Proof. Let $S_2 = S \setminus S_1$. Then, after possibly permuting the states, A can be written in the form:

$$A = \begin{pmatrix} A_{(S_2,S_2)} & A_{(S_2,S_1)} \\ 0 & A_{(S_1,S_1)} \end{pmatrix}.$$

Each class C in S_2 has access to some state in S_1, $spr(A|_{(C,C)}) < \lambda$ by Proposition 2, and $(\lambda I - A|_{(C,C)})^{-1} > 0$ by Theorem 1 item (f). It follows that $(\lambda I - A|_{(S_2,S_2)})^{-1} A_{(S_2,S_1)}$ has a positive element in each row. Then

$$u|_{S_2} = (\lambda I - A|_{(S_2,S_2)})^{-1} A|_{(S_2,S_1)} u|_{S_1} > 0. \qquad \square$$

Lemma 3 (25, Lemma 2.3). *Let A be a nonnegative matrix with spectral radius λ which possesses a strictly positive eigenvector. Then:*

(a) *There exists a nonnegative matrix A^* defined by:*

$$A^* = \lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} \lambda^{-i} A^i.$$

We have $AA^* = A^*A = \lambda A^*$ and $(A^*)^2 = A^*$. Moreover, $a^*_{ij} > 0$ if and only if j belong to a basic class of A and i has access to j under A.

(b) *The matrix $\lambda I - A + A^*$ is nonsingular.*

(c) *If $A^* v = 0$ for some vector $v \geqslant 0$ (or $v \leqslant 0$), then $v_i = 0$ for every state i belonging to a basic class of A.*

(d) *If $Av \geqslant \lambda v$ (or $Av \leqslant \lambda v$) for some vector v then $A^* v \geqslant v$ (respectively, $A^* v \leqslant v$).*

Notice that if A is a (reducible) stochastic matrix then A^* is a limiting transition probability matrix and the inverse of $(I - A + A^*)$ is the so-called fundamental matrix of the respective Markov chain.

Asymptotic behavior of iterations of a nonnegative matrix can be studied through its generalized eigenvectors corresponding to its spectral radius. Let A be a nonnegative matrix with spectral radius λ. The *index* $\eta(A)$ of A with respect to λ is the smallest integer n such that the null spaces of $(A - \lambda I)^n$ and $(A - \lambda I)^{n+1}$ coincide. The elements of $\mathrm{Null}(A - \lambda I)^i \setminus \mathrm{Null}(A - \lambda I)^{i-1}$ are called the *generalized eigenvectors* of order i. It was proved in [17, Theorem 3.1] that $\eta(A) = \nu(A)$. Moreover, it was shown that generalized eigenvectors can be chosen nonnegative with special strictly positive components. More precisely (see also [19,25]):

Theorem 4 (17, Theorem 3.1). *Let A be a nonnegative matrix with spectral radius λ. Let $\{S_0, S_1, \ldots, S_\nu\}$ be the principal partition of S with respect to A. Then there exists a set of nonnegative generalized eigenvectors $v^{(1)}, v^{(2)}, \ldots, v^{(\nu)}$ such that*

$$Av^{(\nu)} = \lambda v^{(\nu)},$$
$$Av^{(i)} = \lambda v^{(i)} + v^{(i+1)}, \quad i = \nu - 1, \ldots, 2, 1.$$

Moreover

$$v_j^{(i)} > 0, \quad j \in \bigcup_{k=i}^{\nu} S_k \quad \text{and} \quad v_j^{(i)} = 0, \quad j \in \bigcup_{k=0}^{i-1} S_k.$$

To give estimates on the growth of $A^n v$ we need the following lemma.

Lemma 4. *For any integer $k \geqslant 0$ and real $\lambda, \beta > 0$ we have asymptotic relation*

$$\sum_{i=0}^{n-1} \lambda^{n-i} i^k \beta^i \sim \begin{cases} n^k \beta^n, & \text{if } \beta > \lambda; \\ n^{k+1}\lambda^n, & \text{if } \beta = \lambda. \end{cases}$$

Proof. The asymptotic relation $\sum_{i=0}^{n-1} i^k \sim n^{k+1}$ is standard. Then for $\lambda = \beta$:

$$\sum_{i=0}^{n-1} \lambda^{n-i} i^k \beta^i = \lambda^n \sum_{i=0}^{n-1} i^k \sim n^{k+1}\lambda^n$$

and for $\beta > \lambda$ we have inequalities:

$$(n-1)^k \beta^{n-1} \leqslant \sum_{i=0}^{n-1} \lambda^{n-i} i^k \beta^i = \lambda^n \sum_{i=0}^{n-1} i^k \left(\frac{\beta}{\lambda}\right)^i \leqslant \lambda^n (n-1)^k \sum_{i=0}^{n-1} \left(\frac{\beta}{\lambda}\right)^i$$

$$\leqslant \lambda^n n^k \frac{\left(\frac{\beta}{\lambda}\right)^n - 1}{\frac{\beta}{\lambda} - 1} \leqslant n^k \frac{\beta^n}{\frac{\beta}{\lambda} - 1}. \quad \square$$

Corollary 5. *Let A be a nonnegative matrix with spectral radius λ. Let $\{S_0, S_1, \ldots, S_\nu\}$ be the principal partition of S with respect to A. Then*

$$(A^n v)_k \sim n^{i-1}\lambda^n, \quad \text{for } k \in S_i, \quad i \geqslant 1$$

for any strictly positive vector v.

Proof. The statement follows from a general result about asymptotic behavior of matrix powers obtained in [18]. We sketch the proof to use it later.

