Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 309 (2009) 834–842
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Bounds on isoperimetric values of trees

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Received 9 July 2007; received in revised form 5 January 2008; accepted 11 January 2008
Available online 4 March 2008

Abstract

Let $G = (V, E)$ be a finite, simple and undirected graph. For $S \subseteq V$, let $\delta(S, G) = \{(u, v) \in E : u \in S \text{ and } v \in V - S\}$ be the edge boundary of S. Given an integer i, $1 \le i \le |V|$, let the edge isoperimetric value of G at i be defined as $b_e(i, G) = \min_{S \subseteq V; |S|=i} |\delta(S, G)|$. The edge isoperimetric peak of G is defined as $b_e(G) = \max_{1 \le j \le |V|} b_e(j, G)$. Let $b_v(G)$ denote the vertex isoperimetric peak defined in a corresponding way. The problem of determining a lower bound for the vertex isoperimetric peak in complete t-ary trees was recently considered in [Y. Otachi, K. Yamazaki, A lower bound for the vertex boundary-width of complete k-ary trees, Discrete Mathematics, in press (doi:10.1016/j.disc.2007.05.014)]. In this paper we provide bounds which improve those in the above cited paper. Our results can be generalized to arbitrary (rooted) trees.

The depth d of a tree is the number of nodes on the longest path starting from the root and ending at a leaf. In this paper we show that for a complete binary tree of depth d (denoted as T_d^2), $c_1 d \le b_e(T_d^2) \le d$ and $c_2 d \le b_v(T_d^2) \le d$ where c_1, c_2 are constants. For a complete t-ary tree of depth d (denoted as T_d^t) and $d \ge c \log t$ where c is a constant, we show that $c_1 \sqrt{t} d \le b_e(T_d^t) \le td$ and $c_2 \frac{d}{\sqrt{t}} \le b_v(T_d^t) \le d$ where c_1, c_2 are constants. At the heart of our proof we have the following theorem which works for an arbitrary rooted tree and not just for a complete t-ary tree. Let $T = (V, E, r)$ be a finite, connected and rooted tree — the root being the vertex r. Define a weight function $w : V \to \mathbb{N}$ where the weight $w(u)$ of a vertex u is the number of its successors (including itself) and let the weight index $\eta(T)$ be defined as the number of distinct weights in the tree, i.e $\eta(T) = |\{w(u) : u \in V\}|$. For a positive integer k, let $\ell(k) = |\{i \in \mathbb{N} : 1 \le i \le |V|, b_e(i, G) \le k\}|$. We show that $\ell(k) \le 2\binom{2\eta+k}{k}$.

Keywords: Isoperimetric problem; Binary trees; t-ary trees; Pathwidth

1. Introduction

Let $G = (V, E)$ be a simple, finite, undirected graph.

Definition 1. For $S \subseteq V$, the edge boundary $\delta(S, G)$ is the set of edges of G with exactly one end point in S. In other words,

$$\delta(S, G) = \{(u, v) \in E : u \in S \text{ and } v \in V - S\}.$$

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0012-365X/$ - see front matter
doi:10.1016/j.disc.2008.01.021

Definition 2. For $S \subseteq V$, the vertex boundary $\phi(S, G)$ is defined similarly.

$$\phi(S, G) = \{v \in V - S : \exists u \in S, \text{ such that } (u, v) \in E\}.$$

Definition 3. Let i be an integer where $1 \le i \le |V|$. For each i define the edge isoperimetric value $b_e(i, G)$ and the vertex isoperimetric value $b_v(i, G)$ of G at i as follows

$$b_e(i, G) = \min_{S \subseteq V; |S| = i} |\delta(S, G)|$$
$$b_v(i, G) = \min_{S \subseteq V; |S| = i} |\phi(S, G)|.$$

Definition 4. For any graph G define the edge and the vertex isoperimetric peaks $b_e(G)$, $b_v(G)$ as,

$$b_e(G) = \max_{1 \le i \le |V|} b_e(i, G)$$
$$b_v(G) = \max_{1 \le i \le |V|} b_v(i, G).$$

The edge (vertex) isoperimetric problem for a graph G is to determine $b_e(i, G)$ ($b_v(i, G)$) respectively for each i, $1 \le i \le |V|$.

