Journal of Combinatorial Theory, Series A **80**, 295–308 (1997)
Article No. TA972827

Multivariable Lagrange Inversion, Gessel-Viennot Cancellation, and the Matrix Tree Theorem

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Communicated by the Managing Editors

Received January 17, 1996

A new form of multivariable Lagrange inversion is given, with determinants

binatorial proof is given by considering functional digraphs, in which one of the principal minors is interpreted as a Matrix Tree determinant, and the other by a form of Gessel-Viennot cancellation.

1. INTRODUCTION

Multivariable Lagrange inversion formulas give a formal power series solution $\mathbf{f} = (f_1, ..., f_m)$ to the system of functional equations, for fixed positive integer m,

$$f_i = x_i g_i(\mathbf{f}), \qquad i = 1, ..., m, \tag{1}$$

where g_i is a formal power series with invertible constant, $i = 1, ..., m$. Gessel [5] gives an excellent account of such formulas, and we refer the reader to this paper for background; where possible, our notation will be consistent with Gessel's. We use $[m] = \{1, ..., m\}$, $\mathbf{g} = (g_1, ..., g_m)$, $\mathbf{x} = (x_1, ..., x_m)$, $\lambda = (\lambda_1, ..., \lambda_m)$ and $\mathbf{n} = (n_1, ..., n_m)$, $\mathbf{x}^\mathbf{n} = x_1^{n_1} \cdots x_m^{n_m}$, $\mathbf{n}! = n_1! \cdots n_m!$for $n_1, ..., n_m$ nonnegative integers. We also write $\mathbf{1} = (1, ..., 1)$, the vector with m 1's, $\mathbf{n} \geqslant \mathbf{1}$ when $n_i \geqslant 1$, $i = 1, ..., m$, and $[A]\, B$ to denote the coefficient of A in B. For $\alpha \subseteq [m]$, $\bar{\alpha}$ denotes the complement of α and $\det(a_{ij})_\alpha$ is the determinant of the submatrix of $(a_{ij})_{m \times m}$ with row and column indices in α.

In this paper we give a combinatorial proof of the following *principal minor* form of multivariable Lagrange inversion, which seems to be new.

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0097-3165/97 $25.00

THEOREM 1.1 (Principal Minor Lagrange Inversion). *Suppose* **f**, **g** *are specified by* (1) *above. Then for* $\alpha \subseteq [m]$, $\mathbf{n} \geqslant \mathbf{1}$ *and formal power series* Φ *we have*

$$\left[\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!}\right] \Phi(\mathbf{f}(\mathbf{x})) \det\left(x_j \frac{\partial f_i}{\partial x_j}\right)_{\alpha}$$
$$= \left[\frac{\lambda^{\mathbf{n}}}{\mathbf{n}!}\right] \Phi(\lambda)\, \mathbf{g}^{\mathbf{n}}(\lambda) \left(\prod_{k \in \alpha} \lambda_k\right) \det\left(\delta_{ij} - \frac{\lambda_j}{g_i} \frac{\partial g_i(\lambda)}{\partial \lambda_j}\right)_{\bar{\alpha}}$$

The best known forms of multivariable Lagrange inversion are the extreme cases $\alpha = \emptyset$, $[m]$ of principal minor Lagrange inversion. These forms are given in the first and second parts of the following result, and we shall refer to them as the *explicit* and *implicit* forms, respectively. In fact, we prove in the following result that principal minor Lagrange inversion and these extreme cases are in fact all equivalent, using Jacobi's theorem for the principal minors of a matrix and its inverse.

COROLLARY 1.1. *Theorem* 1.1 *is equivalent to each of the following*:

(1)

$$\left[\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!}\right] \Phi(\mathbf{f}(\mathbf{x})) = \left[\frac{\lambda^{\mathbf{n}}}{\mathbf{n}!}\right] \Phi(\lambda)\, \mathbf{g}^{\mathbf{n}}(\lambda) \det\left(\delta_{ij} - \frac{\lambda_j}{g_i} \frac{\partial g_i(\lambda)}{\partial \lambda_j}\right)_{m \times m}$$

(2)

$$\left[\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!}\right] \frac{\Phi(\mathbf{f}(\mathbf{x}))}{\det\left(\delta_{ij} - x_j \frac{\partial}{\partial f_j} g_i(\mathbf{f})\right)_{m \times m}} = \left[\frac{\lambda^{\mathbf{n}}}{\mathbf{n}!}\right] \Phi(\lambda)\, \mathbf{g}^{\mathbf{n}}(\lambda)$$

Proof. Theorem 1.1 *implies* (1): This is immediate by taking $\alpha = \emptyset$ in Theorem 1.1.