Using the identities for the generalized eigenvectors $v^{(i)}$ from Theorem 4 and the above lemma, one can get by induction that

$$A^n v^{(i)} \sim \lambda^n \sum_{j=0}^{\nu-i} n^j v^{(i+j)}, \quad \text{for } i = \nu, \ldots, 1 \quad \Rightarrow \quad (A^n v^{(1)})_k \sim n^{i-1}\lambda^n, \text{ for } k \in S_i.$$

Since $spr(A|_{(S_0,S_0)}) < \lambda$ the S_0th components of a strictly positive vector v does not effect the asymptotic behavior of $A^n v|_{S\setminus S_0}$. Hence $A^n v|_{S\setminus S_0} \sim A^n v^{(1)}|_{S\setminus S_0}$. $\square$

Remark. Corollary 5 gives us an algorithm of finding the growth of each component of $A^n v$ (see the detailed analysis in [18]). For indices in S_i for $i \geqslant 1$ it follows directly from the corollary. For $i \in S_0$ we consider the matrix $A|_{(S_0,S_0)}$ and its principal partition and so on. This algorithm can be also described using chains of classes as follows. Take a state i and the corresponding class C_i which contains i. Let β be the maximum of spectral radii of classes C, where C runs through all classes such that C_i has access to C. Consider all possible chains that start at C_i and for each chain count the number of classes C in this chain with spectral radius β. Let k be the maximal among such numbers. Then $(A^n v)_i \sim n^{k-1}\beta^n$ for any strictly positive vector v. In particular, if i belongs to a final class of A then $(A^n v)_i \sim \beta^n$, where β is the spectral radius of the class.

Corollary 5 implies that the components of $A^n v$ are comparable with respect to the partial order $\preceq$. The $\preceq$-minimal possible growth of $(A^n v)_i$ over all indices i is $\sim \gamma^n$, where γ is the spectral radius of some final class C. If state i has access to state j then $(A^n v)_j \preceq (A^n v)_i$. So, $(A^n v)_i \sim (A^n v)_j$ for any two states i and j in the same class of A.

3. Existence of strictly positive eigenvector of $f_{\mathcal{K}}$

Let $\mathcal{K}$ be a finite set of nonnegative matrices. In this section we define the notion of a $\preceq$-minimal matrix and the principal $\preceq$-minimal partition of S with respect to $\mathcal{K}$. Using these notions we give necessary and sufficient conditions for existence of a strictly positive eigenvector for the map $f_{\mathcal{K}}(v) = \min_{A\in\mathcal{K}} Av$.

Note that in general we do not have the property that for every vector v there exists $A \in \mathcal{K}$ such that $f_{\mathcal{K}}(v) = Av$. The following concept eliminates this difficulty (compare with [20, Section 3.1]).

Definition 1. *Let $\mathcal{K}$ be a set of nonnegative $N \times N$ matrices. We say that $\mathcal{K}$ satisfies the product property if for each subset $V \subseteq S$ and for each pair of matrices $A, B \in \mathcal{K}$ the matrix C defined by*

$$C_i := \begin{cases} A_i, & \text{if } i \in V; \\ B_i, & \text{if } i \in S \setminus V. \end{cases}$$

belongs to $\mathcal{K}$.

If we have any finite set $\mathcal{K}_0$ of nonnegative matrices then we can close it with respect to the product property and obtain another finite set $\mathcal{K}$. We just take all possible matrices C obtained as follows: the ith row of C is the ith row of some matrix from $\mathcal{K}$. Then it is easy to see that

$$\min_{A\in\mathcal{K}_0} Av = \min_{A\in\mathcal{K}} Av$$

for any vector v. So we can extend our given set of matrices to a bigger one, which satisfies the product property, without changing the map $f_{\mathcal{K}}$. Moreover, for every v there exists $A = A_v \in \mathcal{K}$ such that $f_{\mathcal{K}}(v) = Av$.

Hence we will always assume that $\mathcal{K}$ possesses the product property.

As was mentioned in introduction, Richard Bellman in [2] considered compact sets $\mathcal{K}$ of positive matrices and proved that $g_{\mathcal{K}}$ has a strictly positive eigenvector. It was generalized to a set of irreducible matrices in [14]. A simple proof of this result was obtained by W.H.M. Zijm [25] following the arguments in [10, Appendix B]. His proof also works for the maps $f_{\mathcal{K}}$.

Proposition 6. *Suppose that every matrix in the set $\mathcal{K}$ is irreducible. Then $f_{\mathcal{K}}$ possesses a strictly positive eigenvector associated with $\lambda_{\mathcal{K}} = \min_{A\in\mathcal{K}} spr(A)$. Moreover, it is unique up to a scalar multiple.*

Proof. Take any $B \in \mathcal{K}$. Let λ_B be the spectral radius of B and let v be the corresponding strictly positive eigenvector. Find $D \in \mathcal{K}$ such that

$$Dv = \min_{A\in\mathcal{K}} Av$$

with $D_i = B_i$ if $(Bv)_i \leqslant (Av)_i$ for all $A \in \mathcal{K}$. If $D = B$ then $f_{\mathcal{K}}(v) = Bv = \lambda_B v$ and we are done. If $D \neq B$ then $Dv \lneqq Bv = \lambda_B v$ and $\lambda_D := spr(D) < \lambda_B$ by Theorem 1 item (e). Apply the same procedure for the matrix D with its strictly positive eigenvector u associated with λ_D. Since $\mathcal{K}$ is finite, after a finite number of steps we will reach a matrix M with spectral radius λ and eigenvector w such that

$$Mw = \min_{A\in\mathcal{K}} Aw = \lambda w.$$

Since $Aw \geqslant \lambda w$ for every $A \in \mathcal{K}$, we get $\lambda = \min_{A\in\mathcal{K}} spr(A)$ by Theorem 1 item (e).

Let $u, v > 0$ be eigenvectors of $f_{\mathcal{K}}$ and let $f_{\mathcal{K}}(v) = Av = \lambda v$ and $f_{\mathcal{K}}(u) = Bu = \lambda u$. Then $spr(A) = spr(B) = \lambda$, $Bv \geqslant f_{\mathcal{K}}(v) = \lambda v$, and $Bv = \lambda v$ by Theorem 1 item (e). Hence by Theorem 1 item (d) the eigenvector of $f_{\mathcal{K}}$ associated with λ is unique up to a scalar multiple. $\square$

The following lemma is an important result for understanding the asymptotic behavior of $f^n_{\mathcal{K}}(v)$. It will be used throughout the paper.

