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Elasticity of factorizations in integral domains

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Abstract

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For an atomic integral domain R, define $\rho(R) = \sup\{m/n \mid x_1 \cdots x_m = y_1 \cdots y_n$, each $x_i, y_j \in R$ is irreducible$\}$. We investigate $\rho(R)$, with emphasis for Krull domains R. When R is a Krull domain, we determine lower and upper bounds for $\rho(R)$; in particular, $\rho(R) \leq \max\{|\mathrm{Cl}(R)|/2, 1\}$. Moreover, we show that for any real number $r \geq 1$ or $r = \infty$, there is a Dedekind domain R with torsion class group such that $\rho(R) = r$.

Introduction

Let R be an integral domain with quotient field K. If R is a UFD, then any two factorizations of a nonzero nonunit of R into the product of irreducible elements have the same length. Of course, this is not true for an arbitrary atomic integral domain (an integral domain R is *atomic* if each nonzero nonunit is the product of irreducible elements (atoms)). Following Zaks [30], we define R to be a *half-factorial domain* (HFD) if R is atomic and whenever $x_1 \cdots x_m = y_1 \cdots y_n$ with each $x_i, y_j \in R$ irreducible, then $m = n$. A UFD is obviously a HFD, but the converse fails since any Krull domain R with divisor class group $\mathrm{Cl}(R) = \mathbb{Z}_2$ is a HFD [31], but not a UFD. In order to measure how far an atomic domain R is from being a HFD, we define $\rho(R) = \sup\{m/n \mid x_1 \cdots x_m = y_1 \cdots y_n$, each $x_i, y_j \in R$ is irreducible$\}$. Thus $1 \leq \rho(R) \leq \infty$, and $\rho(R) = 1$ if and only if R is a

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HFD. $\rho(R)$ is called the *elasticity* of R and was introduced by Valenza [29], who studied $\rho(R)$ for R the ring of integers in an algebraic number field. In particular, he showed that $\rho(R) \leq \max\{h/2, 1\}$, where R has class number h. In an earlier appearing (but later submitted) paper, Steffan [28] studied $\rho(R)$ (without this notation) for a Dedekind domain R with finite divisor class group and showed that $\rho(R) \leq \max\{|\mathrm{Cl}(R)|/2, 1\}$. The purpose of this paper is to study $\rho(R)$ for an arbitrary atomic domain R, but with emphasis on Krull domains. The impetus for much of this study of factorization properties goes back to the study of factorization in rings of algebraic integers, in particular, to the result of Carlitz [13] that the ring of integers in an algebraic number field is a HFD if and only if it has a class number ≤ 2. For other factorization properties and extensive references, see [2], [3], [14], [15], and [16].

As in [2], we define an atomic domain R to be a *bounded factorization domain* (BFD) if for each nonzero nonunit $x \in R$, there is a bound on the lengths of factorizations of x into the product of irreducible elements. A BFD R (even a Dedekind domain) may have $\rho(R) = \infty$; we thus define R to be a *rationally bounded factorization domain* (RBFD) if R is atomic and $\rho(R) < \infty$. For any integral domain R, we have

$$\text{UFD} \Rightarrow \text{HFD} \Rightarrow \text{RBFD} \Rightarrow \text{BFD} \Rightarrow \text{ACCP} \Rightarrow \text{atomic},$$

and none of the implications are reversible (cf. [2, 23]).

In Section 1, we introduce and study semi-length functions on R; these are functions $f : R^* \to \mathbb{R}_+$ which satisfy $f(xy) = f(x) + f(y)$ for all nonzero $x, y \in R$, and $f(x) = 0$ if and only if x is a unit. We completely determine all semi-length functions on R when R is a Krull domain with torsion divisor class group. In the second section, we use semi-length functions to determine lower and upper bounds for $\rho(R)$. We show that for any semi-length function f on an atomic domain R, $\rho(R) \leq M^*/m^*$, where $M^* = \sup\{f(x) \mid x \in R \text{ is irreducible, but not prime}\}$ and $m^* = \inf\{f(x) \mid x \in R \text{ is irreducible, but not prime}\}$. When R is a Krull domain, then $\rho(R) \leq \max\{|\mathrm{Cl}(R)|/2, 1\}$; so a Krull domain with finite divisor class group is a RBFD. We show that a one-dimensional local domain R is a RBFD if and only if R is analytically irreducible. We also find upper bounds for $\rho(R)$ for several classes of integral domains R weaker than Krull domains. In Section 3, we give examples to illustrate the techniques developed in Section 2. In particular, we show that for any real number $r \geq 1$ or $r = \infty$, there is a Dedekind domain R with torsion divisor class group such that $\rho(R) = r$. We also show that for each real number $r \geq 1$, there is a one-dimensional quasilocal atomic domain R with $\rho(R) = r + 1$.

As mentioned above, Steffan [28] and Valenza [29] have studied these ideas in the context of Dedekind domains and rings of algebraic integers, respectively. Recently, Chapman and Smith [16] have also studied $\rho(R)$ for R a Dedekind domain with torsion class group. The correct setting for $\rho(R)$ seems to be for

Krull domains. Many of the proofs for Dedekind domains carry over for Krull domains by replacing the unique factorization of a principal ideal as a product of maximal ideals in a Dedekind domain by its unique factorization as a v-product of height-one prime ideals in a Krull domain. Also, many of the proofs for rings of algebraic integers carry over to Krull domains for which each (nonzero) divisor class contains a prime ideal. Although we do have several results for arbitrary atomic domains, Krull domains are the easiest to work with. In this case, we can relate $\rho(R)$ to properties of the divisor class group $\mathrm{Cl}(R)$ of R. As we will see, $\rho(R)$ depends not only on the group-theoretic properties of $\mathrm{Cl}(R)$, but also on the distribution of the height-one prime ideals in the divisor classes.

General references for any undefined terminology or notation are [19], [21], and [26]. Throughout, R will be an integral domain with (proper) quotient field K, and $\bar{R}$, R^*, and $U(R)$ will denote respectively its integral closure, set of nonzero elements, and group of units. Our general reference for Krull domains will be [19]. If R is a Krull domain, we denote its set of height-one prime ideals by $X(R) = X^{(1)}(R)$, its divisor class group (written additively) by $\mathrm{Cl}(R)$, and the class of $P \in X(R)$ in $\mathrm{Cl}(R)$ by $[P]$. We will use repeatedly the fundamental fact that in a Krull domain R, a nonunit $x \in R^*$ is irreducible if and only if in its v-factorization $xR = (P_1 \cdots P_n)_v$ with each $P_i \in X(R)$, no proper subproduct $(P_{i_1} \cdots P_{i_m})_v$ is principal.

For any partially-ordered abelian group G, G_+ is its submonoid of nonnegative elements. The group of divisibility of R is the abelian group $G(R) = K^*/U(R)$, written additively, and partially ordered by $aU(R) \leq bU(R)$ if and only if $ba^{-1} \in R$. Several examples involve monoid domain constructions; a good reference for monoid domains is [22]. As usual, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ denote respectively the integers, rational numbers, and real numbers.

1. Length functions

If R is a UFD, or more generally a HFD, define $s_R(x)$ to be the length of a(ny) factorization of a nonunit $x \in R^*$ into a product of irreducible elements and $s_R(x) = 0$ if $x \in U(R)$. This defines a function $s_R : R^* \to \mathbb{Z}_+$ such that (i) $s_R(xy) = s_R(x) + s_R(y)$ for all $x, y \in R^*$, (ii) $s_R(x) = 0$ if and only if $x \in U(R)$, and (iii) $s_R(x) = 1$ if and only if x is irreducible. Moreover, R is a HFD if such a length function exists [31, Lemma 1.3]. As in [10], for a nonzero nonunit x in an atomic domain R, we define $l_R(x) = \inf\{n \mid x = x_1 \cdots x_n,$ each $x_i \in R$ is irreducible$\}$ and $L_R(x) = \sup\{n \mid x = x_1 \cdots x_n,$ each $x_i \in R$ is irreducible$\}$. Thus $1 \leq l_R(x) \leq L_R(x) \leq \infty$ for each nonunit $x \in R^*$. We also define $l_R(x) = L_R(x) = 0$ when $x \in U(R)$. Although $l_R(x)$ is always finite, we may have $L_R(x) = \infty$ (note that R is a BFD if and only if $L_R(x) < \infty$ for all $x \in R^*$). These define functions $l_R : R^* \to \mathbb{Z}_+$ and $L_R : R^* \to \mathbb{Z}_+ \cup \{\infty\}$ which satisfy (ii) and (iii). Note that (i) holds (for either l_R or L_R) precisely when R is a HFD, and in this case

$l_R = L_R = s_R$. However, it is easily verified that $l_R(xy) \le l_R(x) + l_R(y)$ and $L_R(xy) \ge L_R(x) + L_R(y)$ for all $x, y \in R^*$. For $x \in R^*$, we define $\rho_R(x) = L_R(x)/l_R(x)$ if x is a nonunit and $\rho_R(x) = 1$ if $x \in U(R)$. Thus $\rho(R) = \sup\{\rho_R(x) \mid x \in R^*\}$.