(1) *implies Theorem* 1.1: Differentiating the functional equations (1) by x_j for all $j = 1, ..., m$ and rearranging, leads to the matrix equation

$$\left(\delta_{ij} - x_i \frac{\partial g_i(\mathbf{f})}{\partial f_j}\right)_{m \times m} \left(\frac{1}{g_j(\mathbf{f})} \frac{\partial f_i}{\partial x_j}\right)_{m \times m} = I_m. \tag{2}$$

Now a result of Jacobi says that if $AB = I$ then $\det(a_{ij})_{\bar{\alpha}} / \det(a_{ij})_{m \times m} = \det(b_{ij})_{\alpha}$, for any $\alpha \subseteq [m]$ (see, e.g. [8], p. 21). Applying this result to Eq. (2), and substituting (1) gives

$$\left(\prod_{k\in\alpha} f_k\right)\det\left(\delta_{ij}-\frac{f_i}{g_i(\mathbf{f})}\frac{\partial g_i(\mathbf{f})}{\partial f_j}\right)_{\bar{\alpha}}\Big/\det\left(\delta_{ij}-\frac{f_i}{g_i(\mathbf{f})}\frac{\partial g_i(\mathbf{f})}{\partial f_j}\right)_{m\times m}$$

$$=\det\left(x_j\frac{\partial f_i}{\partial x_j}\right)_{\alpha}. \tag{3}$$

Now replace $\Phi(\mathbf{f})$ in (1) of this result by $\Phi(\mathbf{f})$ times the expression on either side of (3) to obtain Theorem 1.1 (the f_j arises by dividing the rows by f_i and multiplying the columns by f_j, to leave the determinant unchanged).

Finally, the equivalence of Theorem 1.1 and (2) of this result is obtained by using the determinantal identity corresponding to the choice $\alpha=[m]$ in (3) above. ∎

Gessel [5] gave a combinatorial proof of the implicit form of multivariable Lagrange inversion. The formal power series were interpreted as generating series for functional digraphs, and the determinant on the LHS was treated via the Jacobi identity *det = exp trace log*. Ehrenborg and Méndez [4] gave a combinatorial proof of the explicit form, counting functions in the context of coloured species, with the determinant on the RHS treated via a sign-reversing involution.

In this paper we use Gessel's context of functional digraphs. For an arbitrary functional digraph we shall introduce a graphical substructure called a *colour digraph*, and for functional digraphs that are arborescences we consider another graphical substructure called a *path arborescence*. In Section 2 we establish a combinatorial correspondence between functional digraphs and arborescences such that the colour digraph of the functional digraph is equal to the path arborescence of the arborescence. This correspondence, given as Theorem 2.1, is called the *arborescence substructure bijection*. In Section 3 we prove that the RHS of principal minor Lagrange inversion is the generating function for functional digraphs with certain restrictions on the colour digraph, by interpreting the minor that arises as a Matrix Tree determinant [2]. In Section 4 we prove that the LHS of principal minor Lagrange inversion is the generating function for arborescences with exactly the same restrictions on the path arborescence, in this case by interpreting the minor that arises as a Gessel–Viennot determinant [6]. These pieces are put together in the following way to give the combinatorial proof of principal minor Lagrange inversion.

Combinatorial proof of Theorem 1.1. It follows immediately by combining Theorems 2.1, 3.1 and 4.1: First equate the RHS's of Theorems 3.1 and 4.1. Then multiply by $n_{k+1}\cdots n_m$, and replace $(\partial/\partial f_{k+1})\cdots(\partial/\partial f_m)\,F(\mathbf{f})$ by $\Phi(\mathbf{f})$. This gives the result for α of the form $[k]$, and the result for arbitrary α follows by reindexing. ∎

Note that this proof specializes for the extreme cases $\alpha = \emptyset$, $[m]$ to show that multivariable Lagrange inversion can be deduced from the arborescence substructure bijection and the Matrix Tree Theorem alone in the explicit form, and from the arborescence substructure bijection and Gessel-Viennot cancellation alone in the implicit form.

In the case $m = 1$ our combinatorial proof of the explicit form specializes essentially to Labelle's [9] combinatorial proof of Lagrange inversion in one variable. Note that an arbitrary minor form of Lagrange inversion can also be obtained by the method of proof of Corollary 1.1, but we know of no combinatorial proof of this arbitrary minor result; the Matrix Tree portion of our combinatorial proof can presumably be extended in this case by the various combinatorial interpretations of the arbitrary minor of the Matrix Tree determinant that have appeared (see, e.g., [3], [1]).

The basic combinatorial objects we consider are now described. Let $V(\mathbf{n}) = \{(i, j) : 1 \leqslant j \leqslant n_i,\ 1 \leqslant i \leqslant m\}$, where $m, n_1, ..., n_m$ are positive integers. The element (i, j) in $V(\mathbf{n})$ is said to have *colour* i and *label* j. We denote (i, n_i) by M_i, for $i = 1, ..., m$. Let $V_0(\mathbf{n}) = \{0\} \cup V(\mathbf{n})$, and 0 is said to have colour 0. Let $\mathscr{F}_0(\mathbf{n})$ be the set of functional digraphs of functions from $V(\mathbf{n})$ to $V_0(\mathbf{n})$, and let $\mathscr{A}_0(\mathbf{n}) \subset \mathscr{F}_0(\mathbf{n})$ be the set of arborescences in $\mathscr{F}_0(\mathbf{n})$ (so they must be indirected at 0). A functional digraph in $\mathscr{F}_0(\mathbf{n})$ always has one component that is an arborescence indirected at 0. Any other components consist of a directed cycle of vertices at which arborescences are indirected; these other components are called *cyclic components*. For $D \in \mathscr{F}_0(\mathbf{n})$ we will identify the digraph and the function that it specifies where convenient, for example using $D(u) = v$ for the functional value and $(u, v) \in D$ for the directed edge in the digraph interchangeably.