Lemma 5. *There exists $B \in \mathcal{K}$ such that $B^n v \preceq A^n v$ for any $A \in \mathcal{K}$ and $v > 0$.*

Proof. We use induction on dimension N. For $N = 1$ the statement is obvious. Suppose the lemma is correct for any dimension $< N$. Let us fix $v > 0$.

For each $A \in \mathcal{K}$ and $i \in S$ we can find the asymptotic behavior of $(A^n v)_i$ using Corollary 5. Define the set $\mathcal{K}' \subset \mathcal{K}$ of all matrices B in $\mathcal{K}$ for which there exists $i \in S$ such that $(B^n v)_i \preceq (A^n v)_j$ for all $A \in \mathcal{K}$ and $j \in S$. Note that it follows from Corollary 5 that $(B^n v)_i \sim \lambda^n$ for some real $\lambda \geqslant 0$. For each matrix $B \in \mathcal{K}'$ define

$$S_0(B) = \{j \in S \mid (B^n v)_i \sim (B^n v)_j \sim \lambda^n\}$$

and $S_1(B) = S \setminus S_0(B)$. Suppose some state i in $S_0(B)$ has access to some state j in $S_1(B)$. Then $(B^n v)_j \preceq (B^n v)_i$. Since the asymptotic behavior of $(B^n v)_i$ is $\preceq$-minimal for B, we have $(B^n v)_i \sim (B^n v)_j$. Hence $j \in S_0(B)$ and we have a contradiction. Thus no state in $S_0(B)$ has access to any state in $S_1(B)$, which means that $B|_{(S_0(B),S_1(B))} = 0$.

Observe that the spectral radius of every class of B from $S_0(B)$ is not greater than λ. If a class C from $S_0(B)$ is final then it has spectral radius λ. The converse is also true: a class C from $S_0(B)$ with spectral radius λ is final. Really, suppose it is not final. Then it has access to a final class from $S_0(B)$. Thus there exists a chain which start at C that contains at least two classes with spectral radii λ. So, $(B^n v)_i \succeq n\lambda^n$ for i in C by Corollary 5.

Let us show that $\mathcal{K}'$ contains a matrix B with the biggest set $S_0(B)$, i.e. such that $S_0(B) \supset S_0(A)$ for any $A \in \mathcal{K}'$. It is sufficient to show that for any matrices B and D from $\mathcal{K}'$ there exists $E \in \mathcal{K}'$ such that $S_0(E) \supset S_0(B) \cup S_0(D)$. Define E as follows: $E_i = B_i$ for $i \in S_0(B)$ and $E_i = D_i$ for $i \notin S_0(B)$.

$$E = \begin{pmatrix} D|_{(S_1(B),S_1(B))} & * \\ 0 & B|_{(S_0(B),S_0(B))} \end{pmatrix}$$

Then $E \in \mathcal{K}'$ and $S_0(E) \supset S_0(B)$, because $(E^n v)_i = (B^n v)_i$ for $i \in S_0(B)$. In order to prove that $S_0(E)$ contains $S_0(D)$, it is sufficient to prove that each class C of E, which belong to $S_0(D) \setminus S_0(B)$ with spectral radius λ is final (if it is empty we are done). By construction $E|_{(C,C)} = D|_{(C,C)}$ and C belongs to some class C' of D from $S_0(D)$. If $C \neq C'$ then $spr(D|_{(C',C')}) > spr(D|_{(C,C)}) = spr(E|_{(C,C)}) = \lambda$ by Theorem 1 item (g) and we have contradiction with $C' \subset S_0(D)$. Thus $C = C'$ and $E|_{(C,S\setminus C)} = D|_{(C',S\setminus C')} = 0$. So C is final and our claim is proved.

Choose $B \in \mathcal{K}'$ to be a matrix with the biggest set $S_0(B)$. Denote $S_0 := S_0(B)$ and $S_1 := S_1(B)$.

If $S_0 = S$ then we are done – the matrix B satisfies the condition of the lemma. Suppose that $S_1 \neq \emptyset$. The set $\mathcal{K}|_{(S_1,S_1)}$ satisfies the product property and we can apply induction to it. So there exists $D \in \mathcal{K}$ such that $(D|_{(S_1,S_1)})^n v|_{S_1} \preceq (A|_{(S_1,S_1)})^n v|_{S_1}$ for any $A \in \mathcal{K}$. Define a matrix E in the same way as above: $E_i = B_i$ for $i \in S_0$ and $E_i = D_i$ for $i \notin S_0$. We want to show that it satisfies the condition of the lemma.

Again $(E^n v)_i = (B^n v)_i$ for $i \in S_0$. So $(E^n v)_i \preceq (A^n v)_i$ for any matrix $A \in \mathcal{K}$ for $i \in S_0$. We need to prove the previous inequality for $i \in S_1$.

$$\begin{aligned}(E^n v)|_{S_1} &\leqslant (D|_{(S_1,S_1)})^n v|_{S_1} + \sum_{l=1}^{n-1} (D|_{(S_1,S_1)})^{n-l} D|_{(S_1,S_0)} (B|_{(S_0,S_0)})^l v|_{S_0} \preceq \\ &\preceq (D|_{(S_1,S_1)})^n v|_{S_1} + \sum_{l=1}^{n-1} (D|_{(S_1,S_1)})^{n-l} \lambda^l v|_{S_1} = \sum_{l=0}^{n-1} (D|_{(S_1,S_1)})^{n-l} \lambda^l v|_{S_1}.\end{aligned}$$

Fix i in S_1 and let $((D|_{(S_1,S_1)})^n v|_{S_1})_i \sim n^k \beta^n$. Suppose $\beta < \lambda$. Then there exists $j \in S_1$ such that $((D|_{(S_1,S_1)})^n v|_{S_1})_j \sim \beta^n$. Then:

$$(E^n v)_j \preceq \sum_{l=0}^{n-1} (D|_{(S_1,S_1)})^{n-l} \lambda^l v|_{S_1} \sim \sum_{l=0}^{n-1} \lambda^l \beta^{n-l} \sim \lambda^n$$

and therefore j must be in S_0. We get a contradiction, hence $\beta \geqslant \lambda$.