We will call a function $f : R^* \to \mathbb{Z}_+$ a *length function* on R if it satisfies (i) $f(xy) = f(x) + f(y)$ for all $x, y \in R^*$ and (ii) $f(x) = 0$ if and only if $x \in U(R)$. If $f : R^* \to \mathbb{R}_+$ satisfies (i) and (ii), then we call f a *semi-length function* on R. For $f : R^* \to \mathbb{R}_+$ a semi-length function on an atomic domain R, we define

$$M = M(R, f) = \sup\{f(x) \mid x \in R \text{ is irreducible}\},$$

$$M^* = M^*(R, f) = \sup\{f(x) \mid x \in R \text{ is irreducible, but not prime}\},$$

$$m = m(R, f) = \inf\{f(x) \mid x \in R \text{ is irreducible}\},$$

$$m^* = m^*(R, f) = \inf\{f(x) \mid x \in R \text{ is irreducible, but not prime}\}.$$

If R is a UFD, we set $M^* = m^* = 1$. If R is not a UFD, then $0 \le m(R, f) \le m^*(R, f) \le M^*(R, f) \le M(R, f) \le \infty$. Also, for any integral domain R, $m(R, f) < \infty$ and $0 < M(R, f)$. We say that f is a *bounded semi-length function* if $0 < m(R, f)$ and $M(R, f) < \infty$ (note that the first inequality is automatic for a length function).

If R has a semi-length function f with $f(x) \ge c > 0$ for all nonunits $x \in R^*$, then R is a BFD. Moreover, an atomic domain R which has a semi-length function f with $m(R, f) > 0$ is also necessarily a BFD.

To define a semi-length function f on R, it suffices to define f on $R^* - U(R)$ and then extend f to R^* by defining $f(x) = 0$ for all $x \in U(R)$. This follows from our first result.

Lemma 1.1. *Any function $f : R^* - U(R) \to \mathbb{R}_+ - \{0\}$ which satisfies $f(xy) = f(x) + f(y)$ for all nonunits $x, y \in R^*$ extends to a semi-length function on R by defining $f(x) = 0$ for all $x \in U(R)$.*

Proof. It suffices to show that $f(ux) = f(x)$ for each nonunit $x \in R^*$ and $u \in U(R)$. If this fails, then since $2f(x) = f(x^2) = f(ux) + f(u^{-1}x)$, we may assume that $f(ux) < f(x)$ for some nonunit $x \in R^*$ and $u \in U(R)$. Then the equality $f(u^n x^2) = f(u^n x) + f(x) = f(u^{n-1}x) + f(ux)$ yields that $f(u^n x) - f(u^{n-1}x) = f(ux) - f(x)$ for all integers $n \ge 1$. Thus $f(u^n x) = n[f(ux) - f(x)] + f(x)$ for all integers $n \ge 1$. Hence $f(u^n x) < 0$ for large n, a contradiction. $\square$

Note that a semi-length function f on R extends to a homomorphism $f' : K^* \to \mathbb{R}$ on the set of nonzero elements of the quotient field K of R defined by $f'(x/y) = f(x) - f(y)$ for all $x, y \in R^*$. Since $U(R) \subset \ker f'$, f' induces an order-preserving homomorphism $\bar{f}$ from $G(R)$, the group of divisibility of R, to the additive group $\mathbb{R}$, which satisfies $\bar{f}(x) > 0$ when $x > 0$. Conversely, any order-preserving homomorphism $\bar{f} : G(R) \to \mathbb{R}$ that satisfies $\bar{f}(x) > 0$ when $x > 0$ induces a semi-length function $f : R^* \to \mathbb{R}_+$, which is a length function if im $f \subset \mathbb{Z}_+$. The set $\mathscr{L}(R, \mathbb{R}_+)$ of all semi-length functions $f : R^* \to \mathbb{R}_+$ forms a partially-ordered additive semigroup (with no zero element) which is closed under scalar multiplication by positive real numbers. As observed above, this set is order-isomorphic to the set $\mathscr{G}(R, \mathbb{R})$ of all order-preserving homomorphisms $\bar{f} : G(R) \to \mathbb{R}$ which satisfy $\bar{f}(x) > 0$ when $x > 0$. Note that an integral domain R (even a BFD) need not have an integer- or rational-valued semi-length function. In some cases it is more natural to consider real-valued semi-length functions (see Example 1.3(b)). We next give several examples of semi-length functions that will be used throughout this paper. The first ones are for Krull domains.

Example 1.2. (a) Let R be a Krull domain with $\{v_P \mid P \in X(R)\}$ its set of essential discrete rank-one valuations. Define $V : R^* \to \mathbb{Z}_+$ by $V(x) = \sum v_P(x)$. (Thus $V(x) = n \geq 1$ if and only if $xR = (P_1 \cdots P_n)_v$ for some $P_i \in X(R)$.) Then V defines a length function on R such that $V(x) = 1$ if and only if x is prime. Note that $L_R(x) \leq V(x)$ for each $x \in R^*$ (this observation gives another proof that a Krull domain is a BFD [2, Proposition 2.2]), and $L_R = V$ if and only if R is a UFD. Moreover, $M^*(R, V) = M(R, V)$; while $m^*(R, V) = m(R, V)$ if and only if either R is a UFD or R has no principal primes.

(b) More generally, for a Krull domain R, let $\{r_P \mid P \in X(R)\}$ be any set of positive real numbers. Then $f(x) = \sum r_P v_P(x)$ defines a semi-length function on R. In particular, if R is a UFD and $f = \sum r_P v_P$, then $m = m(R, f) = \inf\{r_P\}$ and $M = M(R, f) = \sup\{r_P\}$. Thus, although we have defined $m^* = M^* = 1$, if $X(R)$ is infinite, then for suitable choices of $\{r_P\}$, m and M may assume any real values such that $0 \leq m \leq M \leq \infty$, $m < \infty$, and $M > 0$.

(c) An important special case of (b) is when $\mathrm{Cl}(R)$ is a torsion group. In this case, let $\mathscr{Z}_R(x) = \sum (n_P)^{-1} v_P(x)$, where n_P is the order of $[P]$ in $\mathrm{Cl}(R)$. Note that $\mathscr{Z}_R(x) = 1$ if x is an irreducible element of the form $xR = (P^n)_v$. Thus $M^*(R, \mathscr{Z}_R) = M(R, \mathscr{Z}_R) \geq 1$ and $m^*(R, \mathscr{Z}_R) = m(R, \mathscr{Z}_R) \leq 1$. This function (in equivalent forms) has been used in [15], [16], [27], and [31]. In [15], $\mathscr{Z}_R(x)$ is called the *Zaks–Skula constant* of x and is defined for Dedekind domains using ideal classes rather than valuations; we will thus call $\mathscr{Z}_R$ the *Zaks–Skula function* of R. $\mathscr{Z}_R$ detects when R is a HFD (see Corollary 2.6).

(d) Even more generally, suppose that $R = \bigcap R_\alpha$ is an intersection of integral domains R_α, each with an associated semi-length function f_α. For any set $\{r_\alpha\}$ of positive real numbers, $f = \sum r_\alpha f_\alpha$ defines a semi-length function on R if the intersection has finite character. An important special case is when each R_α is a valuation domain with f_α its associated (real-valued) valuation.