We also consider the set $\mathscr{A}_m$ of arborescences on $\{0\} \cup [m]$, indirected at 0, and the set $\mathscr{F}_m$ of functional digraphs of functions from $[m]$ to $\{0\} \cup [m]$.

For $D \in \mathscr{F}_0(\mathbf{n})$ and each vertex $v \in V_0(\mathbf{n})$ of D, define $wt_D(v)$ as follows. Suppose j_ℓ elements of colour ℓ are mapped to v for $\ell = 1, ..., m$; if v is of colour i, for some $i = 1, ..., m$, then $wt_D(v) = g_{i,\mathbf{j}}$, and if v is of colour 0 then $wt_D(v) = H_{\mathbf{j}}$, where $\mathbf{j} = (j_1, ..., j_m)$ and $g_{i,\mathbf{j}}$, $H_{\mathbf{j}}$ are indeterminates. Then we define as a weight function the combinatorial monomial associated with D,

$$\Psi(D) = \prod_{v \in V_0(\mathbf{n})} wt_D(v).$$

Now if we let

$$g_i(\mathbf{x}) = \sum_{\mathbf{j} \geqslant \mathbf{0}} g_{i,\mathbf{j}} \frac{\mathbf{x}^{\mathbf{j}}}{\mathbf{j}!}, \qquad i = 1, ..., m,$$

$$H(\mathbf{x}) = \sum_{\mathbf{j} \geqslant \mathbf{0}} H_{\mathbf{j}} \frac{\mathbf{x}^{j}}{\mathbf{j}!}$$

then the solution $\mathbf{f}=(f_1, ..., f_m)$ to the functional equations (1) has a straightforward combinatorial interpretation: f_i is the generating function for arborescences on vertices of colour $1, ..., m$, indirected at a vertex of colour i, with x_j an exponential marker for the (labelled) vertices of colour j, $j=1, ..., m$, with respect to weight Ψ. This combinatorial connection with multivariable Lagrange inversion is the basis of the proof that we develop for principal minor Lagrange inversion.

2. PATH ARBORESCENCES AND COLOUR DIGRAPHS

We begin with a combinatorial construction on arborescences.

CONSTRUCTION 2.1. *Given an arborescence $A \in \mathscr{A}_0(\mathbf{n})$, carry out the following for $j=1, ..., m$:*

- *Let $\mu=\max([m]-\{\sigma_1, ..., \sigma_{j-1}\})$.*
- *Find the dipath ρ in A from M_μ to the first vertex x whose colour is in $\{0\} \cup \{\sigma_1, ..., \sigma_{j-1}\}$.*
- *On ρ let the previous vertex to x be y, and let the vertex of the same colour as y that is closest to M_μ (or is equal to M_μ) be z. (Thus y can equal z, and z can equal M_μ when y is of colour μ).*
- *Define σ_j to be the colour of vertex y (and z), and let $K_{\sigma_j}=x$, $F_{\sigma_j}=z$. Let π_{σ_j} be the dipath in ρ from z to y inclusive.*

Before considering the properties of this construction, we give an example.

EXAMPLE 2.1. For the arborescence E given in Fig. 1, with $m=4$, $n_1=\cdots=n_4=5$, Construction 2.1 yields:

For $j=1$, $\mu=4$, $M_4=(4, 5)$, $x=0$, $y=(2, 4)$, $z=(2, 2)$, so $\sigma_1=2$, $K_2=0$, $F_2=(2, 2)$, and $\pi_2=((2, 2), (2, 1), (3, 4), (1, 2), (2, 3), (1, 1), (2, 4))$, where the dipath is specified by listing its vertices, in order.

For $j=2$, $\mu=4$, $M_4=(4, 5)$, $x=(2, 2)$, $y=(4, 4)$, $z=(4, 5)$, so $\sigma_2=4$, $K_4=(2, 2)$, $F_4=(4, 5)$, and $\pi_4=((4, 5), (4, 3), (1, 4), (4, 4))$.

For $j=3$, $\mu=3$, $M_3=(3, 5)$, $x=(2, 3)$, $y=(1, 2)$, $z=(1, 5)$, so $\sigma_3=1$, $K_1=(2, 3)$, $F_1=(1, 5)$, and $\pi_1=((1, 5), (1, 3), (3, 4), (1, 2))$.

For $j=4$, $\mu=3$, $M_3=(3, 5)$, $x=(1, 5)$, $y=(3, 3)$, $z=(3, 5)$, so $\sigma_4=3$, $K_3=(1, 5)$, $F_3=(3, 5)$, and $\pi_3=((3, 5), (3, 3))$.

In Construction 2.1 it is clear that $\sigma=\sigma_1 \cdots \sigma_m$ is a permutation of $[m]$. For example, for the arborescence E above we obtain $\sigma=2413$. Now define $P(A) \in \mathscr{F}_m$ to be the digraph with edges directed from σ_j to the colour of K_{σ_j}, for $j=1, ..., m$. For Example $P(E)$ is given in Fig. 1 for the

