A combined mixed and discontinuous Galerkin method for compressible miscible displacement problem in porous media

Mingrong Cui

School of Mathematics and System Sciences, Shandong University, Jinan 250100, Shandong, PR China

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Abstract

In this paper, we present a numerical scheme for solving the coupled system of compressible miscible displacement problem in porous media. The flow equation is solved by the mixed finite element method, and the transport equation is approximated by a discontinuous Galerkin method. The scheme is continuous in time and a priori hp error estimates is presented.

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1. Introduction

We consider the compressible miscible model problem, which is given by the following equations with boundary and initial conditions [16]:

\[
\begin{align*}
\frac{d(c)}{dt} + \nabla \cdot u &= d(c) \frac{\partial p}{\partial t} - \nabla \cdot (a(c) \nabla p) = q, \quad (x, t) \in \Omega \times J, \\
\phi \frac{\partial c}{\partial t} + b(c) \frac{\partial p}{\partial t} + u \cdot \nabla c - \nabla \cdot (D \nabla c) &= (\hat{c} - c)q, \quad (x, t) \in \Omega \times J, \\
u \cdot v &= 0, \quad (x, t) \in \partial \Omega \times J, \\
D \nabla c \cdot v &= 0, \quad (x, t) \in \partial \Omega \times J, \\
p(x, 0) &= p_0(x), \quad x \in \Omega, \\
c(x, 0) &= c_0(x), \quad x \in \Omega.
\end{align*}
\]

(1.1)

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Tel.: +86 0531 8837 8967; fax: +86 531 8856 4652.

E-mail address: mrcui@math.sdu.edu.cn.

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Here \( \Omega \) is a polygonal domain in \( \mathbb{R}^d \) \((d = 2, 3), J = (0, T) \). The fluid pressure is denoted by \( p \), the Darcy velocity
\( u = -a(c) \nabla p \), \( \phi = \phi(x) \) is the porosity, \( c = c(x, t) \) is the solvent (volumetric) concentration, and \( q \) is the external volumetric flow rate. The permeability of the medium is denoted by \( k(x) \), and \( \mu(c) \) is the viscosity.

In model (1.1), we confine ourselves to a two component displacement problem just for clarity of presentation. However, the numerical methods that we shall introduce and analyze below can be applied to the \( n \) component model. The coefficients appearing in (1.1) can be stated as follows:

\[
c = c_1 = 1 - c_2,
\]
\[
a(c) = a(x, c) = k(x)\mu(c)^{-1},
\]
\[
b(c) = b(x, c) = \phi(x)c_1 \left\{ z_1 - \sum_{j=1}^{2} z_j c_j \right\},
\]
\[
d(c) = d(x, c) = \phi(x) \sum_{j=1}^{2} z_j c_j,
\]

where \( z_j \) is the “constant compressibility” factor for the \( j \)th component.

In problem (1.1), the matrix \( D = D(x) = \phi(x) d_m I \), and the notation \( \hat{c} \) denotes the specified \( c_w \) at sources \((q > 0)\) and the resident concentration at sinks \((q < 0)\). If we put \( q^+ = \max(q, 0) \) and \( q^- = \min(q, 0) \), then \( q = q^+ + q^- \). We assume that the flow rate \( q \) is smoothly distributed in order to imply that the solution of the problem is smooth.

For problem (1.1), we need the following hypotheses (H):

1. The mixture viscosity \( \mu(c) \) has positive lower and upper bounds, and its derivative is uniformly Lipschitz continuous.
2. There exist positive constants \( k_s, k^*, \phi_s, \phi^*, D_s, D^*, b^*, d_s \) and \( d^* \) such that
\[
0 < k_s \leq k(x) \leq k^*, \quad 0 < \phi_s \leq \phi(x) \leq \phi^*, \quad 0 < D_s \leq D(x) \leq D^*, \quad |b(k)| \leq b^*, \quad 0 < d_s \leq d(k) \leq d^*, \quad \kappa \in \mathbb{R}^1.
\]
3. There are two positive constants \( K_1 \) and \( K_2 \) such that
\[
|q| \leq K_1, \quad \left| \frac{\partial q}{\partial t} \right| \leq K_2.
\]

We make a few remarks for (2) in hypotheses (H). First, in real computations, once an approximate solution \( C \) for \( c \) is obtained, then we truncate \( C \) to \([0, 1] \), i.e., we use \( C^* = \min(\max(C, 0), 1) \) instead of \( C \) [32]. For the brevity of presentation, we simply use \( \kappa \in \mathbb{R}^1 \) instead of \( \kappa \in [0, 1] \). Secondly, the above assumptions made for \( b(c) \) and \( d(c) \) are reasonable, as it is easy to check that \( \min_{j} z_j \leq \sum_{j=1}^{2} z_j c_j \leq \max_{j} z_j \) as \( \sum_{j=1}^{2} c_j = 1 \) and \( c_j \geq 0 \) \((j = 1, 2)\). Under the above assumptions, we know that \( \partial a/\partial c \) is uniformly bounded and Lipschitz continuous with respect to \( c \).