If $\beta > \lambda$ then

$$(E^n v)_i \preceq \sum_{l=0}^{n-1} (D|_{(S_1,S_1)})^{n-l} \lambda^l v|_{S_1} \sim \sum_{l=0}^{n-1} \lambda^l (n-l)^k \beta^{n-l} \sim n^k \beta^n \sim ((D|_{(S_1,S_1)})^n v|_{S_1})_i \preceq$$
$$\preceq ((A|_{(S_1,S_1)})^n v|_{S_1})_i \preceq (A^n v)_i$$

for every $A \in \mathcal{K}$.

Now suppose that $\beta = \lambda$. Let C_i be the class of $D|_{(S_1,S_1)}$ that contains i. Then λ is the maximum of spectral radii of $D|_{(S_1,S_1)}|_{(C,C)}$, where C runs through all classes of $D|_{(S_1,S_1)}$ such that C_i has access to C. Also the maximal number of classes C with $spr(D|_{(S_1,S_1)}|_{(C,C)}) = \lambda$ in chains that start at C_i is k. If the maximum of spectral radii of $D|_{(C,C)}$, where C_i has access to C, is greater than λ, then $(D^n v)_i \succeq n^{k+1}\lambda^n$. If not then the maximal number of classes C of B with $spr(D|_{(C,C)}) = \lambda$ in a chain that starts at C_i is at least $k+1$, otherwise there exists a state j in S_1 with $(E^n v)_j \sim \lambda^n$. Thus, $(D^n v)_i \succeq n^{k+1}\lambda^n$. Notice that the above statement is true for any matrix $A \in \mathcal{K}$, i.e. if $((A|_{(S_1,S_1)})^n v|_{S_1})_i \sim n^k \lambda^n$ then $(A^n v)_i \succeq n^{k+1}\lambda^n$. Then

$$(E^n v)_i \preceq \sum_{l=0}^{n-1} (D|_{(S_1,S_1)})^{n-l} \lambda^l v|_{S_1} \sim \sum_{l=0}^{n-1} \lambda^l (n-l)^k \lambda^{n-l} \sim n^{k+1}\lambda^n \preceq (A^n v)_i$$

for any $A \in \mathcal{K}$. So $(E^n v)_i \preceq (A^n v)_i$ for all i and $A \in \mathcal{K}$. □

By similar arguments one can show that there exists $C \in \mathcal{K}$ such that $A^n v \preceq C^n v$ for every $A \in \mathcal{K}$.

Definition 2. *A matrix $B \in \mathcal{K}$ which satisfies Lemma 5 will be called $\preceq$-minimal for $\mathcal{K}$.*

There is a simple (but not effective) algorithm to find all $\preceq$-minimal matrices. We find the asymptotic behavior of $(A^n v)_i$ for all $A \in \mathcal{K}$ by Corollary 5 and take matrices with $\preceq$-minimal growth (such matrices exist by Lemma 5).

If the spectral radius of a matrix A is zero, then A is nilpotent and asymptotic behavior of $A^n v$ is trivial. Consideration of such matrices is elementary but does not fit precisely in the discussion below. To avoid these unnecessary complications and without loss of generality in the sequel all considered matrices have spectral radius > 0.

The principal partitions, spectral radii and degrees of every two $\preceq$-minimal matrices coincide, which follows from Corollary 5 and from the fact that $A^n v \sim B^n v$ for any $\preceq$-minimal matrices $A, B \in \mathcal{K}$. Notice that the spectral radius of a $\preceq$-minimal matrix is equal to $\min_{A\in\mathcal{K}} spr(A)$. Denote $\lambda = spr(B)$ and $\nu = \nu(B)$ for a $\preceq$-minimal matrix $B \in \mathcal{K}$.

Definition 3. *The principal partition $\{S_0, S_1, \ldots, S_\nu\}$ of a $\preceq$-minimal matrix is called the principal $\preceq$-minimal partition of S with respect to $\mathcal{K}$.*

The following proposition gives a sufficient condition for existence of a strictly positive eigenvector for the map $f_{\mathcal{K}}$. Moreover, it will follow from Corollary 12 that this condition is also necessary, and what is more important for the subject of this paper that it is equivalent to the property that all components of the iterations $f_{\mathcal{K}}^n(v)$ have the same growth.

Proposition 7. *Suppose that some $\preceq$-minimal matrix possesses a strictly positive eigenvector. Then $f_{\mathcal{K}}$ possesses a strictly positive eigenvector associated with λ.*

Proof. Note that by Proposition 2 if one $\preceq$-minimal matrix possesses a strictly positive eigenvector then all $\preceq$-minimal matrices do.

Let B be a $\preceq$-minimal matrix with strictly positive eigenvector v. Apply the same procedure as in the proof of Proposition 6. Find $D \in \mathcal{K}$ such that

$$Dv = \min_{A \in \mathcal{K}} Av$$

with $D_i = B_i$ if $(Bv)_i \leqslant (Av)_i$ for all $A \in \mathcal{K}$. Then $Dv \leqslant \lambda v$ and $D^n v \leqslant \lambda^n v = B^n v$. Thus D is $\preceq$-minimal, has strictly positive eigenvector and $spr(D) = \lambda$. Since each final class of D is basic, $(Dv)_i = (\lambda v)_i$ for all i in final classes by Theorem 1 item (*e*). Hence $D_i = B_i$ for i in the final classes of D and the set of final classes of B contains the set of final classes of D.

By Proposition 2 each nonfinal class of D is nonbasic. Let $S_1 \subset S$ be the union of all final classes and let $S_2 = S \setminus S_1$. Then, after possibly permuting the states

$$D = \begin{pmatrix} D|_{(S_2,S_2)} & E \\ 0 & B|_{(S_1,S_1)} \end{pmatrix}$$

with $spr(D|_{(S_1,S_1)}) = \lambda$ and $spr(D|_{(S_2,S_2)}) < \lambda$. Define

$$u|_{S_1} = v|_{S_1} \quad \text{and} \quad u|_{S_2} = (\lambda I - D|_{(S_2,S_2)})^{-1} E v|_{S_2}. \tag{1}$$

Then $Du = \lambda u$ and thus $u > 0$ by Lemma 2. Suppose $u_i > v_i$ for some $i \in S$. Then it follows from $Du = \lambda u$ and $Dv \leqslant \lambda v$ that

$$D|_{(S_2,S_2)} \left[u|_{S_2} - v|_{S_2} \right] \geqslant \lambda \left[u|_{S_2} - v|_{S_2} \right].$$

This contradicts $spr(D|_{(S_2,S_2)}) < \lambda$ by Lemma 1. Hence $v \geqslant u > 0$.