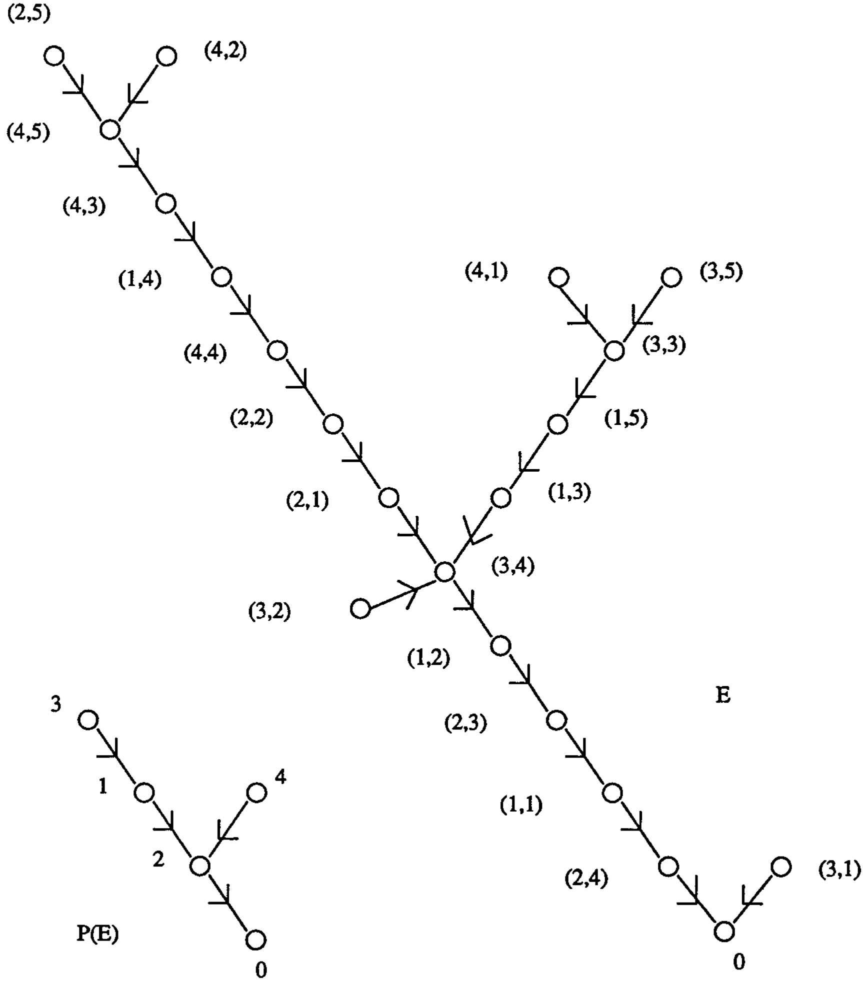

Fig. 1. An arborescence indirected at 0 and its path arborescence.

arborescence E above. Let $P^{(j)}(A)$ be the subgraph of $P(A)$ induced by vertices $\{0\} \cup \{\sigma_1, ..., \sigma_j\}$, for $j=0, 1, ..., m$. In particular $P^{(0)}(A)$ is the single vertex $\{0\}$, and $P^{(m)}(A) = P(A)$.

Various properties of the construction are recorded next, including an appealing combinatorial relationship between σ and $P(A)$, expressed in terms of a subset of elements of the permutation σ called *rlmax* elements. These are σ_i such that $\sigma_i = \max\{\sigma_i, ..., \sigma_m\}$.

PROPOSITION 2.1. *If Construction* 2.1 *is applied to an arborescence* $A \in \mathscr{A}_0(\mathbf{m})$, *then, for* $j = 1, ..., m$,

(1) *if* σ_j *is an rlmax in* σ *then* $F_{\sigma_j} = M_{\sigma_j}$, *otherwise if* σ_j *is not an rlmax in* σ *then* $F_{\sigma_j} = K_{\sigma_{j+1}}$,

(2) $P^{(j)}(A)$ *is an arborescence indirected at* 0, *and the smallest leaf in* $P^{(j)}(A)$ *is* σ_j,

(3) *the dipath* π_{σ_j} *contains no vertex of colour in* $\{0\} \cup \{\sigma_1, ..., \sigma_{j-1}\}$.

Proof. (1) For μ chosen at step j of the construction, if $\sigma_j = \mu$ then σ_j is an rlmax in σ since $\mu = \max\{\sigma_j, \sigma_{j+1}, ..., \sigma_m\}$, and in this case we must choose $F_{\sigma_j} = M_{\sigma_j}$. Otherwise, if $\sigma_j \neq \mu$ then σ_j is not an rlmax in σ, and in this case μ will be unchanged at step $j+1$, which forces the choice $K_{\sigma_{j+1}} = F_{\sigma_j}$.

(2) By construction, the colour of K_{σ_j} lies in $\{0\} \cup \{\sigma_1, ..., \sigma_{j-1}\}$ for $j = 1, ..., m$, so it follows by induction that $P^{(j)}(A)$ is an arborescence, with σ_j as a leaf. Now if σ_{j-1} is not an rlmax in σ, then from (1) of this result, we have $K_{\sigma_j} = F_{\sigma_{j-1}}$, so the colour of K_{σ_j} equals σ_{j-1} (since F_{σ_j} is, by construction, of colour σ_j for all $j = 1, ..., m$). Thus, in this case, σ_{j-1} is not a leaf in $P^{(j)}(A)$. It follows by induction that any $\sigma_1, ..., \sigma_{j-1}$ that is not an rlmax in σ cannot be a leaf in $P^{(j)}(A)$. But any $\sigma_1, ..., \sigma_{j-1}$ that is an rlmax in σ must be larger than σ_j, so σ_j is the smallest leaf in σ.