Example 1.3. (a) Suppose that R is a subdomain of an integral domain T and let f be a (semi-) length function defined on T. It is easily verified that $f|_R$ is a (semi-)

length function on R if and only if $U(T) \cap R = U(R)$. The situation when $U(T) \cap R = U(R)$ has been studied extensively in [2, Section 6], where it is shown that R is a BFD if T is a BFD. (Two important cases when $U(T) \cap R = U(R)$ are when either $R \subset T$ is an integral extension or (T, M) and (R, N) are each quasilocal with $N = R \cap M$.) For example, if T is a HFD, $U(T) \cap R = U(R)$, and $f = s_T$, then $f|_R$ is a length function on R. In this case R is a BFD, but possibly with irreducibles of length >1 and $M^*(R, f) = \infty$. As another application, if $T = \bar{R}$ is a Krull domain (in particular, if R is Noetherian), then $V|_R$ defines a length function on R.

(b) Let F be a field and T an additive submonoid of $\mathbb{R}_+$. Let $A = F[X; T] = \{\sum a_t X^t \mid a_t \in F, t \in T\}$ be the monoid domain. Then $S = \{h \in A \mid h(0) \neq 0\}$ is a saturated multiplicatively closed subset of A. Finally, let $R = A_S$. Then R is quasilocal. If T is a nonzero submonoid of $\mathbb{Q}_+$, then A and R are each one-dimensional [22, Theorems 21.4 and 17.1]. Note that neither A nor R is necessarily atomic. But in either case, $g : A^* \to \mathbb{R}_+$ and $f : R^* \to \mathbb{R}_+$, given by $g(\sum a_i X^{t_i}) = \max\{t_i \mid a_i \neq 0\}$ and $f(\sum a_i X^{t_i}/s) = \min\{t_i \mid a_i \neq 0\}$, define semi-length functions on A and R, respectively. Thus A and R are both BFD's if T has a least positive element. The domain R will be investigated in more detail in Example 3.3.

In Example 1.2(b), we saw that for a Krull domain R, $f = \sum r_P v_P$ defines a semi-length function for any set $\{r_P \mid P \in X(R)\}$ of positive real numbers. We next show that when R has torsion divisor class group the converse is also true. In this case, $\mathscr{L}(R, \mathbb{R}_+)$ is thus order-isomorphic to $\prod \{\mathbb{R}_P \mid P \in X(R)\}$, where each $\mathbb{R}_P = \{x \in \mathbb{R} \mid x > 0\}$.

Proposition 1.4. *Let R be a Krull domain with torsion divisor class group* $\mathrm{Cl}(R)$ *and f any semi-length function on R. Then $f = \sum r_P v_P$, where $r_P = f(x_P)/n_P$, n_P is the order of $[P]$ in* $\mathrm{Cl}(R)$, *and $x_P R = (P^{n_P})_v$. Moreover, the r_P's are uniquely determined by f.*

Proof. Note that each x_P is irreducible and $v_P(x_Q) = n_P \delta_{PQ}$. Let $r_P = f(x_P)/n_P$. Suppose that $x \in R^*$ is a nonunit with $xR = (P_1^{m_1} \cdots P_s^{m_s})_v$ for some $P_i \in X(R)$ and integers $m_i \geq 1$. Let $x_i = x_{P_i}$, $n_i = n_{P_i}$, $v_i = v_{P_i}$, $r_i = r_{P_i}$, $N = n_1 \cdots n_s$, and $N_i = N/n_i$. Since $x^N R = (P_1^{m_1 N} \cdots P_s^{m_s N})_v$, $x^N = u x_1^{m_1 N_1} \cdots x_s^{m_s N_s}$ for some $u \in U(R)$ and each $v_i(x^N) = m_i N$. We have $Nf(x) = f(x^N) = \sum m_i N_i f(x_i) = \sum (f(x_i)/n_i) m_i N = \sum r_i v_i(x^N) = N(\sum r_i v_i(x))$. Thus $f = \sum r_P v_P$. Moreover, if $f = \sum \alpha_P v_P$ for any $\alpha_P \in \mathbb{R}$, then $\alpha_P = f(x_P)/n_P = r_P$ by above. Hence the r_P's are uniquely determined by f. $\square$

When the Krull domain R has torsion divisor class group, the Zaks–Skula function $\mathscr{Z}_R$ is the unique semi-length function f on R for which $f(x) = 1$ for each irreducible x of the form $xR = (P^n)_v$ for some $P \in X(R)$. When $\mathrm{Cl}(R)$ is not torsion, it is possible to have $f = \sum r_P v_P$ be a semi-length function on R with some $r_P = 0$. We plan to investigate this case in a later paper.

Our next result shows that in some cases a semi-length function may be extended to a larger domain. A special instance of our next proposition is when $A \subset B$ is a *root extension* of integral domains, that is, for each $x \in B$, $x^n \in A$ for some integer $n \geq 1$.

Proposition 1.5. *Let $A \subset B$ be an extension of integral domains with $U(B) \cap A = U(A)$ such that for each nonunit $b \in B^*$, $b^n \in A$ for some integer $n \geq 1$. If $f : A^* \to \mathbb{R}_+$ is a semi-length function on A, then there is a (unique) semi-length function $g : B^* \to \mathbb{R}_+$ on B with $f = g|_A$.*

Proof. By Lemma 1.1, we need only define g on the set of nonunits of B^*. For such an x, $x^n \in A$ for some integer $n \geq 1$. Define $g(x) = f(x^n)/n$. It is easily verified that g is well defined and $g(xy) = g(x) + g(y)$ for all nonunits $x, y \in B^*$. If $g(x) = 0$, then $f(x^n) = 0$. Thus $x^n \in U(A)$, and hence $x \in U(B)$. Thus g defines a semi-length function on B which clearly restricts to f on A. Note that g is uniquely determined by f. □

Corollary 1.6. *Let $A \subset B$ be an extension of integral domains with $U(B) \cap A = U(A)$ and B a Krull domain with torsion divisor class group such that for each nonunit $b \in B^*$, $b^n \in A$ for some integer $n \geq 1$. If f is a semi-length function on A, then $f = g|_A$, where $g = \sum r_P v_P$ for a unique set $\{r_P \mid P \in X(R)\}$ of positive real numbers.*

Proof. Combine Proposition 1.4 and Proposition 1.5. □

We next define the *Davenport constant*, denoted by $D(G)$, for an abelian group G, and relate it to $M^*(R, V)$. For a finite abelian group G, $D(G)$ is the least positive integer d such that for each sequence $S \subset G$ with $|S| = d$, some nonempty subsequence of S has sum 0. If G is infinite, we set $D(G) = \infty$; this is consistent with the definition of $D(G)$ for G finite. (In [28], $D(G)$ is denoted by $l(G)$, and in [29] it is denoted by $\sigma(G)$ and is called the sequential depth of G.) For any abelian group G, $D(G) \leq |G|$. If $G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ with $1 < n_1 \mid \cdots \mid n_r$, then define $M(G) = 1 + (n_1 - 1) + \cdots + (n_r - 1)$. It is always true that $D(G) \geq M(G)$, with equality if $r \leq 2$, but not in general. Also, there seems to be no general formula for $D(G)$. (For the above facts, other references on $D(G)$, and a summary of known results, see [20].) The importance of $D(G)$ is that if R is the ring of integers in an algebraic number field with ideal class group G, then $D(G)$ is the maximal number of prime ideals (counted with multiplicity) which can occur in the prime ideal factorization of an irreducible element of R (more generally, this holds for any Krull domain in which each (nonzero) divisor class contains a prime ideal). Another important class of Krull domains for which each divisor class contains a prime ideal are polynomial rings $R[X]$ [19, Theorem 14.3]. We collect these observations in our next proposition.

Proposition 1.7. *Let R be a Krull domain and $V = \sum v_P$. Then $1 \le M^*(R, V) = M(R, V) \le D(\mathrm{Cl}(R)) \le |\mathrm{Cl}(R)|$. If each (nonzero) divisor class contains a prime ideal, then $M^*(R, V) = D(\mathrm{Cl}(R))$.*

Proof. We may assume that $\mathrm{Cl}(R)$ is finite. We have already observed that $M^*(R, V) = M(R, V)$ and $D(\mathrm{Cl}(R)) \le |\mathrm{Cl}(R)|$. Suppose that $s > D(\mathrm{Cl}(R))$. Then for any $P_1, \ldots, P_s \in X(R)$, some proper subsum of $[P_1] + \cdots + [P_s]$ is 0. Thus if $s = V(x)$, then $xR = (P_1 \cdots P_s)_v$ is properly contained in a principal ideal and hence is not irreducible. Thus if x is irreducible, then $V(x) \le D(\mathrm{Cl}(R))$; so $M(R, V) \le D(\mathrm{Cl}(R))$.