Discrete isoperimetric inequalities form a very useful and important subject in graph theory and Combinatorics. See [6], Chapter 16 for a brief introduction on isoperimetric problems. For a detailed treatment see the book by Harper [17]. See also the surveys by Leader [21] and by Bezrukov [3,2] for a comprehensive overview of work in the area. The edge (vertex) problem is NP-hard for an arbitrary graph. The NP hardness of the edge version can be seen by observing that if we know $b_e(i, G)$ for all i, $1 \le i \le |V|$ we can easily find solutions to the bisection width problem [13] and the sparsest cut problem [24]. Isoperimetric problems are typically studied for graphs with special (usually symmetric) structure and the edge and vertex versions of the problem are considered separately as they require different techniques. Probably the earliest example is Harper's work [14]: He studied the edge isoperimetric problem for d-dimensional hypercubes. Hart [18] also found the same result separately. Harper later worked on the vertex version [15]. Simpler proofs were discovered for his result by Katona [20] and independently by Frankl and Füredi, see [6], Chapter 16. The edge isoperimetric problem in the grid i.e. the cartesian product of paths was considered by Bollabas and Leader [7]. Since then many authors have considered the isoperimetric problems in graph cartesian products. See for example [11]. The isoperimetric problem for the cartesian product of two Markov chains is studied in [19]. Recently Harper considered the isoperimetric problem in Hamming graphs [16].

The isoperimetric properties of graphs with respect to eigen values of their adjacency or Laplacian matrices is considered by many authors, for example see [1]. The isoperimetric properties of a graph is very closely related to its expansion properties. A graph G is called an expander graph if for every positive integer $i \le \epsilon|V|$, $b_v(i, G) \ge \epsilon' i$, where ϵ and ϵ' are predefined constants. A great deal of effort has gone into explicitly constructing expander graphs — the first construction of an infinite family was due to Margulis [23]. See [26] for a recent construction.

The importance of isoperimetric inequalities lies in the fact that they can be used to give lower bounds for many useful graph parameters. For example it can be shown that pathwidth$(G) \ge b_v(G)$ [8], bandwidth$(G) \ge b_v(G)$ [14] and cutwidth$(G) \ge b_e(G)$ [4]. In [10], it is shown that given any j (where $1 \le j \le |V|$), treewidth$(G) \ge \min_{j/2 \le i \le j} b_v(i, G) - 1$ and in [9] it is shown that carving-width$(G) \ge \min_{j/2 \le i \le j} b_v(i, G)$, where $1 \le j \le |V|$ and in [14] it is shown that wirelength$(G) \ge \sum_{i=1}^{|V|} b_e(i, G)$.

2. Our results

Let $T = (V, E, r)$ be a finite,connected rooted tree rooted at r. Consider the natural partial order $\preceq_T$ induced by the rooted tree on the vertices.

Definition 5. In a rooted tree $T = (V, E, r)$ for any two vertices u, v, $u \preceq_T v$ if and only if there is a path from the root to v with u in the path. In particular $u \preceq_T u$ for any vertex u.

Definition 6. For a rooted tree $T = (V, E, r)$ we define a weight function $w_T : V \to 1, 2, \dots, |V|$ as follows: $w_T(u) = |\{v \in V : u \preceq v\}|$ (i.e. the number of successors of u, including u). Let us define the weight index of the rooted tree $T = (V, E, r)$ as $\eta(T) = |\{w_T(u) : u \in V\}|$. Note that this is the number of distinct weights. When there is no confusion let $\eta(T)$ be abbreviated by η.

Definition 7. For any graph G let, $\ell_G(k) = |\{i \in \mathbb{N} : 1 \leq i \leq |V|, b_e(i, G) \leq k\}|$ where k is a positive integer.