(3) This is immediate from the construction. ∎

In view of (2) of this result, we call $P(A) = P^{(m)}(A)$ the *path arborescence* of A.

Now given a functional digraph $D \in \mathscr{F}_0(\mathbf{n})$ suppose the colour of $D(M_j)$ is c_j for $j = 1, ..., m$. Then the digraph in $\mathscr{F}_m$ with edges directed from j to c_j, for $j = 1, ..., m$ is called the *colour digraph* of D, denoted $C(D)$. Of course, in general $C(D)$ is not an arborescence.

The main result of this section is a bijection, given as Theorem 2.1 below, between arborescences A and functional digraphs D, in the case that $C(D)$ is an arborescence; in fact the bijection is such that $C(D) = P(A)$. For convenience, we first consider separately two combinatorial operations on functional digraphs.

For $u, v \in V(\mathbf{n})$ of the same colour, and $D \in \mathscr{F}_0(\mathbf{n})$, suppose $D' \in \mathscr{F}_0(\mathbf{n})$ is given by $D'(u) = D(v)$, $D'(v) = D(u)$, and $D'(x) = D(x)$ for all other $x \in V(\mathbf{n})$. Then we say that D has been *switched* at u and v to obtain D'; we do not insist that u and v are distinct, though when $u = v$ we simply obtain $D' = D$. Clearly $\Psi(D') = \Psi(D)$, since u and v are of the same colour, and we thus say that switching is *Ψ-preserving*.

Next consider $u, v \in V(\mathbf{n})$ of the same colour, say c, and $D \in \mathscr{F}_0(\mathbf{n})$ such that D contains a directed path π from u to v, with u, v distinct. Now move back along the dipath from v to $D(u)$ and find all vertices of colour c whose label is largest among all vertices of colour c encountered to that point. Call these vertices $w_1, ..., w_k$ in order, beginning with $w_1 = v$ so $k \geqslant 1$. (In fact the labels of $w_1, ..., w_k$ are rlmax in the sequence of labels of the vertices of colour c on the dipath.) Suppose $D' \in \mathscr{F}_0(\mathbf{n})$ is given by $D'(u) = D(v)$, $D'(w_k) = D(u)$, and $D'(w_i) = D(w_{i+1})$, for $i = 1, ..., k-1$, with $D'(x) = D(x)$ for all other $x \in V(\mathbf{n})$. Then we say that the dipath π has been *c-peeled* from D to obtain D'. If $u = v$, then we define $D' = D$. In both cases $\Psi(D') = \Psi(D)$, so peeling is also Ψ-preserving.

We can now give the main result of this section, involving a bijection with a sequence of steps, alternately peeling and switching.

THEOREM 2.1 (Arborescence Substructure Bijection). *For each* $\mathbf{n} \geqslant \mathbf{1}$ *and* $T \in \mathscr{A}_m$,

$$\sum_{\begin{cases} A \in \mathscr{A}_0(\mathbf{n}) \\ P(A) = T \end{cases}} \Psi(A) = \sum_{\begin{cases} D \in \mathscr{F}_0(\mathbf{n}) \\ C(D) = T \end{cases}} \Psi(D)$$

Proof. Consider $A \in \mathscr{A}_0(\mathbf{n})$ and apply Construction 2.1 to obtain, say, $P(A) = T$. Now start with A, considered as a functional digraph, and perform the following pair of operations in succession, for $j = 1, ..., m$:

(a) σ_j-peel π_{σ_j},

(b) switch at M_{σ_j} and F_{σ_j}.

Our claim is that, after these $2m$ operations are performed, A has been transformed to a functional digraph $D \in \mathscr{F}_0(\mathbf{n})$, such that $C(D) = T$ and $\Psi(A) = \Psi(D)$, and moreover that this is a bijection, which would establish the result. (Note that these operations may be fixed points, when π_{σ_j} is a single vertex, and $M_{\sigma_j} = F_{\sigma_j}$, respectively.) To prove our claim, first note that at stage j, the only vertices whose functional values are changed are of colour σ_j (we call this the *disjoint colour property* of the mapping). Thus by Proposition 2.1(3) the dipath π_{σ_j} is not affected by the operations at stages $1, ..., j-1$, so the mapping is well-defined, and since peeling and switching are both Ψ-preserving, we immediately obtain $\Psi(A) = \Psi(D)$. Second, at stage j, after operation (a) the functional value of F_{σ_j} is K_{σ_j}, and hence after operation (b) the functional value of M_{σ_j} is K_{σ_j}; thus from the disjoint colour property we conclude that

$$D(M_{\sigma_j}) = K_{\sigma_j}, \qquad \text{for} \quad j = 1, ..., m, \tag{$*$}$$

so that $C(D) = T$.