Next, suppose that each (nonzero) divisor class contains a prime ideal. Then there exist $d = D(\mathrm{Cl}(R))$ prime ideals $P_1, \ldots, P_d \in X(R)$ with $[P_1] + \cdots + [P_d] = 0$, but no proper subsum is 0. Let $xR = (P_1 \cdots P_d)_v$. Then $V(x) = d$, and x is irreducible since no proper subsum of the $[P_i]$'s is 0. Hence $D(\mathrm{Cl}(R)) = M(R, V)$. □

We close this section with some examples and remarks to show that the inequalities in the above proposition may be strict.

Remark 1.8. (a) Let R be a Krull HFD with $\mathrm{Cl}(R) = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ for $1 < n_1 \mid \cdots \mid n_r$. Then $M(R, V) \le n_r \le M(\mathrm{Cl}(R)) \le D(\mathrm{Cl}(R))$. Thus $M(R, V) < M(\mathrm{Cl}(R)) \le D(\mathrm{Cl}(R))$ if $r > 1$.

Proof. Let $t = M(R, V)$ and $xR = (P_1 \cdots P_t)_v$ for x irreducible. Then $x^{n_r}R = (P_1^{n_r} \cdots P_t^{n_r})_v$ and each $(P_i^{n_r})_v$ is principal. Thus x^{n_r}, a product of n_r irreducible elements, is also a product of t nonunits. Hence $t \le n_r$ since R is a HFD. □

(b) For a Krull domain R with $\mathrm{Cl}(R) = G$ finite, we may have (i) $M(R, V) < M(G)$, (ii) $M(R, V) = M(G)$, or (iii) $M(R, V) > M(G)$.

Proof. For (i), let R be a Dedekind HFD with $G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ for $1 < n_1 \mid \cdots \mid n_r$ (such domains exist for any G [31, Example 9]). If $r > 1$, then $M(R, V) < M(\mathrm{Cl}(R))$ by (a) above. For (ii), let $G = \mathbb{Z}_p$, p prime. Then $p = M(R, V) = D(\mathrm{Cl}(R)) = M(\mathrm{Cl}(R))$. For (iii), let R be a Krull domain with $\mathrm{Cl}(R) = G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ for $1 < n_1 \mid \cdots \mid n_r$. Then $\mathrm{Cl}(R[X]) = G$ and each divisor class contains a prime ideal. Hence $M(R[X], V) = D(\mathrm{Cl}(R[X])) = D(G)$ by Proposition 1.7. But there are finite abelian groups G with $D(G) > M(G)$; so (iii) may hold. □

2. Bounds on $\rho(R)$

In this section, we use semi-length functions on an atomic domain R to determine lower and upper bounds for $\rho(R)$. Our first theorem, although quite elementary, is the basis for much of our later work. As usual, we define $\infty/a = a/0 = \infty/0 = \infty$ for any real number $a > 0$.

Theorem 2.1. *Let R be an atomic integral domain and f a semi-length function on R. Then $1 \leq \rho(R) \leq M^*(R, f)/m^*(R, f)$. Thus if R has a bounded semi-length function, then R is a RBFD.*

Proof. We may assume that R is not a UFD and that $0 < m^* = m^*(R, f) \leq M^* = M^*(R, f) < \infty$. Since R is atomic, for a nonunit $x \in R^*$, we may write $x = yz$, where $y = p_1 \cdots p_s$ is a product of primes and z has no prime factors. Then $L_R(x) = s + L_R(z)$ and $l_R(x) = s + l_R(z)$. Thus $\rho_R(x) = L_R(x)/l_R(x) \leq L_R(z)/l_R(z) = \rho_R(z)$. Hence it suffices to show that $\rho_R(x) \leq M^*/m^*$ when x has no prime factors. Suppose that $x = x_1 \cdots x_s$, where each x_i is irreducible, but not prime. Then $m^* \leq f(x_i) \leq M^*$; so $sm^* \leq f(x) \leq sM^*$. Hence $f(x)/M^* \leq s \leq f(x)/m^*$. Thus $f(x)/M^* \leq l_R(x) \leq L_R(x) \leq f(x)/m^*$, and hence $\rho_R(x) = L_R(x)/l_R(x) \leq M^*/m^*$. Thus $\rho(R) \leq M^*/m^*$. □

We next use Theorem 2.1 to show that $\rho(R) \leq \max\{\mathrm{Cl}(R)|/2, 1\}$ if R is a Krull domain. Our next theorem generalizes a result of Steffan [28, Proposition 1] for Dedekind domains with finite divisor class group and Valenza [29, Proposition 4] for rings of integers in an algebraic number field. The Dedekind domain case of Corollary 2.3(b) is also due to Steffan [28, Proposition 2].

Theorem 2.2. *Let R be a Krull domain which is not a UFD and $V = \sum v_P$. Then*

$$\begin{aligned} 1 &\leq \rho(R) \leq M^*(R, V)/m^*(R, V) \\ &\leq M^*(R, V)/2 \leq D(\mathrm{Cl}(R))/2 \leq |\mathrm{Cl}(R)|/2\,. \end{aligned}$$

In particular, a Krull domain R with finite divisor class group has $\rho(R) \leq \max\{|\mathrm{Cl}(R)|/2, 1\} < \infty$, and thus R is a RBFD.

Proof. If R is not a UFD, then $m^*(R, V) \geq 2$. The inequalities then follow from Theorem 2.1 and Proposition 1.7. □

Corollary 2.3. (a) *Let R be a Krull domain with $\mathrm{Cl}(R) = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ for $1 < n_1 \mid \cdots \mid n_r$. Then $M^*(R, V)/n_r \leq \rho(R) \leq M^*(R, V)/m^*(R, V)$. Thus $\rho(R) = M^*(R, V)/2$ when $\mathrm{Cl}(R)$ is 2-elementary.*

(b) *Let R be a Krull domain which is not a UFD. If each (nonzero) divisor class contains a prime ideal, then $\rho(R) = M^*(R, V)/2 = D(\mathrm{Cl}(R))/2$.*

(c) (Zaks [31, Theorem 1.4]) *A Krull domain R is a HFD if $\mathrm{Cl}(R) = \mathbb{Z}_2$. If each (nonzero) divisor class contains a prime ideal, then R is a HFD if and only if $|\mathrm{Cl}(R)| \leq 2$.*

Proof. The proof of (a) is similar to that of Remark 1.8(a). The proof of (b) is similar to that of [28, Proposition 2]. Part (c) follows from parts (a) and (b). □

Theorem 2.2 is far from the best possible since for any finite abelian group G, there is a Dedekind HFD R with $\mathrm{Cl}(R) = G$ [31, Example 9]. For a Krull domain R, $\rho(R)$ thus depends not just on $|\mathrm{Cl}(R)|$, but also on the distribution of the height-one prime ideals in the divisor classes (cf. [14–16, 24, 25]). Other semi-length functions besides V may give more accurate results (see Corollary 2.6 and Example 3.3). Theorem 2.1 and Theorem 2.2 give upper bounds for $\rho(R)$; we next determine a lower bound for $\rho(R)$. This is a trivial modification to Krull domains of a result of Chapman and Smith [16] for Dedekind domains (also, cf. [29, Proposition 5]).