In other words $\ell_G(k)$ is the number of integers i such that the edge isoperimetric value of G at i is at most k. The main Theorem in this paper is as follows:

Theorem 1.

We use the above result to show the following interesting corollaries.

Corollary 1. *Let T_d^2 be the complete binary tree of depth d. Then $c_1 d \leq b_e(T_d^2) \leq d$ and $c_2 d \leq b_v(T_d^2) \leq d$ where c_1 and c_2 are constants.*

Corollary 2. *Let T_d^t be the complete t-ary tree of depth d with $t \geq 2$ and $d \geq c \log t$ where c is a suitable chosen constant. Then, $c_1 \sqrt{t} d \leq b_e(T_d^t) \leq (t-1)d$ and $c_2 \frac{d}{\sqrt{t}} \leq b_v(T_d^t) \leq d$ where c_1 and c_2 are appropriate constants.*

We would like to point that recently Otachi and Yamazaki have considered the problem of determining the vertex isoperimetric peak in complete t-ary trees [25]. They prove that $d \geq b_v(T_d^t) \geq \frac{d \log t - (t+6+2\log d)}{(t+6+2\log d)}$. Asymptotically our results are better as we prove $b_v(T_d^t) \geq c_2 \frac{d}{\sqrt{t}}$ where c_2 is a constant. The best bound that can be obtained from their result for the edge isoperimetric peak is $b_e(T_d^t) \geq \frac{d \log t - (t+6+2\log d)}{(t+6+2\log d)}$ while we show that $b_e(T_d^t) \geq c_1 \sqrt{t} d$ where c is a constant. Similarly in the special case of a complete binary tree their result implies $b_v(T_d^2) \geq \frac{d \log 2 - (8+2\log d)}{(8+2\log d)} \approx \frac{cd}{\log d}$ where c is a constant. In contrast we give a tight result showing that $b_e(T_d^2) \geq c_1 d$ and $b_v(T_d^2) \geq c_2 d$ where c_1 and c_2 are constants. Moreover our proof techniques are such that the above results can be extended to arbitrary (rooted) trees. The proofs in this paper are also comparatively simpler. As consequences of the above results we have the following theorems. We just mention the theorems here. The necessary definitions and detailed discussions are available in the corresponding sections (Sections 5.1 and 5.2)

Theorem 2. *There exists an increasing function f such that for any graph G if* pathwidth$(G) \geq k$ *then there exists a minor G' of G such that $b_v(G') \geq f(k)$.*

Theorem 3. *For the complete binary tree on T_d^2 on n vertices* thinness$(T_d^2) = \Omega(\log n)$. *This means that there exist trees with arbitrarily large thinness.*

3. Upper bounds on the isoperimetric peak of a tree

A depth first traversal is one in which all the subtrees of the given rooted tree are recursively visited before visiting the root. Perform such a traversal of the tree and list the vertices in the order in which they appear in the traversal. This gives an ordering of the vertices. Let us choose S_i as the first i vertices as they appear in this ordering. It can be very easily verified that $b_e(i, T) \leq |\delta(S_i, G)| \leq (\Delta - 1)d$ where d is the depth of the tree and Δ is the maximum degree of a vertex in T. Using the same technique we can prove that $b_v(T) \leq d$. For a t-ary tree of depth d this implies $b_e(T_d^t) \leq td$ and $b_v(T_d^t) \leq d$.