By construction $u = v$ if and only if $D = B$. We can apply the same procedure to D and u. On each step the set of final classes of the new matrix is contained in the set of final classes of the previous matrix and the next eigenvector coincides with the previous one on the states from final classes of the new matrix. Since $\mathcal{K}$ is finite, after some steps all received matrices will have the same set of final classes and all received eigenvectors are the same on this set. Now suppose this process will never stabilize. It means that all received eigenvectors are different. Since $\mathcal{K}$ is finite, some matrix appears in this process at least two times with different strictly positive eigenvectors that coincide on the final classes of this matrix. But by (1) eigenvector of a matrix is uniquely defined by its coordinates from the final classes of this matrix. We get a contradiction. Thus after a finite number of steps we will reach a $\preceq$-minimal matrix $M \in \mathcal{K}$ with strictly positive eigenvector w such that

$$Mw = \min_{A \in \mathcal{K}} Aw = \lambda w. \quad \square$$

Corollary 8. *Under the conditions of Propositions 6 or 7 the asymptotic relation*

$$(f^n_{\mathcal{K}}(u))_i \sim \lambda^n$$

holds for any strictly positive vector u and $i \in S$.

So, if there exists a $\preceq$-minimal matrix with strictly positive eigenvector, then the growth exponent of each component of $f^n_{\mathcal{K}}(v)$ is equal to the spectral radius of this $\preceq$-minimal matrix.

The next proposition with $\nu = 1$ gives the basis of induction for Lemma 7.

Proposition 9. *Let $\{S_0, S_1, \ldots, S_\nu\}$ be the principal $\preceq$-minimal partition of the state space S with respect to $\mathcal{K}$. Then there exists a nonnegative vector w with $w|_{S_\nu} > 0$ such that*

$$\min_{A \in \mathcal{K}} Aw = \lambda w.$$

Proof. Let B be any $\preceq$-minimal matrix. Since $\{S_0, S_1, \ldots, S_\nu\}$ is the principal partition of B, the matrix $B|_{(S_\nu,S_\nu)}$ possesses a strictly positive eigenvector v associated with λ. The set $\mathcal{K}|_{S_\nu} = \{A|_{(S_\nu,S_\nu)}, A \in \mathcal{K}\}$ also satisfies the product property and $B|_{S_\nu}$ is $\preceq$-minimal for it. We can apply Proposition 7 for $\mathcal{K}|_{S_\nu}$. There exists a strictly positive vector u defined on S_ν such that

$$\min_{A \in \mathcal{K}} A|_{(S_\nu,S_\nu)} u = \lambda u.$$

Take w such that $w|_{S_\nu} = u$ and $w|_{S \setminus S_\nu} = 0$. Then w satisfies the condition of the proposition. $\square$

It was shown in [25,22] that a stronger result holds for $g_{\mathcal{K}}$, which proves existence of a simultaneous block-triangular representation of the matrices in $\mathcal{K}$ and allows one to define the "principal partition" of S with respect to $\mathcal{K}$. This partition plays a fundamental role in those papers. This result does not hold for $f_{\mathcal{K}}$.

4. Generalized eigenvectors of $f_{\mathcal{K}}$

We prove in this section two lemmata from which the main result follows immediately. The first lemma proves existence of a solution of a set of "nested" functional equations. As it was noticed in [25], it can be viewed as a generalization of the Howard's policy iteration procedure [9].

Let t be an integer greater than 1. Suppose that for each $A \in \mathcal{K}$ we have a sequence of vectors $r_i(A), i = 1, \ldots, t-1$.

Lemma 6. *Assume that the set of rectangular matrices*

$$\{(A, r_1(A), r_2(A), \ldots, r_{t-1}(A)) \mid A \in \mathcal{K}\}$$

satisfies the product property. Suppose that there exists (for all) a $\preceq$-minimal matrix $B \in \mathcal{K}$ with a strictly positive eigenvector v. Suppose furthermore $B^* r_{t-1}(B) > 0$ for any $\preceq$-minimal matrix B (here B^* is defined in Lemma 3). Then there exists a solution $\{v^{(1)}, \ldots, v^{(t)}\}$ of the set of functional equations:

$$\min_{A \in \mathcal{K}} A v^{(t)} = \lambda v^{(t)}$$
$$\min_{A \in \mathcal{K}_i} \left\{ A v^{(i-1)} + r_{i-1}(A) \right\} = \lambda v^{(i-1)} + v^{(i)}, \quad i = 2, \ldots, t,$$

where $\mathcal{K}_i$ is defined recursively by

$$\mathcal{K}_t := \{A \mid A \in \mathcal{K}, A v^{(t)} = \lambda v^{(t)}\},$$
$$\mathcal{K}_i := \{A \mid A \in \mathcal{K}_{i+1}, A v^{(i)} + r_i(A) = \lambda v^{(i)} + v^{(i+1)}\}, \quad i = 2, \ldots, t-1.$$

Furthermore $v^{(t)} > 0$.

Proof. The set of equations

$$\begin{aligned} &B v^{(t)} = \lambda v^{(t)}, \\ &B v^{(i)} + r_i(B) = \lambda v^{(i)} + v^{(i+1)}, \quad i = 1, \ldots, t-1, \\ &B^* v^{(1)} = 0 \end{aligned} \tag{2}$$

has a unique solution

$$v^{(t)} = B^* r_{t-1}(B),$$
$$v^{(i)} = (\lambda I - B + B^*)^{-1} [r_i(B) + B^* r_{i-1}(B) - v^{(i+1)}], \quad i = 2, \ldots, t-1,$$
$$v^{(1)} = (\lambda I - B + B^*)^{-1} [r_1(B) - v^{(2)}].$$

Moreover $v^{(t)} > 0$. Since we have the "extended" product property, there exists a matrix $D \in \mathcal{K}$ such that

$$D v^{(t)} = \min_{A \in \mathcal{K}} A v^{(t)},$$
$$D v^{(i)} + r_i(D) = \min_{A \in \mathcal{H}_{i+1}} \left\{ A v^{(i)} + r_i(A) \right\}, \quad i = 1, \ldots, t-1,$$

where $\mathcal{H}_i \subset \mathcal{K}$ denotes the set of matrices which minimize the right hand side of ith equation above. We choose $D = B$ if B satisfies above equations, i.e. if $B \in \mathcal{H}_1$.