Theorem 2.4 (Chapman and Smith [16, Theorem 1.6]). *Let R be a Krull domain with torsion divisor class group and $\mathscr{Z}_R$ its Zaks–Skula function. Then* $\max\{\mathscr{Z}_R(x), \mathscr{Z}_R(x)^{-1}\} \leq \rho(R)$ *for all irreducible $x \in R$.*

Proof. Suppose that x is irreducible and $xR = (P_1^{m_1} \cdots p_r^{m_r})_v$ for some $P_i \in X(R)$ and integers $m_i \geq 1$. As in the proof of Proposition 1.4, let $x_i = x_{P_i}$, $n_i = n_{P_i}$, $N = n_1 \cdots n_r$, and $N_i = N/n_i$. Since $x^N R = (P_1^{m_1 N} \cdots P_r^{m_R N})_v$, $x^N = u x_1^{m_1 N_1} \cdots x_r^{m_r N_r}$ for some $u \in U(R)$. Thus both $N/(m_1 N_1 + \cdots + m_r N_r)$ and $(m_1 N_1 + \cdots + m_r N_r)/N \leq \rho(R)$. The result now follows since $(m_1 N_1 + \cdots + m_r N_r)/N = m_1/n_1 + \cdots + m_r/n_r = \mathscr{Z}_R(x)$. □

Corollary 2.5 (Chapman and Smith [16, Corollary 1.7]). *Let R be a Krull domain with torsion divisor class group. Then* $\max\{M^*(R, \mathscr{Z}_R), m^*(R, \mathscr{Z}_R)^{-1}\} \leq \rho(R)$. □

We note that Corollary 2.5 does not hold if $\mathscr{Z}_R$ is replaced by $f = V$. However, Corollary 2.3(a) does give a lower bound for $\rho(R)$ in terms of $M^*(R, V)$, and $M^*(R, V)/n_r \leq M^*(R, \mathscr{Z}_R)$. By combining Theorem 2.1 with Corollary 2.5, we recover the following result of Zaks [31, Theorem 3.3] and Skula [27, Theorem 3.1]. Thus the Zaks–Skula function detects when a Krull domain R with torsion divisor class group is a HFD.

Corollary 2.6 (Zaks [31, Theorem 3.3] and Skula [27, Theorem 3.1]). *Let R be a Krull domain with torsion divisor class group. Then R is a HFD if and only if* $m^*(R, \mathscr{Z}_R) = M^*(R, \mathscr{Z}_R) = 1$. □

We next consider another case in which $\rho(R) = M^*/m^*$.

Theorem 2.7. *Let (R, M) be a quasilocal domain having a DVR $(D, \pi D)$ centered on M. Let v be the associated valuation on D, $m^* = m^*(R, f)$, and $M^* =$*

$M^*(R, f)$, where $f = v|_R$. Then $\rho(R) \leq M^*/m^*$. Moreover, if $U(D)/U(R)$ is a torsion group, then $\rho(R) = M^*/m^*$.

Proof. We first note that R is a BFD (and hence atomic) and f is a length function on R since $U(D) \cap R = U(R)$. The first inequality then follows from Theorem 2.1. For the reverse inequality, we may assume that R is not a UFD and $U(D)/U(R)$ is a torsion group. Pick $x \in R$ irreducible with $v(x) = m^*$. Let $y \in R$ be irreducible and not prime with $v(y) = j$; so $m^* \leq j \leq M^*$ and $j < \infty$. Then $v(y^{m^*}) = m^*j = v(x^j)$; so $y^{m^*} = ux^j$ for some $u \in U(D)$. (Here we use the fact that v is actually a valuation, and not just a length function on D.) Since $U(D)/U(R)$ is torsion, $u^k \in R$ for some integer $k \geq 1$. Thus $y^{m^*k} = u^k x^{jk}$. Let $z = y^{m^*k}$. Then $l_R(z) \leq m^*k$ and $L_R(z) \geq jk$. Hence $\rho_R(z) = L_R(z)/l_R(z) \geq jk/m^*k = j/m^*$. Thus $\rho(R) \geq M^*/m^*$, and hence $\rho(R) = M^*/m^*$. □

Remark 2.8. Theorem 2.7 may be generalized as follows (the proof is similar and will be omitted):

Theorem. *Let (R, M) be a quasilocal atomic domain having a rank-one valuation ring (D, P) centered on M. Let v be the associated valuation on D, $m^* = m^*(R, f)$, and $M^* = M^*(R, f)$, where $f = v|_R$. Then $\rho(R) \leq M^*/m^*$. Moreover, if $U(D)/U(R)$ is a torsion group and D is rational, then $\rho(R) = M^*/m^*$.* □

We next give an example to show that in the above theorem we may have $M^* = \infty$ when $U(D)/U(R)$ is a torsion group, and hence $\rho(R) = \infty$.

Example 2.9. Let $X = \{X_j \mid 1 \leq j < \infty\}$ and Y be indeterminates. Let $k_0 = \mathbb{Z}_2(X)$, $k_i = k_0(X_1^{1/2}, \ldots, X_i^{1/2})$ for each integer $i \geq 1$, and $K = \bigcup k_i$. Then $R = k_0 + k_1Y + k_2Y^2 + \cdots = \prod k_iY^i \subset K[[Y]]$ is a one-dimensional quasilocal domain, $D = K[[Y]]$ is a DVR centered on the maximal ideal of R, and $f^2 \in R$ for each $f \in K[[Y]]$. Thus $U(D)/U(R)$ is a torsion group. However, note that each $x_j^{1/2}Y^j$ is irreducible in R and has degree j. Thus $M^* = \infty$ and $\rho(R) = \infty$.

We next make a few remarks about the hypotheses in Theorem 2.7. Recall that we use 'local' to mean a Noetherian quasilocal ring.

Remark 2.10. Let (D, M) be a quasilocal integral domain with subring R.

(a) $U(D) \cap R = U(R)$ if and only if R is quasilocal with maximal ideal $R \cap M$.

(b) If D is a DVR and $U(D) \cap R = U(R)$, then R is a BFD, but not necessarily a RBFD (Example 2.9).

(c) If $U(D)/U(R)$ is a torsion group, then D is integral over R, and hence D and R have the same Krull dimension. Thus, if in addition D is a DVR, then R is necessarily one-dimensional.

(d) Any local domain (R, M) does have a DVR overring centered on M [17]; but by (c) above, $U(D)/U(R)$ cannot be a torsion group if $\dim R > 1$.

Theorem 2.11. *Let (R, M) be a quasilocal domain with $(D, \pi D)$ a DVR centered on M with associated valuation v. If $M^*(R, v|_R) = n < \infty$, then $\hat{R}$, the M-adic completion of R, is an integral domain.*

Proof. Suppose that $x \in R$ with $v(x) \geq kn$, for an integer $k \geq 1$. If $x = x_1 \cdots x_s$ with each $x_i \in R$ irreducible, then $s \geq k$ since each $v(x_i) \leq n$. Hence $x \in M^k$. Thus $\pi^{kn}D \cap R \subset M^k$. Let $I_i = \pi^i D \cap R$. Thus $\{I_i\}$ and $\{M^i\}$ induce the same topology on R. Thus $\hat{R} = \varprojlim R/I_i = \{(r_i + I_i) \in \prod R/I_i \mid r_i + I_m = r_m + I_m$ for $1 \leq m \leq i\}$. Also, $\hat{D} = \varprojlim D/\pi^i D$ is an integral domain, in fact, a DVR. We may consider R/I_i as a subring of $D/\pi^i D$ via the map $r + I_i \rightarrow r + \pi^i D$. Thus $\prod R/I_i$ is a subring of $\prod D/\pi^i D$. Let $(r_i + I_i) \in \varprojlim R/I_i$. Then $r_i + I_m = r_m + I_m$ for $1 \leq m \leq i$. Hence $r_i - r_m \in I_m \subset \pi^m D$. Thus $r_i + \pi^m D = r_m + \pi^m D$, i.e., $(r_i + \pi^m D) \in \varprojlim D/\pi^i D$. Thus $\hat{R} = \varprojlim R/I_i$ is a subring of $\hat{D} = \varprojlim D/\pi^i D$, and hence $\hat{R}$ is an integral domain. $\square$

Let (R, M) be the domain in Example 2.9. Then $M^*(R, v|_R) = \infty$, so the converse of Theorem 2.11 is false. Nevertheless, the proof of Theorem 2.11 shows that $\hat{R}$ is a subring of $K[[Y]]$. In fact, it is easily seen that for this example $\hat{R} = R$.

If (R, M) is a one-dimensional local domain, then $\hat{R}$ need not equal R. But in this case, we obtain a very satisfactory characterization (Theorem 2.12) of when R is a RBFD.

Recall that a local domain R is said to be *analytically irreducible* if $\hat{R}$ is a domain. The key observation is the following well-known exercise from Nagata [26, Exercise 1, p. 122]: a one-dimensional local domain R is analytically irreducible if and only if $\bar{R}$ is a finitely generated R-module and $\bar{R}$ is a DVR.