4. Lower bounds on the isoperimetric peak of a tree

Definition 8. Let $T = (V, E, r)$ be a rooted tree with $|V| = n$ and root r, and let $S \subseteq V$. Then we define the function $f_{S,T} : E \cup \{r\} \to \{w_T(u) : u \in V\} \cup \{0\}$ as follows:

$$
\begin{aligned}
f_{S,T}(r) &= 0 \quad \textbf{if } r \in V - S \\
&= w_T(r) = n \quad \textbf{if } r \in S \\
f_{S,T}(e) &= 0 \quad \textbf{if } e \in E - \delta(S, T).
\end{aligned}
$$

Finally if $e = (u, v) \in \delta(S, T)$, without loss of generality assume that u is a child of v in T. Then,

$$\begin{aligned} f_{S,T}(e) = f_{S,T}(u, v) &= w_T(u) \quad \textbf{if } u \in S \\ &= -w_T(u) \quad \textbf{if } u \in V - S. \end{aligned}$$

Lemma 1. *Let $T = (V, E, r)$ be a tree with root r and let $S \subseteq V$. Then, $f_{S,T}(r) + \sum_{e \in E(T)} f_{S,T}(e) = |S|$.*

Proof. We use induction on the number of vertices $|V| = n$. For a rooted tree $T' = (V', E', r)$ with $|V'| = 1$, it is trivial to verify the Lemma. Let the Lemma be true for any rooted tree $T' = (V'', E'', r'')$ on at most $n - 1$ vertices (where $n \geq 2$) and for all possible subsets of V''. Let S be an arbitrary subset of V. Let $v_1, v_2, \ldots, v_k$ be the children of r in T. We denote by $T_i = (V_i, E_i, v_i)$ the subtree of T rooted at v_i. Let $S_i = S \cap V_i$ for $1 \leq i \leq k$. Also, let f denote the function $f_{S,T} : \{r\} \cup E \to \{w_T(u) : u \in V\} \cup \{0\}$, let f^i denote the function $f_{S_i,T_i} : \{v_i\} \cup E_i \to \{w_{T_i}(u) : u \in V_i\} \cup \{0\}$. By the induction assumption we have,

$$f^i(v_i) + \sum_{e \in E_i} f^i(e) = |S_i| \quad \text{for } 1 \leq i \leq k. \tag{1}$$

Noting that for any edge $e \in E(T) \cap E(T_i)$, $f(e) = f^i(e)$ we have:

$$\begin{aligned} \sum_{e \in E(T)} f(e) &= \sum_{i=1}^{k} \sum_{e \in E_i} f^i(e) + \sum_{i=1}^{k} f(r, v_i) \\ &= \sum_{i=1}^{k} |S_i| - f^i(v_i) + \sum_{i=1}^{k} f(r, v_i). \end{aligned} \tag{2}$$

By the definitions of the functions f and f^i (see Definition 8) we have:

$$f(r, v_i) - f^i(v_i) = 0 \quad \textbf{if } r \in V - S \tag{3}$$

$$f(r, v_i) - f^i(v_i) = -w_{T_i}(v_i) = -w_T(v_i) \quad \textbf{if } r \in S. \tag{4}$$

Now substituting Eqs. (3) and (4) in Eq. (2), we get

$$f(r) + \sum_{e \in E} f(e) = \sum_{i=1}^{k} |S_i| = |S| \quad \textbf{if } r \in V - S$$

and

$$\begin{aligned} f(r) + \sum_{e \in E} f(e) &= \sum |S_i| + w_T(r) - \sum_{i=1}^{k} w_T(v_i) \\ &= \sum_{i=1}^{k} |S_i| + 1 = |S| \quad \textbf{if } r \in S \end{aligned}$$

as required. ■

We need the following lemma to prove the corollaries of the next theorem.

Lemma 2. *For any graph $G = (V, E)$, $b_e(G) \geq b_v(G) \geq \frac{b_e(G)}{\Delta}$.*

Proof. The first part of the inequality is obvious. Let the edge isoperimetric peak occur at i and the vertex isoperimetric peak at j. Since Δ is the maximum degree, $\Delta b_v(i, G) \geq b_e(i, G) = b_e(G)$ (Every vertex can have atmost Δ edges incident on it). But $b_v(G) = b_v(j, G) > b_v(i, G)$. Therefore $\Delta b_v(G) = \Delta b_v(j, G) \geq b_e(G)$. ■

Theorem 1. *For any rooted tree $T = (V, E, r)$, with weight index η, $\ell_T(k) \leq 2\binom{2\eta+k}{k}$.*