Then $Dv^{(t)} \leqslant \lambda v^{(t)}$ and thus D is $\preceq$-minimal and possesses a strictly positive eigenvector. As above, the set of equations (2) with the matrix D instead of B has a unique solution $\{u^{(1)}, \ldots, u^{(t)}\}$ with $u^{(t)} > 0$ and so on. We want to show that this process will eventually stop. It is easy to see that if $v^{(i)}$ and $u^{(i)}$ satisfy the following properties

(a) $u^{(t)} \leqslant v^{(t)}$;
(b) if $u^{(i)} = v^{(i)}$ for $i = k+1, \ldots, t$ then $u^{(k)} \leqslant v^{(k)}$;
(c) if $u^{(i)} = v^{(i)}$ for all $i = 1, \ldots, t$ then $D = B$,

then, since $\mathcal{K}$ is finite, after a finite number of steps we will reach a matrix which stays intact under application of this process. The corresponding solution of (2) will satisfy the conditions of the lemma.

Let us prove (a), (b) and (c). Let $C \subset S$ be the union of all final classes of D.

(a) Using Lemma 3 and construction of $u^{(i)}$ and $v^{(i)}$ several times we get

$$u^{(t)} = D^* u^{(t)} = D^*[Du^{(t-1)} - \lambda u^{(t-1)} + r_{t-1}(D)] = D^* r_{t-1}(D) \leqslant$$
$$\leqslant D^*[\lambda v^{(t-1)} + v^{(t)} - Dv^{(t-1)}] = D^* v^{(t)} \leqslant v^{(t)}.$$

(b) Now suppose $u^{(i)} \leqslant v^{(i)}$ for $i = k+1, \ldots, t$. Define vectors $\psi^{(i)}, i = 1, \ldots, t$, such that:

$$Dv^{(t)} = \lambda v^{(t)} + \psi^{(t)},$$
$$Dv^{(i)} + r_i(D) = \lambda v^{(i)} + v^{(i+1)} + \psi^{(i)}.$$

From (2) for the matrix D and the previous equations we get:

$$D[v^{(i)} - u^{(i)}] = \lambda[v^{(i)} - u^{(i)}] + [v^{(i+1)} - u^{(i+1)}] + \psi^{(i)}. \tag{3}$$

Thus, $\psi^{(i)} = 0$ and $Dv^{(i)} + r_i(D) = Bv^{(i)} + r_i(B)$ for $i = k+1, \ldots, t$. Hence $B \in \mathcal{H}_{k+1}$. It follows that $\psi^{(k)} \leqslant 0$ and

$$D[v^{(k)} - u^{(k)}] = \lambda[v^{(k)} - u^{(k)}] + \psi^{(k)} \qquad \Rightarrow \qquad \text{(applying } D^*) \tag{4}$$
$$D^*\psi^{(k)} = 0.$$

Hence $\psi_i^{(k)} = 0$ for $i \in C$ by Lemma 3 item (c).

Consider the case $k \geqslant 2$. Then $\psi_i^{(k-1)} \leqslant 0$ for $i \in C$ and hence $D^*\psi^{(k-1)} \leqslant 0$ by Lemma 3 item (a). Applying D^* to $(k-1)$st equation of (3) we obtain:

$$0 = D^*[v^{(k)} - u^{(k)}] + D^*\psi^{(k-1)}, \quad \text{but} \quad D[v^{(k)} - u^{(k)}] \leqslant \lambda[v^{(k)} - u^{(k)}].$$

Hence $[v^{(k)} - u^{(k)}] \geqslant D^*[v^{(k)} - u^{(k)}] = -D^*\psi^{(k-1)} \geqslant 0$, because $\psi_i^{(k-1)} \leqslant 0$ for $i \in C$.

For $k = 1$ we have $B \in \mathcal{H}_2$ and since $\psi_i^{(1)} = 0$ for $i \in C$ we may choose $D_i = B_i$ for $i \in C$. In this case $D_i^* = B_i^*$ and $u_i^{(1)} = v_i^{(1)} = 0$ for $i \in C$. Thus $D^* v^{(1)} = 0$. It follows from (4) that

$$[v^{(1)} - u^{(1)}] \geqslant D^*[v^{(1)} - u^{(1)}] = 0.$$

(c) As above, $\psi^{(i)} = 0$ for all i and hence $B \in \mathcal{H}_1$. Thus $D = B$ by construction. $\square$

Lemma 7. *Let $\{S_0, S_1, \ldots, S_\nu\}$ be the principal $\preceq$-minimal partition with respect to $\mathcal{K}$. There exists a set of nonnegative vectors $v^{(1)}, v^{(2)}, \ldots, v^{(\nu)}$ such that*

$$\min_{A \in \mathcal{K}} Av^{(\nu)} = \lambda v^{(\nu)}, \tag{5}$$

$$\min_{A\in\mathcal{K}_{i+1}} Av^{(i)} = \lambda v^{(i)} + v^{(i+1)}, \quad i = \nu-1,\dots,2,1;$$

where

$$\mathcal{K}_\nu := \{A \mid A\in\mathcal{K}, Av^{(\nu)} = \lambda v^{(\nu)}\},$$
$$\mathcal{K}_i := \{A \mid A\in\mathcal{K}_{i+1}, Av^{(i)} = \lambda v^{(i)} + v^{(i+1)}\}, \quad i = 2,\dots,\nu-1.$$

Moreover

$$v_j^{(i)} > 0, \quad j\in\bigcup_{k=i}^{\nu} S_k \quad \text{and} \quad v_j^{(i)} = 0, \quad j\in\bigcup_{k=0}^{i-1} S_k. \tag{6}$$

Proof. By induction on ν. For $\nu = 1$ the result follows from Proposition 9. Suppose that the lemma holds for $\nu < t$ and let now $\nu = t$.