To establish that this is a bijection, we now show that it is reversible, by considering $D \in \mathscr{F}_0(\mathbf{n})$ with, say, $C(D) = T$. Thus we begin by setting $P(A) = T$, and by successively removing the smallest leaves from T we recover σ, by Proposition 2.1(2). Then determine K_{σ_j} and F_{σ_j} for each σ_j, $j = 1, ..., m$ by (*) and Proposition 2.1(1), respectively (note that M_{σ_j} is the fixed element of colour with the largest label). Moving backwards, as j is reduced from m to 1, clearly operation (b) is now reversible at each stage, with the result that the functional value of F_{σ_j} is K_{σ_j}. To reverse operation (a), find the cyclic components in the functional digraph with at least one vertex of colour σ_j on the cycle, but no vertices of colour $0, \sigma_1, ..., \sigma_{j-1}$. The σ_j-peeling is now easily reversed by cutting each of these cycles after their largest element of colour σ_j, and placing them between F_{σ_j} and K_{σ_j} by decreasing order of these largest elements (this works because, moving forward, every cycle introduced at stage j must have an element of colour σ_j on it, by the disjoint colour property; also by Proposition 2.1(3), the cycles introduced in (a) have no vertices of colour $0, \sigma_1, ..., \sigma_{j-1}$ on them). ∎

As an example of this bijection, the arborescence E given in Fig. 1 and considered in Example 2.1 corresponds to the functional digraph given in Fig. 2. Note that, in this case, $M_i = (i, 5)$, for $i = 1, 2, 3, 4$, so the colour digraph is indeed equal to $P(E)$, given in Fig. 1.

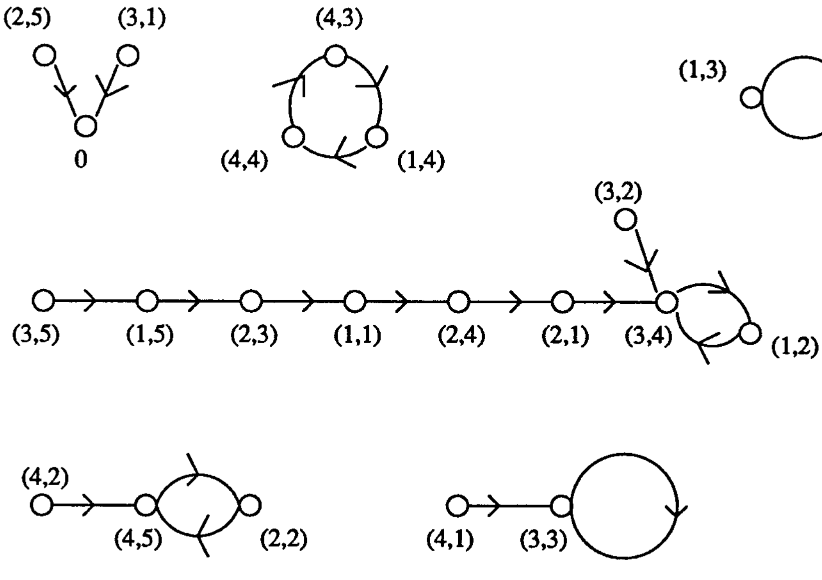

Fig. 2. The functional digraph corresponding to arborescence E.

3. THE MATRIX TREE THEOREM AND COLOUR DIGRAPHS

In this section we consider the RHS of the arborescence substructure bijection, and begin by expressing it directly in terms of the generating series H and $g_1, ..., g_m$, in a form reminiscent of the RHS of the various multivariable Lagrange inversion formulas given in the Introduction. In this result, for $T \in \mathscr{F}_m$, $S_\ell(T)$ denotes the set of vertices i such that $(i, \ell) \in T$, for each $\ell = 0, 1, ..., m$.

LEMMA 3.1.

$$\sum_{\begin{cases} D \in \mathscr{F}_0(\mathbf{n}) \\ C(D) = T \end{cases}} \Psi(D) = \left[\frac{\lambda^{\mathbf{n}-\mathbf{1}}}{(\mathbf{n}-\mathbf{1})!}\right] \left\{ \left(\prod_{i \in S_0(T)} \frac{\partial}{\partial \lambda_i} \right) H(\lambda) \right\} \times \prod_{\ell=1}^{m} \left\{ \left(\prod_{i \in S_\ell(T)} \frac{\partial}{\partial \lambda_i} \right) g_\ell^{n_\ell}(\lambda) \right\}$$

Proof. The summation of the left side of this result is the generating function for all functional digraphs with colour digraph T, with the preimage of each element marked by wt_D.

Now, without considering the colour digraph, the generating function for each element of colour ℓ is $g_\ell(\lambda)$, where λ_j is an exponential marker for the elements of colour j in the preimage, so the generating function for the n_ℓ elements of colour ℓ is $g_\ell^{n_\ell}(\lambda)$. But if $C(D) = T$, then the elements M_i for $i \in S_\ell(T)$ must appear in the preimages of the n_ℓ elements of colour ℓ, and the generating function for this is

$$\left(\prod_{i \in S_\ell(T)} \frac{\partial}{\partial \lambda_i} \right) g_\ell^{n_\ell}(\lambda), \qquad \ell = 1, ..., m.$$

Similarly the generating function of the preimage of element 0, with the restriction that M_i must appear, for $i \in S_0(T)$, is

$$\left(\prod_{i \in S_0(T)} \frac{\partial}{\partial \lambda_i} \right) H(\lambda).$$

We multiply these generating functions together and take the coefficient of $\lambda^{\mathbf{n}-\mathbf{1}}/(\mathbf{n}-\mathbf{1})!$, since the location of $M_1, ..., M_m$ has been fixed by applying $\partial/\partial\lambda_1, ..., \partial/\partial\lambda_m$, and thus there remains only $n_\ell - 1$ labelled elements of colour ℓ to be distributed among the preimages, for $\ell = 1, ..., m$. This gives the required result. ∎

Now we use the Matrix Tree Theorem in the directed case to sum the above result over all T in $\mathscr{T}^{(k)}$, where $\mathscr{T}^{(k)} \subseteq \mathscr{A}_m$ consists of those arborescences containing edges $(k+1, 0), \ldots, (m, 0)$, for each fixed $k = 0, 1, \ldots, m$. (The adjacency matrix of the Matrix Tree Theorem appears in a differential theoretic manner.) This produces, in the next result, an expression in terms of $H, g_1, \ldots, g_m$ that leads directly to the RHS of principal minor Lagrange inversion.