Proof. Let $i \leq |V|$ be a positive integer such that $b_e(i, T) = k' \leq k$. Then there exists a subset $S_i \subseteq V$ such that $|\delta(S_i, T)| = k'$ and $|S_i| = i$. Let $\delta(S_i, T) = \{e_1, e_2, \ldots, e_{k'}\}$. We define $k+1$ variables $t_0, t_1, \ldots, t_k$ as follows. Let $t_0 = f_{S_i,T}(r)$ and let $t_i = f_{S_i,T}(e_i)$ for $1 \leq i \leq k'$. If $k' < k$, then let $t_i = 0$ for $k' < i \leq k$. By Lemma 1, we have $\sum_{e \in E} f_{S_i,T}(e) = |S_i| = i$. Recalling Definition 8, for an edge e, $f_{S_i,T}(e) \neq 0$ only when $e \in \delta(S_i, T)$. Thus we have:

$$t_0 + t_1 + \cdots + t_k = i.$$

How many distinct positive integers can be expressed as $\sum_{i=0}^{k} t_i$? This will clearly give an upper bound for $\ell(k)$. Let $W = \{w_1, \ldots, w_\eta\}$ where η is the weight index of the tree, denote the set of distinct weights. Then t_i can take the values 0 or $\pm w_j$, $1 \leq j \leq \eta$. Considering the k variables t_i $(1 \leq i \leq k)$ as k unlabeled balls and imagining the $2\eta + 1$ distinct possible values they can take as $2\eta + 1$ labeled boxes, it is easy to see that the number of distinct integers expressible as $\sum_{i=1}^{k} t_i$ is bounded above by the number of ways of arranging k unlabeled balls in $2\eta + 1$ labeled boxes, i.e. $\binom{2\eta+k}{k}$. Recalling that t_0 can take only two possible values, we get:

$$\ell_T(k) \leq 2\binom{2\eta + k}{k}. \quad \blacksquare$$

Definition 9. For any graph G with weight index η, define p as the minimum value of k such that $2\binom{2\eta+k}{k} \geq n$.

Lemma 3. *For any rooted tree $T = (V, E, r)$, $b_e(G) \geq p$.*

Proof. Assume $b_e(G) < p$. Let $b_e(G) = q$. Then by the definition of p we have $2\binom{2\eta+q}{q} < n$. But by Definition 7 $\ell_T(q) = n$ a contradiction. $\blacksquare$

Corollary 1. Let T_d^2 be the complete binary tree of depth d. Then $c_1 d \leq b_e(T_d^2) \leq d$ and $c_2 d \leq b_v(T_d^2) \leq d$ where c_1 and c_2 are constants.

Proof. Let the number of vertices in T_d^2 be denoted by n. We need only prove that $b_e(T_d^2) \geq c_1 d$ for some constant c_1 as the upper bound follows from Section 3. Note that $\eta(T_d^2) = d$ so, $\ell(k) \leq 2\binom{2d+k}{k}$ where k is a positive integer. Now let $k = \lfloor \frac{d}{5} \rfloor = \lfloor 0.2d \rfloor$. Then we have (discarding the floor symbol),

$$\begin{aligned} 2\binom{2d + k}{k} &= 2\binom{2.2d}{0.2d} \\ &= \frac{2(2.2d)!}{0.2d!2d!} \\ &= \frac{c}{\sqrt{d}}\left(\frac{(2.2)^{2.2}}{(0.2)^{0.2}2^2}\right)^d \\ &\leq \frac{c'}{\sqrt{d}}(1.96)^d. \end{aligned}$$

Here we have used Stirling's approximation, $c''\sqrt{2\pi n}n^n e^{-n} \leq n! \leq c'''\sqrt{2\pi n}n^n e^{-n}$. This means that for a sufficiently large value of d, $\ell(k) < n$ when $k = \frac{d}{5}$ which implies that $b_e(T_d^2) \geq c_1 d$ where c_1 is a constant. Again this implies $b_v(T_d^2) \geq c_2 d$ where c_2 is a constant, as $\Delta = 3$ for a complete binary tree. $\blacksquare$

The reader may note that the above proof shows that for almost all integers i, $1 \leq i \leq k b_e(i, T_d^2) \geq .2d$. More precisely $\lim_{d\to\infty} \frac{\ell_{T_d^2}(\frac{d}{5})}{n} \to 0$.