Theorem 2.12. *The following conditions are equivalent for a one-dimensional local domain (R, M).*

(1) *R is a RBFD, that is, $\rho(R) < \infty$.*

(2) *R is analytically irreducible.*

(3) *For every DVR $(D, \pi D)$ centered on M with associated valuation v, $M^*(R, v|_R) < \infty$.*

(4) *There is a DVR $(D, \pi D)$ centered on M with associated valuation v such that $M^*(R, v|_R) < \infty$.*

(5) *Every semi-length function f on R is bounded.*

(6) *There is a bounded semi-length function f on R.*

Moreover, if $\rho(R) < \infty$, then $\rho(R)$ is rational.

Proof. (1)$\Rightarrow$(2) Suppose that $\rho(R) = r < \infty$. Thus $L_R(z) \leq r l_R(z)$ for any $z \in R^*$. Given a nonzero $x \in M$, $M^{n_0} \subset xR$ for some integer $n_0 \geq 1$. Hence for each integer $k \geq 1$, $M^{kn_0} \subset x^kR$. Thus $L_R(y) \geq k$ for each nonzero $y \in M^{kn_0}$. Let $a,b \in \hat{R}$ with $ab = 0$. Then $a = \lim x_i$ and $b = \lim y_i$, where $x_i, y_i \in R^*$. Since $ab = 0$, $\lim x_iy_i = 0$ in R. Since $\lim x_iy_i = 0$, for each integer $k \geq 1$, there is an integer $n_k \geq 1$ such that $x_iy_i \in M^{2[r+1]kn_0}$ for all integers $i \geq n_k$. For each integer $i \geq n_k$, $rl_R(x_iy_i) \geq L_R(x_iy_i) \geq 2[r+1]k$; so $l_R(x_i) + l_R(y_i) \geq l_R(x_iy_i) \geq 2[r+1]k/r \geq 2k$. Thus either $l_R(x_i) \geq k$ or $l_R(y_i) \geq k$. But then either $x_i \in M^k$ or $y_i \in M^k$ for each integer $i \geq n_k$. Thus either $\{x_i\}$ or $\{y_i\}$ has a subsequence which is eventually in M^k. Hence either $a \in M^k\hat{R}$ or $b \in M^k\hat{R}$. Suppose that $a \neq 0$; so $a \in M^j\hat{R} - M^{j+1}\hat{R}$ for some integer $j \geq 0$. Then $b \in M^i\hat{R}$ for all integers $i \geq j+1$; hence $b \in \bigcap M^i\hat{R} = 0$. Thus $\hat{R}$ is an integral domain.

(2) $\Rightarrow$ (1) Since R is analytically irreducible, $\bar{R}$ is a DVR and $[R : \bar{R}] = \pi^n\bar{R}$ for some integer $n \geq 0$. Thus $f = v|_R$ defines a bounded length function on R since $u\pi^k$ is not irreducible in R for $k \geq 2n$. By Theorem 2.7 (or Theorem 2.1), $\rho(R) < \infty$.

The proof of (2) $\Rightarrow$ (1), together with Corollary 1.6, shows that (2) implies (3), (4), (5), and (6). By Theorem 2.1 and Remark 2.10(d), (3), (4), (5), and (6) all imply (1).

For the 'moreover' statement, assume that $\rho(R) < \infty$. *By* (2) above, R is analytically irreducible, and hence $\bar{R}$ is a DVR with maximal ideal $\pi\bar{R}$ and $[R : \bar{R}] = \pi^n\bar{R}$ for some integer $n \geq 0$. If $n = 0$, then $R = \bar{R}$ is a DVR and $\rho(R) = 1$. Thus we may assume that $n \geq 1$. Let v be the associated valuation on $\bar{R}$, $M^* = M^*(R, v|_R) < \infty$ and $m^* = m^*(R, v|_R) > 0$. Then $\rho(R) \leq M^*/m^*$ by Theorem 2.1. Let $u\pi^{m^*}$ and $\lambda\pi^{M^*}$ be irreducible elements of R with $u, \lambda \in U(\bar{R})$. For each integer $k \geq 1$, let $z_k = (\lambda^{m^*k}\pi^n)(u\pi^{m^*})^{M^*k} = (u^{M^*k}\pi^n)(\lambda\pi^{M^*})^{m^*k}$. Then $\lambda^{m^*k}\pi^n$, $u^{M^*k}\pi^n$, $z_k \in R$, and $L_R(z_k) \geq M^*k + 1$ and $l_R(z_k) \leq m^*k + n$. Hence $\rho_R(z_k) = L_R(z_k)/l_R(z_k) \geq (M^*k+1)/(m^*k+n) = (M^* + 1/k)/(m^* + n/k)$. Thus $\rho(R) \geq M^*/m^*$, and hence $\rho(R) = M^*/m^*$. □

As in [5], we say that an integral domain R is a *Cohen–Kaplansky domain* (CK domain) if it is atomic and has only a finite number of nonassociate atoms. An integral domain R is a CK domain if and only if R is a one-dimensional semilocal Noetherian domain and for each nonprincipal maximal ideal M of R, R/M is finite and R_M is analytically irreducible [5, Theorem 2.4] (this result is a special case of [1, Theorem 2]). In [5], Mott and the first author studied these domains extensively.

Corollary 2.13. *If R is a CK domain, then $\rho(R) < \infty$. Thus a CK domain is a RBFD.*

Proof. By the above comments, R has only a finite number of maximal ideals, say $M_1, \ldots, M_s$, and each $R_i = R_{M_i}$ is one-dimensional and analytically irreducible. Thus each $\rho(R_i) < \infty$. Let $r = \max\{\rho(R_i)\}$. For $x \in R^*$,

$$\rho_R(x) = L_R(x)/l_R(x) = \left(\sum L_{R_i}(x/1)\right)\Big/\left(\sum l_{R_i}(x/1)\right)$$
$$\le \left(\sum rl_{R_i}(x/1)\right)\Big/\left(\sum l_{R_i}(x/1)\right) = r$$

[5, Theorem 3.2]. Thus $\rho(R) \le r < \infty$. □

We say that an integral domain R is a *weakly Krull domain* if $R = \bigcap \{R_P \mid P \in X^{(1)}(R)\}$ is of finite character. Such domains, although not called weakly Krull domains there, were the subject of [6]. Note that a Noetherian domain R is weakly Krull if and only if every grade-one prime ideal of R has height-one; thus a one-dimensional Noetherian domain is weakly Krull. Also, as in [4], we say that R is a *weakly factorial domain* if each nonunit of R is a product of primary elements. A CK domain R is certainly also weakly factorial. A weakly Krull domain R is weakly factorial if and only if $\mathrm{Cl}_t(R) = 0$ [7, Theorem]. Here $\mathrm{Cl}_t(R)$ is the *t-class group* of R, the group of t-invertible t-ideals of R modulo its subgroup of principal fractional ideals. The t-class group is thus defined for any integral domain R. This definition is due to Bouvier and Zafrullah [11]. When R is a Krull domain $\mathrm{Cl}_t(R) = \mathrm{Cl}(R)$, while $\mathrm{Cl}_t(R) = \mathrm{Pic}(R)$ for any one-dimensional integral domain R. For these and other results, the reader is referred to [9] or [11].

Theorem 2.14. *Let R be an atomic weakly Krull domain. Then*

$$\rho(R) \le |\mathrm{Cl}_t(R)| \sup\{\rho(R_P) \mid P \in X^{(1)}(R)\} .$$

In particular, if R is a one-dimensional Noetherian domain, then

$$\rho(R) \le |\mathrm{Pic}(R)| \sup\{\rho(R_P) \mid P \in X^{(1)}(R)\} .$$

Proof. We may assume that $|\mathrm{Cl}_t(R)| = n < \infty$ and $\sup\{\rho(R_P) \mid P \in X^{(1)}(R)\} = r < \infty$. For a nonunit $x \in R^*$, define $L_R^Q(x) = \sup\{k \mid xR = (Q_1 \cdots Q_k)_t,\ Q_i$ is an (product) irreducible t-invertible primary ideal$\}$ and $l_R^Q(x)$ similarly, but with 'sup' replaced by 'inf' (these make sense by [6, Theorem 3.1]). Note that $L_R^Q(x) = \sum L_{R_P}^Q(x/1) = \sum L_{R_P}(x/1)$ and $l_R^Q(x) = \sum l_{R_P}^Q(x/1) = \sum l_{R_P}(x/1)$, where each sum is indexed over $P \in X^{(1)}(R)$. Since $L_{R_P}(x/1) \le rl_{R_P}(x/1)$ for each $P \in X^{(1)}(R)$, $L_R^Q(x) = \sum L_{R_P}(x/1) \le \sum rl_{R_P}(x/1) = rl_R^Q(x)$. Suppose that $l_R(x) = s$; thus $x = x_1 \cdots x_s$, where each $x_i \in R$ is irreducible. Factor each x_iR into a product of (product) irreducible P-primary t-invertible ideals. Each x_iR can have at most n such factors; for otherwise x_iR would be properly contained in a principal ideal since $\mathrm{D}(\mathrm{Cl}_t(R)) \le |\mathrm{Cl}_t(R)| \le n$. Thus $l_R^Q(x) \le ns = nl_R(x)$. Hence $L_R(x) \le L_R^Q(x) \le rl_R^Q(x) \le rnl_R(x)$. Thus $\rho_R(x) = L_R(x)/l_R(x) \le rn$ for each nonunit $x \in R^*$, and hence $\rho(R) \le rn$. □