THEOREM 3.1.

$$\sum_{\left\{\substack{D \in \mathscr{F}_0(\mathbf{n}) \\ C(D) \in \mathscr{T}^{(k)}}\right.} \Psi(D) = \frac{1}{n_{k+1} \cdots n_m} \left[\frac{\lambda^{\mathbf{n}}}{\mathbf{n}!}\right] \lambda_{k+1} \cdots \lambda_m \left(\frac{\partial}{\partial \lambda_{k+1}} \cdots \frac{\partial}{\partial \lambda_m} H(\lambda)\right)$$

$$\times \mathbf{g}^{\mathbf{n}}(\lambda) \det\left(\delta_{ij} - \frac{\lambda_j}{g_i} \frac{\partial g_i(\lambda)}{\partial \lambda_j}\right)_{[k]}$$

Proof. The Matrix Tree Theorem [2] for indirected arborescences gives

$$\sum_{T \in \mathscr{A}_m} \prod_{(i, j) \in T} b_{ij} = \det\left(\delta_{ij}\left(b_{i0} + \sum_{\alpha=1}^{m} b_{i\alpha}\right) - b_{ij}\right)_{m \times m}$$

from which we immediately obtain

$$\sum_{T \in \mathscr{T}^{(k)}} \prod_{(i, j) \in T} b_{ij} = b_{k+1\,0} \cdots b_{m0} \det\left(\delta_{ij}\left(b_{i0} + \sum_{\alpha=1}^{m} b_{i\alpha}\right) - b_{ij}\right)_{[k]} \tag{4}$$

Now for T in Lemma 3.1 we have $(i, \ell) \in T$ meaning that $i \in S_\ell(T)$, so we apply result (4) to Lemma 3.1 with b_{ij} identified with $(\partial/\partial\lambda_i)\, g_j^{n_j}(\lambda)$ for $j = 1, \ldots, m$, and b_{i0} identified with $(\partial/\partial\lambda_i)\, H(\lambda)$, and products $b_{ij}\, b_{\ell j}$ identified with $(\partial/\partial\lambda_i)(\partial/\partial\lambda_\ell)\, g_j^{n_j}(\lambda)$ for $j \neq 0$, and $(\partial/\partial\lambda_i)(\partial/\partial\lambda_\ell)\, H(\lambda)$ for $j = 0$. Under this identification, the effect of the diagonal term $b_{i0} + \sum_{\alpha=1}^{m} b_{i\alpha}$ is, by the product rule, to simply apply $\partial/\partial\lambda_i$ to $H(\lambda) \prod_{\ell=1}^{m} g_\ell^{n_\ell}(\lambda)$, and this differential operator can be replaced by multiplying by n_i/λ_i, since we are taking the coefficient of $\lambda^{n_i - 1}/(n_i - 1)!$ Thus

$$\sum_{\left\{\substack{D \in \mathscr{F}_0(\mathbf{n}) \\ C(D) \in \mathscr{T}^{(k)}}\right.} \Psi(D) = \left[\frac{\lambda^{\mathbf{n}-\mathbf{1}}}{(\mathbf{n}-\mathbf{1})!}\right] \left\{\left(\prod_{i=k+1}^{m} \frac{\partial}{\partial \lambda_i}\right) H(\lambda)\right\}$$

$$\times \mathbf{g}^{\mathbf{n}}(\lambda) \det\left(\delta_{ij} \frac{n_i}{\lambda_i} - \frac{1}{g_j^{n_j}(\lambda)} \frac{\partial}{\partial \lambda_i} g_j^{n_j}(\lambda)\right)_{[k]}$$

and the result follows. ∎

4. A GESSEL-VIENNOT DETERMINANT AND PATH ARBORESCENCES

In this section we consider the LHS of the arborescence substructure bijection. In this case we have been unable to find an analogue of Lemma 3.1, but by a cancellation argument similar to Gessel-Viennot cancellation [6] for lattice paths, we are able to obtain the analogue of Theorem 3.1. (The cancellation acts on a set of dipaths in an arborescence, one dipath for each colour, defined below.) This is given as Theorem 4.1, and produces an expression that leads directly to the LHS of principal minor Lagrange inversion.

We require the following notation. For an arborescence $A \in \mathscr{A}_0(\mathbf{n})$, find the dipath from M_i to 0 and let the previous vertex to 0 on this dipath be called $N_i(A)$; let $\tau_i(A)$ be the dipath from M_i to N_i, $i=1, ..., m$. For any j for which there is a vertex of colour j in $\tau_i(A)$, let $L_j(\tau_i)$ be the vertex of colour j in $\tau_i(A)$ that is closest to N_i (this will be N_i itself when it is of colour j), $j=1, ..., m$.