Corollary 2. Let T_d^t be the complete t-ary tree of depth d with $t \geq 2$ and $d \geq c \log t$ where c is a suitable chosen constant. Then, $c_1\sqrt{t}d \leq b_e(T_d^t) \leq td$ and $c_2 d \frac{d}{\sqrt{t}} \leq b_v(T_d^t) \leq d$ where c_1 and c_2 are constants.

Proof. The upper bound follows from Section 3. We will assume that $t \geq 9$ initially and $d \geq 30$. Note that for a t-ary tree of depth d, $\eta(T_d^t) = d$. For a positive integer k, by Theorem 1 we have

$$\ell(k) \leq 2\binom{2d+k}{k}.$$

Now let $k = \lfloor m\sqrt{t}d \rfloor$ where $0 < m < 2(\frac{1}{e} - \frac{1}{3})$ is a constant. Then we have (discarding the floor symbol),

$$\begin{aligned} 2\binom{2d+k}{k} &= 2\binom{(2+m\sqrt{t})d}{m\sqrt{t}d} \\ &= \frac{2((2+m\sqrt{t})d)!}{(m\sqrt{t}d)!(2d)!} \\ &\leq \frac{c'\sqrt{(2+m\sqrt{t})d}}{\sqrt{2d}\sqrt{m\sqrt{t}d}}\left(\frac{(2+m\sqrt{t})^{2+m\sqrt{t}}}{(m\sqrt{t})^{m\sqrt{t}}2^2}\right)^d \\ &\leq c''\left(\frac{(2+m\sqrt{t})^{2+m\sqrt{t}}}{(m\sqrt{t})^{m\sqrt{t}}2^2}\right)^d \end{aligned} \tag{5}$$

as $\frac{\sqrt{(2+m\sqrt{t})d}}{\sqrt{2d}\sqrt{m\sqrt{t}d}} < 1$ for $d \geq 30$ and $t \geq 9$ with m being chosen appropriately. Now consider,

$$\begin{aligned} c''\left(\frac{(2+m\sqrt{t})^{2+m\sqrt{t}}}{(m\sqrt{t})^{m\sqrt{t}}2^2}\right)^d &= c''\left(\frac{(2+m\sqrt{t})^{m\sqrt{t}}}{(m\sqrt{t})^{m\sqrt{t}}}\frac{(2+m\sqrt{t})^2}{2^2}\right)^d \\ &= c''\left(\left(\left(1+\frac{2}{m\sqrt{t}}\right)^{\frac{m\sqrt{t}}{2}}\right)^2\frac{(2+m\sqrt{t})^2}{2^2}\right)^d \\ &\leq c''\left(e^2\frac{(2+m\sqrt{t})^2}{2^2}\right)^d. \end{aligned} \tag{6}$$

Here we have used the fact that $(1+x)^{\frac{1}{x}} \leq e$ for $x > 0$. Let the number of nodes in T_d^t be $n = \frac{(t^d-1)}{(t-1)} \geq t^{(d-1)}$. Therefore from Eqs. (5) and (6) we have

$$\begin{aligned} \frac{2\binom{2d+k}{k}}{n} &\leq \frac{2\binom{2d+k}{k}}{t^{(d-1)}} \\ &\leq \frac{c''\left(e^2\frac{(2+m\sqrt{t})^2}{2^2}\right)^d}{t^{(d-1)}} \\ &\leq c''t\left(e^2\left(\frac{1}{\sqrt{t}}+\frac{m}{2}\right)^2\right)^d \\ &= S\,(say). \end{aligned}$$