The above proof shows that $|\mathrm{Cl}_t(R)|$ (resp., $|\mathrm{Pic}(R)|$) may be replaced by $D(\mathrm{Cl}_t(R))$ (resp., $D(\mathrm{Pic}(R))$) in the statement of Theorem 2.14. Our next result sharpens Corollary 2.13 when R is a CK domain.

Corollary 2.15. *If R is an atomic weakly factorial domain, then* $\rho(R) = \sup\{\rho(R_P) \mid P \in X^{(1)}(R)\}$.

Proof. By [7, Theorem], a weakly Krull domain R is weakly factorial if and only if $\mathrm{Cl}_t(R) = 0$. This gives '$\le$'. For the reverse inequality, let $P \in X^{(1)}(R)$ and $0 \ne z \in P_P$. Then $zR_P \cap R = yR$ for some $y \in R^*$ [7, Theorem (6)]. Also, $L_R(y) = L_{R_P}(z)$ and $l_R(y) = l_{R_P}(z)$ [7, Theorem (5)]; so $\rho_R(y) = \rho_{R_P}(z)$. Hence $\sup\{\rho(R_P) \mid P \in X^{(1)}(R)\} \le \rho(R)$ and we have equality. $\square$

We close this section with a conjecture which reduces to Theorem 2.2 or '$\le$' of Corollary 2.15 when R is respectively a Krull domain or an atomic weakly factorial domain. Note that the conjecture does hold if $|\mathrm{Cl}_t(R)| = 1$ or ∞, or $\sup\{\rho(R_P) \mid P \in X^{(1)}(R)\} = \infty$.

Conjecture. If R is an atomic weakly Krull domain, then

$$1 \le \rho(R) \le \max\{|\mathrm{Cl}_t(R)|/2, 1\} \sup\{\rho(R_P) \mid P \in X^{(1)}(R)\} .$$

3. Examples

We first show that for any real number $r \ge 1$ or $r = \infty$, there is a Dedekind domain R with torsion divisor class group such that $\rho(R) = r$. We then give several other examples to illustrate the theory developed in Sections 1 and 2.

Our first example is based on a theorem of Claborn [18, Theorem 2.1]. (A more general version is in [19, Theorem 15.18].) For completeness, we state Claborn's theorem and the necessary terminology. Let $F = \bigoplus \mathbb{Z}e_n$ be the free abelian group on $\{e_n \mid 1 \le n < \infty\}$ and let F_+ be its subset of nonnegative elements under the usual product order. A subset P of F_+ is *finitely dense* if for each finite sequence $n_1, \ldots, n_k$ of nonnegative integers, there is an $f = \sum a_i e_i \in P$ with $a_i = n_i$ for $1 \le i \le k$. (This is Claborn's condition (α) in [18].)

Theorem 3.1 (Claborn [18, Theorem 2.1]). *Let F be the free abelian group on $\{e_n \mid 1 \le n < \infty\}$ and P a finitely dense subset of F. Then there is a Dedekind domain R with nonzero prime ideals $\{M_n \mid 1 \le n < \infty\}$ such that* $\mathrm{Cl}(R)$ *is isomorphic to $F/\langle P\rangle$ under the correspondence that sends $[M_n]$ to $\bar{e}_n$.* $\square$

The above theorem just states that for such an F and $P \subset F_+$, there is a Dedekind domain R with maximal ideals $\{M_n\}$ such that the isomorphism

$\varphi : \mathrm{Div}(R) \to F$ given by $\varphi(M_i) = e_i$ also sends $\mathrm{Prin}(R)$ onto $H = \langle P \rangle$, and hence induces an isomorphism $\bar{\varphi}$ of $\mathrm{Cl}(R) = \mathrm{Div}(R)/\mathrm{Prin}(R)$ onto $G = F/H$. For $f = \sum a_i e_i \in F$, we define $V(f) = \sum a_i$; this is consistent with our earlier definition of V since $V(\varphi(xR)) = V(x)$ for $x \in R^*$.

Theorem 3.2. *Let $r \geq 1$ be a real number or $r = \infty$. Then there is a Dedekind domain R with torsion class group such that $\rho(R) = r$. Moreover, if r is rational, we may choose* $\mathrm{Cl}(R)$ *to be finite.*

Proof. We break the proof down into several subproofs.

(I) Let m and n be integers with $1 < n \leq m$. Define $u_k = e_k + \cdots + e_{n+k-1}$ for each integer $k \geq 1$. Let $H = \langle me_1, \ldots, me_n, \{u_k\} \rangle \subset F = \bigoplus \mathbb{Z}e_k$. Clearly $\langle \{u_k\} \rangle_+$ is finitely dense because $v_{ij} = e_i - e_j \in H$ whenever $i \equiv j \pmod n$. Thus H_+ is also finitely dense. Let R be the Dedekind domain given by Theorem 3.1. Note that $\mathrm{Cl}(R) = F/H$ is finite.

We say that $f \in H$ with $f > 0$ is *irreducible* if there do not exist $f_1, f_2 \in H$ with $f_1 > 0$, $f_2 > 0$, and $f = f_1 + f_2$. Note that $x \in R^*$ is irreducible if and only if $\varphi(xR)$ is irreducible in H_+.

(II) If $0 < f = a_1e_1 + \cdots + a_ne_n \in H_+$ is irreducible, then f is either $me_1, \ldots, me_n$, or u_1.

Proof. Suppose that $f = b_1(me_1) + \cdots + b_n(me_n) + cu_1$ for some $b_1, \ldots, b_n$, $c \in \mathbb{Z}$. Then $c = a_i - b_i m$ for each $1 \leq i \leq n$. If all $a_i > 0$, then $f = u_1$. Otherwise, some $a_i > 0$ and $a_j = 0$. Clearly each $0 \leq a_i \leq m$. Then $c = 0 - b_j m$ implies that $m \mid a_i$. Hence $a_i = m$, so $f = me_i$.

(III) Choose $x_i \in R^*$ with $\varphi(x_iR) = me_i$ and $y \in R^*$ with $\varphi(yR) = u_1$. Then $x_1 \cdots x_n = vy^m$ for some $v \in U(R)$. Hence $\rho_R(y^m) \geq m/n$, and thus $\rho(R) \geq m/n$.

(IV) Let $f \in H_+$ with $f > 0$ and $g = e_i - e_j \in H$ with $i \equiv j \pmod n$. If $f + g \in H_+$, then f is irreducible if and only if $f + g$ is irreducible.

Proof. Since $-g \in H$, we need only show that f is reducible implies that $f + g$ is reducible. Suppose that $f = f_1 + f_2$ with $f_1, f_2 \in H_+$ and $0 < f_1, f_2 < f$. Then $f + g = f_1 + f_2 + (e_i - e_j) \in H_+$ implies that either $f_1 + g \in H_+$ or $f_2 + g \in H_+$ since either f_1 or f_2 must have a positive jth coefficient. Say $f_1 + g \in H_+$. Clearly $V(f_1 + g) = V(f_1)$. Thus $f_1 > 0$ implies $f_1 + g > 0$. Thus $f + g = (f_1 + g) + f_2$ is not irreducible.

(V) If $0 \neq f \in H_+$ is irreducible, then $V(f) = m$ or $V(f) = n$.

Proof. By (IV), there is a $g \in H$ with $V(g) = 0$ such that $f + g$ is irreducible and $f + g \in (\mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n)_+$. By (II), $V(f + g) = V(f) + V(g) = m$ or n. Since $V(g) = 0$, thus $V(f) = m$ or n.