Notice that

$$\mathcal{K}_t = \{A \mid A\in\mathcal{K}, Av^{(t)} = \lambda v^{(t)} \text{ and } A|_{(S\setminus S_t, S_t)} = 0\}$$

for any given $v^{(t)}$ such that $v^{(t)}|_{S\setminus S_t} = 0$. Define the set of matrices

$$\mathcal{H} = \{A|_{(S\setminus S_t, S\setminus S_t)}, A\in\mathcal{K}_t\}.$$

Clearly $\mathcal{H}$ also satisfies the product property and $B|_{(S\setminus S_\nu, S\setminus S_\nu)}$ is a $\preceq$-minimal matrix for $\mathcal{H}$ for any $\preceq$-minimal matrix B for $\mathcal{K}$. Thus $S_0, S_1,\dots,S_{\nu-1}$ is the principal $\preceq$-minimal partition of $\mathcal{H}$. By the induction hypothesis there exist nonnegative vectors $u^{(1)}, u^{(2)},\dots,u^{(t-1)}$ defined on $S\setminus S_\nu$ such that $u_i^{(t-1)} > 0$ for $i\in S_{t-1}$ and

$$\min_{A\in\mathcal{H}} Au^{(t-1)} = \lambda u^{(t-1)},$$
$$\min_{A\in\mathcal{H}_{i+1}} Au^{(i)} = \lambda u^{(i)} + u^{(i+1)}, \quad i = t-2,\dots,2,1.$$

Now we need to find vectors $v^{(1)}, v^{(2)},\dots,v^{(t)}$ such that (5) holds. Let us take

$$v_j^{(i)} = u_j^{(i)} \quad \text{and} \quad v_j^{(t)} = 0 \quad \text{for } j\in S\setminus S_t.$$

Then $\mathcal{K}_i \subset \{A \mid A\in\mathcal{K}_t, A|_{(S\setminus S_t, S\setminus S_t)}\in\mathcal{H}_i\}$ for $i = 1,\dots,t-1$, and the vectors $v^{(i)}$, independent of their coordinates on S_t, satisfy (5) for states in $S\setminus S_t$. It remains to determine $v_j^{(i)}$ for $j\in S_t$, $i = 1,\dots,t$. The conditions on $v^{(i)}|_{S_t}$ are the following:

$$\min_{A\in\mathcal{K}} A|_{(S_t,S_t)} v^{(t)}|_{S_t} = \lambda v^{(t)}|_{S_t},$$
$$\min_{A\in\mathcal{K}_{i+1}} \left\{ A|_{(S_t,S_t)} v^{(i)}|_{S_t} + \sum_{j=i}^{t-1} A|_{(S_t,S_j)} w^{(i)}|_{S_j} \right\} = \lambda v^{(i)}|_{S_t} + v^{(i+1)}|_{S_t}, \quad i = t-1,\dots,2,1.$$

Since $\{S_0, S_1,\dots,S_t\}$ is the principal partition of any $\preceq$-minimal matrix $B\in\mathcal{K}$, the matrix $B|_{(S_t,S_t)}$ possesses a strictly positive eigenvector associated with λ. Moreover $u^{(t-1)}|_{S_{t-1}} > 0$. Each final class of $B|_{(S_t,S_t)}$ has access to some state in $B|_{(S_{t-1},S_{t-1})}$. Thus

$$(B|_{(S_t,S_{t-1})} u^{(t-1)}|_{S_{t-1}})_i > 0$$

for some i in every final class of $B|_{(S_t,S_t)}$. Then $B|^*_{(S_t,S_t)} B|_{(S_t,S_{t-1})} u^{(t-1)}|_{S_{t-1}} > 0$ for any $\preceq$-minimal B by Lemma 3 item (a). We can now apply Lemma 6 and find $v^{(i)}|_{S_t}$.

It may happened that $v^{(i)}$ does not satisfy the nonnegativity constrains (6) on S_t (they satisfy it on $S\setminus S_t$ by induction). In this case consider

$$\begin{aligned} w^{(t)} &= v^{(t)}, \\ w^{(i)} &= v^{(i)} + \alpha v^{(i+1)}, \quad i = 1, \ldots, t-1. \end{aligned} \tag{7}$$

They also satisfy (5) and we can choose α large enough so that $w_j^{(i)} > 0$ for all $j \in S_t, i = 1, \ldots, t$. □

Now we are ready to prove the main result.

Theorem 10. *Let $\{S_0, S_1, \ldots, S_\nu\}$ be the principal $\preceq$-minimal partition with respect to $\mathcal{K}$. Then there exists a set of nonnegative vectors $v^{(1)}, v^{(2)}, \ldots, v^{(\nu)}$ such that*

$$\begin{aligned} \min_{A \in \mathcal{K}} Av^{(\nu)} &= \lambda v^{(\nu)}, \\ \min_{A \in \mathcal{K}} Av^{(i)} &= \lambda v^{(i)} + v^{(i+1)}, \quad i = \nu - 1, \ldots, 2, 1. \end{aligned}$$

Moreover

$$v_j^{(i)} > 0, \quad j \in \bigcup_{k=i}^{\nu} S_k \quad \text{and} \quad v_j^{(i)} = 0, \quad j \in \bigcup_{k=0}^{i-1} S_k.$$

Proof. Use Lemma 7 to find solutions $v^{(1)}, v^{(2)}, \ldots, v^{(\nu)}$ of the corresponding system (5). Now consider the vectors $w^{(1)}, w^{(2)}, \ldots, w^{(\nu)}$ from (7). It is easy to see that for α large enough

$$\min_{A \in \mathcal{K}_{i+1}} Aw^{(i)} = \min_{A \in \mathcal{K}_{i+2}} Aw^{(i)} = \ldots = \min_{A \in \mathcal{K}} Aw^{(i)}, \quad i = 1, \ldots, \nu.$$

Hence for α large enough the vectors $w^{(1)}, w^{(2)}, \ldots, w^{(\nu)}$ satisfy the conditions of the theorem. □

Corollary 11. *Let $\{S_0, S_1, \ldots, S_\nu\}$ be the principal $\preceq$-minimal partition with respect to $\mathcal{K}$. Then*

$$(f_{\mathcal{K}}^n(v))_i \sim n^{k-1}\lambda^n, \quad \text{where } i \in S_k,$$

for any strictly positive vector v and $k \geqslant 1$. Moreover, for any $\preceq$-minimal matrix $B \in \mathcal{K}$

$$(f_{\mathcal{K}}^n(v))_i \sim (B^n v)_i$$

for any strictly positive vector v and $i \in S$.