Clearly for large enough d i.e $d \geq c\log t$, $S < 1$ as $e^2(\frac{1}{\sqrt{t}} + \frac{m}{2})^2 < 1$ for the chosen value of m. This means that for large enough d, $\binom{2d+k}{k} < n$ which implies $b_e(T_d^t) \geq p \geq k \geq m\sqrt{t}d$. In our proof we have assumed that $t \geq 9$. This assumption can be removed by noting that for all values of $t < 9$ we can prove $b_e(T_d^t) > c'''d$ for some constant c''' using the same techniques as in the proof for the binary tree. So we can show $b_e(T_d^t) \geq c''''\sqrt{t}d$ by taking $c'''' = \frac{c'''}{3}$ since in this case $\sqrt{t} < 3$. This completes the proof that $b_e(T_d^t) \geq c_1\sqrt{t}d$ for all $t \geq 2$ where c_1 is an appropriately chosen constant. $\Delta = (t+1)$ in T_d^t. Therefore $b_v(T_d^t) \geq \frac{b_v(T_d^t)}{(t+1)} \geq \frac{c_1\sqrt{t}d}{(t+1)} \geq \frac{c_2 d}{\sqrt{t}}$. ■

These results can be generalized to an arbitrary tree.

Corollary 3. *Let $T = (V, E, r)$ be a rooted tree with $|V| = n$ and weight index η and $p \geq 2$. Then, $b_e(T) \geq c_1\eta(n^{(\frac{1}{2\eta})} - c_2)$ and $b_v(T) \geq \frac{c_1\eta(n^{(\frac{1}{2\eta})} - c_2)}{\Delta}$ where c_1 and c_2 are constants.*

Proof. We have, $n \leq \binom{2\eta+p}{p}$. Let $p = \omega\eta$. Then,

$$\begin{aligned} n &\leq 2\binom{2\eta+\omega\eta}{\omega\eta} \\ &= 2\binom{(2+\omega)\eta}{\omega\eta} \\ n &\leq \frac{2c\sqrt{2\pi(2+\omega)\eta}((2+\omega)\eta)^{(2+\omega)\eta}e^{-(2+\omega)\eta}}{(c'\sqrt{2\pi\omega\eta}(\omega\eta)^{\omega\eta}e^{-\omega\eta})(c'\sqrt{4\pi\eta}(2\eta)^{2\eta}e^{-2\eta})} \\ &\leq \frac{c''((2+\omega)\eta)^{(2+\omega)\eta}}{((\omega\eta)^{\omega\eta})((2\eta)^{2\eta})} \\ &\leq c'''\left(1+\frac{2}{\omega}\right)^{\omega\eta}\left(\frac{\omega}{2}+1\right)^{2\eta} \end{aligned}$$

where c, c', c'' and c''' are suitably chosen constants. We have used the fact that $2\eta + \omega\eta \leq 2\eta\omega\eta$ (which follows from the fact that $2\eta \geq 2$ and $p = \omega\eta \geq 2$). Simplifying this yields,

$$n^{\frac{1}{2\eta}} \leq c''''\left(1+\frac{2}{\omega}\right)^{\frac{\omega}{2}}\left(\frac{\omega}{2}+1\right).$$

Since $(1+x)^{\frac{1}{x}} \leq e$ for $x > 0$,

$$c_1 n^{\frac{1}{2\eta}} - 2 \leq \omega.$$

Since $b_e(T) \geq p$, $b_e(T) \geq c_1\eta(n^{(\frac{1}{2\eta})} - c_2)$ where c_1 and c_2 are suitably chosen constants. Therefore $b_v(T) \geq \frac{c_1\eta(n^{(\frac{1}{2\eta})} - c_2)}{\Delta}$ and the corollary follows. ■

Comment: It is interesting to study for what values of η the above result would be useful. A simple observation is that $n^{(\frac{1}{2\eta})} > c_2$. An analysis of the proof for the above result shows that $c_1 \geq \frac{1}{e}$ and thus $c_2 \leq 2e$. We note that $n = e^{\log n}$. For a tree T with $\eta \leq \frac{\log n}{4}$ we would have $b_e(T) \geq c\eta$ and $b_v(T) \geq \frac{c\eta}{\Delta}$ for a constant c. Similarly for a tree T with $\eta = k$ a constant we have $b_e(T) \geq c' n^{\frac{1}{2k}}$ and $b_v(T) \geq \frac{c' n^{\frac{1}{2k}}}{\Delta}$ for a constant c'.