For example if E is the arborescence given in Fig. 1 and considered in Example 2.1, then $M_1=(1,5)$, so $N_1(E)=(2,4)$, $L_1(\tau_1)=(1,1)$, $L_2(\tau_1)=(2,4)$, $L_3(\tau_1)=(3,4)$, and $L_4(\tau_1)$ is not defined, since there is no vertex of colour 4 on the dipath τ_1 from $(4,5)$ to $(2,4)$ in E. We also have $N_2(E)=N_3(E)=N_4(E)=(2,4)$ in this example.

For fixed $k=0, 1, ..., m$, let $\mathscr{S}^{(k)}$ consist of arborescences $A \in \mathscr{A}_0(\mathbf{n})$ for which

- for $i=k+1, ..., m$, $\tau_i(A)$ are pairwise disjoint,
- for $i=k+1, ..., m$, $N_i(A)$ is of colour $\kappa_i(A)$, for some permutation κ of $[m]-[k]$.

(Thus, for example, E in Example 2.1 is not in $\mathscr{S}^{(k)}$ for any $k=0, ..., 3$, though $E \in \mathscr{S}^{(4)}$ trivially.) Let $\mathscr{R}^{(k)}$ be the subset of $\mathscr{S}^{(k)}$ in which $\tau_i(A)$ has no vertex of colour larger than i, for $i=k+1, ..., m$, and let $\mathscr{U}^{(k)}=\mathscr{S}^{(k)}-\mathscr{R}^{(k)}$. Note that for $A \in \mathscr{R}^{(k)}$, this condition forces $\kappa(A)$ to be the identity permutation on $[m]-[k]$.

THEOREM 4.1.

$$\sum_{\substack{A \in \mathscr{A}(\mathbf{n}) \\ P(A) \in \mathscr{T}^{(k)}}} \Psi(A) = \frac{1}{n_{k+1} \cdots n_m} \left[\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!} \right] \left\{ \frac{\partial}{\partial f_{k+1}} \cdots \frac{\partial}{\partial f_m} H(\mathbf{f}) \right\}$$

$$\times \det \left(x_j \frac{\partial f_i}{\partial x_j} \right)_{[m]-[k]}$$

Proof. First, from Construction 2.1, $\mathscr{R}^{(k)}$ is precisely the set of arborescences $A \in \mathscr{A}_0(\mathbf{n})$ such that $P(A) \in \mathscr{T}^{(k)}$, so

$$\sum_{\begin{cases} A \in \mathscr{A}_0(\mathbf{n}) \\ P(A) \in \mathscr{T}^{(k)} \end{cases}} \Psi(A) = \sum_{A \in \mathscr{R}^{(k)}} \Psi(A) = \sum_{A \in \mathscr{R}^{(k)}} sgn(\kappa(A))\, \Psi(A) \tag{5}$$

since $\kappa(A)$ must be the identity permutation for $A \in \mathscr{R}^{(k)}$.

Now we prove that

$$\sum_{A \in \mathscr{U}^{(k)}} sgn(\kappa(A))\, \Psi(A) = 0 \tag{6}$$

by considering the mapping ϕ on $\mathscr{U}^{(k)}$, where for $U \in \mathscr{U}^{(k)}$,

- let γ be the largest j such that $\tau_j(U)$ has a vertex of colour larger than j,
- let β be the largest colour of vertex appearing in $\tau_\gamma(U)$,
- switch U at $L_\beta(\tau_\gamma(U))$ and $L_\beta(\tau_\beta(U))$ to obtain $\phi(U)$.

If $\phi(U) = U'$, then applying ϕ to U' gives the same choices of γ, β, $L_\beta(\tau_\gamma)$ and $L_\beta(\tau_\beta)$, so ϕ is an involution. Moreover, $\kappa(U')$ is obtained from $\kappa(U)$ by applying the transposition (γ, β), and by construction $\beta > \gamma$, so $sgn(\kappa(U)) = -sgn(\kappa(U'))$, and ϕ is sign-reversing. Finally, $\Psi(U) = \Psi(U')$ since switching is Ψ-preserving, so ϕ is weight-preserving, and we have established (6) above.

Combining (5) and (6) gives

$$\sum_{\begin{cases} A \in \mathscr{A}_0(\mathbf{n}) \\ P(A) \in \mathscr{T}^{(k)} \end{cases}} \Psi(A) = \sum_{A \in \mathscr{S}^{(k)}} sgn(\kappa(A))\, \Psi(A) \tag{7}$$

But the arborescences $A \in \mathscr{S}^{(k)}$ with fixed choice of κ are arborescences indirected at 0 in which one neighbour of the root is a distinguished vertex of colour κ_j for each $j = k+1, ..., m$ (this vertex is N_{κ_j}). Thus, to construct such A, we place an arborescence indirected at N_{κ_j}, with the restriction that vertex M_j must appear in this arborescence, for $j = k+1, ..., m$. The contribution to the sum in (7) from these arborescences is

$$sgn(\kappa) \frac{1}{n_{k+1} \cdots n_m} \left[\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!} \right] \left\{ \frac{\partial}{\partial f_{k+1}} \cdots \frac{\partial}{\partial f_m} H(\mathbf{f}) \right\} x_{k+1} \cdots x_m \prod_{j=k+1}^{m} \left(\frac{\partial f_{\kappa_j}}{\partial x_j} \right)$$

and the result follows by summing this over all κ. ∎

ACKNOWLEDGMENT

This work was supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

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