Proof. The proof of the first part is the same as for a single matrix (see Corollary 5). Thus $(f_{\mathcal{K}}^n(v))_i \sim (B^n v)_i$ for $i \notin S_0$. We need to prove this asymptotic relation for $i \in S_0$.

The upper bound $f_{\mathcal{K}}^n(v) \preceq B^n v$ is obvious. Define

$$\mathcal{H} = \{A \in \mathcal{K} \mid A|_{(S_0, S \setminus S_0)} = 0\} \quad \text{and} \quad f_{\mathcal{K}}|_{S_0} = \min_{A \in \mathcal{H}} A|_{(S_0,S_0)}.$$

Then $B|_{(S_0,S_0)}$ is $\preceq$-minimal for $\mathcal{H}|_{S_0}$ for any matrix B $\preceq$-minimal for $\mathcal{K}$. Let $\beta = spr(B|_{(S_0,S_0)})$ for $\preceq$-minimal B (notice that $\beta < \lambda$) and let $\{S'_0, S'_1, \ldots, S'_{\nu'}\}$ be the principal $\preceq$-minimal partition of S_0 with respect to $\mathcal{H}|_{S_0}$. By Theorem 10 there exist nonnegative vectors $w^{(1)}, \ldots, w^{(\nu')}$ defined on S_0 such that

$$\begin{aligned} f_{\mathcal{K}}|_{S_0}(w^{(\nu')}) &= \beta w^{(\nu')} \\ f_{\mathcal{K}}|_{S_0}(w^{(i)}) &= \beta w^{(i)} + w^{(i+1)}, \quad i = \nu' - 1, \ldots, 2, 1, \end{aligned}$$

and with specified nonnegative constrains. Notice that then $(B^n v)_i \sim n^{k-1}\beta^n$ for $i \in S'_k, k \geqslant 1$.

Let v be a strictly positive vector defined on $S \setminus S_0$ such that $A|_{(S\setminus S_0, S\setminus S_0)} v \geqslant \lambda v$ (take for example $v^{(1)}|_{S \setminus S_0}$). Define vectors $u_{\alpha_i}^{(i)}, i = \nu', \ldots 2, 1$, such that $u_{\alpha_i}^{(i)}|_{S \setminus S_0} = \alpha_i v$ and $u_{\alpha_i}^{(i)}|_{S_0} = w^{(i)}$. Then

$$f_{\mathcal{K}}(u^{(i)}_{\alpha_i}) = \min_{A\in\mathcal{K}} \begin{pmatrix} \alpha_i A|_{(S\setminus S_0,S\setminus S_0)} v + A|_{(S\setminus S_0,S_0)} w^{(i)} \\ \alpha_i A|_{(S_0,S\setminus S_0)} v + A|_{(S_0,S_0)} w^{(i)} \end{pmatrix} = \min_{A\in\mathcal{H}} A u^{(i)}_{\alpha_i}, \quad i = \nu', \dots 2, 1,$$

for α_i large enough. Moreover we can additionally choose α_i such that $\alpha_i\lambda v \geqslant \alpha_i\beta v + \alpha_{i+1} v$ for $i = \nu' - 1, \dots 2, 1$. Then

$$f_{\mathcal{K}}(u^{(\nu')}_{\alpha_{\nu'}}) \geqslant \begin{pmatrix} \alpha_{\nu'}\lambda v \\ f_{\mathcal{K}}|_{S_0}(w^{(\nu')}) \end{pmatrix} = \begin{pmatrix} \alpha_{\nu'}\lambda v \\ \beta w^{(\nu')} \end{pmatrix} \geqslant \begin{pmatrix} \alpha_{\nu'}\beta v \\ \beta w^{(\nu')} \end{pmatrix} = \beta u^{(\nu')}_{\alpha_{\nu'}},$$

$$f_{\mathcal{K}}(u^{(i)}_{\alpha_i}) \geqslant \begin{pmatrix} \alpha_i\lambda v \\ f_{\mathcal{K}}|_{S_0}(w^{(i)}) \end{pmatrix} = \begin{pmatrix} \alpha_i\lambda v \\ \beta w^{(i)} + w^{(i+1)} \end{pmatrix} \geqslant \begin{pmatrix} \alpha_i\beta v + \alpha_{i+1} v \\ \beta w^{(i)} + w^{(i+1)} \end{pmatrix} = \beta u^{(i)}_{\alpha_i} + u^{(i+1)}_{\alpha_{i+1}},$$

for $i = \nu' - 1, \dots, 1$. It follows that $(f^n_{\mathcal{K}}(u^{(1)}_{\alpha_1}))_i \succeq n^{k-1}\beta^n$ for $i \in S'_k, k \geqslant 1$, and the lower bound is proved for i in $S \setminus S_0$ and $S_0 \setminus S'_0$. We can now do the same for the states in S'_0. $\square$

Corollary 12. *The following conditions are equivalent*:

(a) *Function $f_{\mathcal{K}}$ has a strictly positive eigenvector*.
(b) *Some (every) $\preceq$-minimal matrix has a strictly positive eigenvector*.
(c) $(f^n_{\mathcal{K}}(v))_i \sim \lambda^n$ *for all i and for some (every) vector* $v > 0$.
(d) $(f^n_{\mathcal{K}}(v))_i \sim (f^n_{\mathcal{K}}(v))_j$ *for all i, j and for some (every) vector* $v > 0$.

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