5. Applications

5.1. Pathwidth

Pathwidth and Path decomposition are important concepts in graph theory and computer science. For the definition and several applications see [5]. It is not difficult to show that pathwidth$(G) \geq b_v(G)$ (see [8]). An obvious question is whether the reason for the high pathwidth of a graph G, is the "good" isoperimetric property of an induced subgraph or minor of G. More precisely if pathwidth$(G) \geq k$ is it possible to find an induced subgraph or minor G' of G such that $b_v(G') \geq f(k)$ for some function f, where $f(k)$ increases with k. Let us first consider whether such an induced subgraph always exists. The answer is negative: Given any integer k, it is possible to demonstrate a graph G (on arbitrarily large number of vertices) such that pathwidth$(G) \geq k$, but $b_v(G')$ for any induced subgraph of G is bounded above by a constant. For example, one can start with a complete binary tree of sufficiently large depth. The pathwidth of such a tree is $\Omega(d)$, where d is the depth. Now we can replace each edge of the binary tree with a path of appropriately chosen length, to make sure that for any induced subgraph T' of the resulting tree $b_v(T') \leq c$, where c is some constant. On the other hand, reader can easily verify that by replacing an edge with a path (i.e. by subdividing an edge) we can not decrease the pathwidth of the original graph. Thus the resulting tree will have pathwidth as much as that of the original. (We leave the rigorous proof of the above as an exercise to the reader.) But when we ask the same question with respect to minors, the answer is positive. Robertson and Seymour proved the following result. (See [12], Chapter 12.) *If* pathwidth$(G) \geq k$, *then there exists a function* g *such that every tree on at most* $g(k)$ *vertices is a minor of* G. Then clearly there exists a minor of G which is isomorphic to a complete binary tree T on at least $\frac{g(k)}{2}$ vertices. By our result (Corollary 1) $b_v(T) \geq c \log n = c' \log(g(k))$ where c and c' are appropriate constants. Thus we have the following result:

There exists a function f *such that if the pathwidth of a graph* G *is at least* k, *then there exists a minor* G' *of* G *such that* $b_v(G') \geq f(k)$.

5.2. Thinness

A new graph parameter *thinness*, is defined in [22] which attempts to generalize certain properties of interval graphs. The thinness of a graph $G = (V, E)$ is the minimum positive integer k such that there exists an ordering $v_1, v_2, \ldots, v_n$ (where $n = |V|$) of the vertices of G and a partition $V_1, V_2, \ldots, V_k$ of V into k disjoint sets, satisfying the following condition: For any triple (r, s, t) where $r < s < t$, if v_r and v_s belong to the same set V_i and if v_t is adjacent to v_r then v_t is adjacent to v_s also. The motivation for studying this parameter was the observation that the maximum independent set problem can be solved in polynomial time, if a family of graphs has bounded thinness. The applications of *thinness* for the Frequency Assignment Problems in GSM networks are explained in [22]. One interesting aspect of thinness is that for a graph G, thinness$(G) \leq$ pathwidth(G). A natural question which arose in connection with our study of thinness was the following: *Are trees of bounded thinness? In other words, is there a family of trees for which the thinness grows with the number of vertices?* It is proved in a later paper by the authors of [22] that for any graph G, thinness$(G) \geq \frac{b_v(G)}{\Delta}$ where Δ is the maximum degree of G. Combining this lower bound with our earlier observations, we can infer that the thinness of a complete binary tree on n vertices is $\Omega(\log n)$.

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