(VI) $\rho(R) = m/n$ (this is the case when r is rational).

Proof. (V) shows that $M^*(R, V) = m$ and $m^*(R, V) = n$. Thus $\rho(R) \leq m/n$ by Theorem 2.1. Hence $\rho(R) = m/n$ by (III).

(VII) Let $r > 1$ be a real number or $r = \infty$. Choose an increasing sequence $\{r_k\} \subset \mathbb{Q}_+$ with $r_k \to r$, where each $r_k = m_k/n_k$ with $1 < n_k < m_k$. For each r_k, construct H_k as above so that the Dedekind domain R_k associated with H_k has $\rho(R_k) = r_k$. Let $H = \bigoplus H_k$. Then H is finitely dense since each H_k is finitely dense. Let R be the Dedekind domain associated with H. Note that each R_k, and hence R, has torsion class group.

(VIII) $\rho(R) = r$.

Proof. In each $(H_k)_+$, there is an f_k with $L(f_k) = m_k$ and $l(f_k) = n_k$ (here $L(\ )$ and $l(\ )$ have the obvious meanings). Thus $\rho(R) \geq m_k/n_k$ for each integer $k \geq 1$. Hence $\rho(R) \geq r$. Conversely, let $f \in H_+$. Then $f \in (H_1 \oplus \cdots \oplus H_k)_+$ for some integer $k \geq 1$. Since any irreducible in H_+ must be in some $(H_j)_+$, we have $L(f) = a_1 + \cdots + a_k$ and $l(f) = b_1 + \cdots + b_k$, with each $a_j/b_j \leq m_j/n_j \leq m_k/n_k$. Thus $\rho(f) = L(f)/l(f) = (a_1 + \cdots + a_k)/(b_1 + \cdots + b_k) \leq m_k/n_k < r$. Hence $\rho(R) \leq r$ and we have equality. □

Questions. Theorem 3.2 motivates the following two questions.

(1) If R is a Krull domain and $\mathrm{Cl}(R)$ is finite, is $\rho(R)$ rational?

(2) If R is a Krull domain, $\mathrm{Cl}(R)$ is finite, and $\rho(R)$ is rational, does $\rho(R) = \rho_R(x)$ for some nonunit $x \in R^*$?

We remark that (2) has a negative answer if $\mathrm{Cl}(R)$ is not assumed to be finite. The proof (part (VIII)) of Theorem 3.2 yields a Dedekind domain R with infinite torsion divisor class group such that $\rho(R)$ is rational and $\rho_R(x) < \rho(R)$ for each nonunit $x \in R^*$. Also note that the Dedekind domain R constructed in Theorem 3.2 has no principal primes.

We next use monoid domains and the semi-length functions from Example 1.3(b) to give another class of atomic domains for which $\rho(R) = M^*/m^*$.

Example 3.3. Let F be a field and $r \geq 1$ a real number. Let T be the additive submonoid of $\mathbb{R}_+$ generated by $\{1\} \cup [r, \infty)$. Then $R = F[X; T]_S$, where $S = \{h \in F[X; T] \mid h(0) \neq 0\}$, is an infinite-dimensional quasilocal RBFD with $\rho(R) = r + 1$. (We could also use $T' = T \cap \mathbb{Q}_+$; in this case R is a one-dimensional quasilocal RBFD.)

Proof. Let f be defined as in Example 1.3(b). Clearly $m^*(R, f) = 1$ and $M^*(R, f) = r + 1$. Thus $1 \leq \rho(R) \leq M^*/m^* = r + 1$ by Theorem 2.1. For each integer $n \geq 1$, we can choose a rational number a/b with $r + 1 - 1/n < a/b < r + 1$ so that $a/b \in T$ and $X^{a/b}$ is irreducible. Since X is also irreducible and $X^a = (X)^a = (X^{a/b})^b$, we have $l_R(X^a) \leq b$ and $L_R(X^a) \geq a$. Thus $\rho_R(X^a) \geq a/b$; so $\rho(R) \geq a/b > r + 1 - 1/n$. Hence $\rho(R) = r + 1$. □

Example 3.4. Let F be a field and $R_n = F[X^n, XY, Y^n]$ for each integer $n \geq 1$. Then R_n is a two-dimensional Noetherian Krull domain with $\mathrm{Cl}(R_n) = \mathbb{Z}_n$ [8]. If $n = 1$, then of course R_n is a UFD. By Theorem 2.2, $\rho(R_n) \leq n/2$ for each integer $n \geq 2$. Note that X^n, Y^n, and XY are all irreducible in R_n and $(XY)^n = X^nY^n$. Hence $\rho_{R_n}(X^nY^n) = n/2$, and thus $\rho(R_n) = n/2$. (Thus R_n is a HFD if and only if either $n = 1$ or 2.) One may also see that $\rho(R_n) = n/2$ by using $\mathscr{X}_{R_N}$ and Corollary 2.5. □

Example 3.5. Let $k \subset K$ be a proper extension of finite fields. For integers $1 \leq m \leq n$, let $R_{m,n} = k + kX^m + \cdots + kX^{n-1} + X^n K[[X]]$. Then $\rho(R_{m,n}) = (n + m - 1)/m$. Thus for integers $m \geq 1$ and $i \geq 2m - 1$, there is a local CK domain S with $\rho(S) = i/m$.

Proof. Let $R = R_{m,n}$, $D = \bar{R} = K[[X]]$, and $v(h) = \text{ord}\, h$. Clearly $m^*(R, v|_R) = m$, and $M^*(R, v|_R) = n + m - 1$ since bX^{n+m-1} is irreducible in R for each $b \in K - k$ and $X^m | h$ in R if $\text{ord}\, h \geq n + m$. Since K is finite, $U(D)/U(R)$ is a torsion group. Hence $\rho(R_{m,n}) = M^*/m^* = (n + m - 1)/m$ by Theorem 2.7 □

Example 3.6. Let $k \subset K$ be a proper extension of fields and $R_{m,n}$ be as in Example 3.5. Then $\rho(R_{m,n}) = (n + m - 1)/m$. For $k = K$ and $m = n$, let $R = K + X^n K[[X]]$. Then $\rho(R) = (2n - 1)/n$.

Proof. As in Example 3.5, $\rho(R_{m,n}) \leq (n + m - 1)/m$ by Theorem 2.7. To show that $\rho(R_{m,n}) \geq (n + m - 1)/m$, we need only find $b_1, \ldots, b_m \in K - k$ with $b_1 \cdots b_m = 1$. For then $(X^m)^{n+m-1} = (b_1 X^{n+m-1}) \cdots (b_m X^{n+m-1})$, and each of these factors is irreducible in $R_{m,n}$. For m even, pick $b_1 \in K - k$, let $b_2 = b_1^{-1}$, and pick the remaining even number of b_i's in a similar manner. For m odd, pick $b_1 \in K - k$, let $b_2 = b_1$ if $b_1^2 \in K - k$ and $b_2 = b_1(1 + b_1)$ otherwise, and let $b_3 = (b_1 b_2)^{-1}$. The remaining even number of b_i's may then be chosen as in the case when m is even. For the second example, just note that X^n and X^{2n-1} are both irreducible and of respectively lowest and highest order. □

Example 3.7. Let T be a quasilocal integral domain of the form $K + M$, where M is the nonzero maximal ideal of T and K is a subfield of T. Let D be a subring of K and $R = D + M$. This '$D + M$' construction has been used extensively since it has proven to be an excellent technique for constructing counterexamples (cf. [2] and [12]). Here, we let $D = k$ be a subfield of K. Up to multiplication by a $\alpha \in K^*$ (resp., $\alpha \in k^*$), each element of T (resp., R) has the form m or $1 + m$ for some $m \in M$. Since each of these elements is irreducible in R if and only if it is irreducible in T, we have that R is atomic if and only if T is atomic [2, Proposition 1.2], and in this case $\rho(k + M) = \rho(K + M)$. This construction yields atomic domains with different ring-theoretic properties, but with the same elasticity. □

Note added in proof. The two questions after Theorem 3.2 have been answered affirmatively by S. Chapman, W.W. Smith, and the two authors in "Rational elasticity of factorizations in Krull domains" to appear in Proc. Amer. Math. Soc.

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