In this paper, we consider the numerical solutions for the above coupled equations. First, we consider the numerical methods for the flow equation. To obtain a velocity by differencing or differentiating the resulting pressure determined by standard finite difference and finite element method then multiplying it by the rough coefficient will result in a rough and inaccurate velocity which will reduce the accuracy of numerical simulation of the fluid flow in porous media [17]. Mixed finite element method has the advantages that both the pressure and the velocity can have the same optimal order of convergence, and this method has been widely used in the numerical simulation for porous media problems since the early period of 1980s [14,15]. The \( p \) approximation results using the Raviart–Thomas–Nédélec spaces were given in [21,36], and the hp-version was presented in [2,20,22].

Now we turn to the approximation schemes for the concentration equation. Discontinuous Galerkin finite element methods (DGFEMs) have become very popular in the science and engineering community now. They were introduced in the early seventies in the last century for solving the neutron transport equation [26]. In the paper written by Cockburn et al. [11], a general survey and a historical review were provided. In 1998, Oden et al. [24] presented an extension of the discontinuous Galerkin method for diffusion problems. Rivière et al. discussed DG methods for elliptic problems...
and time-dependent convection–diffusion systems by introducing a penalty term on the jumps of the solution across the element interfaces [27,29]. The framework to represent various types of DGFEMs for elliptic problems was discussed in [4]. Mixed hp-DGEM for Stokes problems was given in paper [33]. Mixed discontinuous Galerkin for computational electromagnetics appeared in [18].

DGFEMs have several advantages over other types of finite element methods. For example, no continuity constraints are explicitly imposed on the trial and test functions across the finite element interfaces, thus the spaces are easy to construct, and the use of highly nonuniform and unstructured meshes is permitted.

To our knowledge, there are only a few articles (see, for example, [28]) on the DG methods for miscible displacement problems up to now. In paper [34] continuous in time scheme consisting of the mixed finite element and nonsymmetric interior penalty Galerkin method for incompressible miscible displacement problem in porous media was given, and in recent paper [35], continuous in time schemes of primal discontinuous Galerkin methods with interior penalty for incompressible miscible displacement problem were proposed. The scheme and numerical comparisons made between DG and other locally conservative methods were given in [28]. Compressible case was discussed in paper [8], but without any proof for the induction hypothesis. In paper [12] we present discontinuous Galerkin method for both flow and transport equations for compressible case with error estimates.

In this paper, we present a mixed finite element method and a discontinuous Galerkin method to treat the flow equation and the transport equation, respectively. The features of this paper are in two aspects. First, we use the induction hypothesis as a tool in our proof, instead of the cut-off operator used in papers [34,8], thus we avoid the difficulty for the proper choice for the positive constant appearing in the operator (which should be large enough). Secondly, we use local spaces of different polynomial orders for the transport equation, thus, they are ideal for hp-adaptivity. Rate of convergence for concentration in $L^2(H^1)$ norm and $L^\infty(L^2)$ rate of convergence for velocity are proved. We present the hp convergence results.

The paper is organized as follows. In Section 2 we give the mixed finite element method for the flow equation and a discontinuous Galerkin method for the transport equation. Error estimates are given in Section 3. We use the induction hypothesis and the Cauchy–Schwarz inequality in our proof.

Throughout this paper, the symbols $K$, $K_i$ ($i \in N$) and $L$ will denote generic positive constants, independent of $x$, $t$ and all mesh parameters. They may take different values at different occurrences. The symbol $\varepsilon$ will denote a generic small positive constant.

2. The mixed finite element and discontinuous Galerkin scheme

In this section we give the mixed finite element scheme to solve the flow equation and a discontinuous Galerkin scheme for the concentration equation. We begin with the subdivision of $\Omega$ and a few notations (see, for example, [30]).

Let us consider a quasi-uniform family $\{\mathcal{T}_h\}$ of $\Omega$ with $\mathcal{T}_h = \{E_1, E_2, \ldots, E_{N_h}\}$, the element $E_i$ is triangle or quadrilateral for two dimensional domain ($d = 2$), or tetrahedron for three dimensional domain ($d = 3$). The partition of $\Omega$ should be conforming, as both the flow and transport equations will use the same mesh, and mixed finite element spaces (standard Raviart–Thomas–Nédélec spaces) will be used. We assume that $\bigcup_{i=1}^{N_h} E_i = \Omega$, as $\Omega$ is a polygonal domain. Each $E \in \mathcal{T}_h$ is an affine image of a fixed master element $\hat{E}$; i.e., $E = F_E(\hat{E})$ for all $E \in \mathcal{T}_h$, where $\hat{E}$ is either the open unit simplex or the open unit hypercube in $R^d$. The regularity requirement is that the element is convex and there exists $\rho > 0$ such that if $h_i$ is the diameter of $E_i \in \mathcal{T}_h$, then for each of the sub-triangles (for $d = 2$) or sub-tetrahedra (for $d = 3$) of element $E_i$ contains a ball of radius $\rho h_i$ in its interior. We set, as usual, $h = \max_{E_i \in \mathcal{T}_h} h_i$. The set of all interior edges ($d = 2$) or faces ($d = 3$) for $\mathcal{T}_h$ is denoted by $\Gamma_h$.

The usual Sobolev inner product is denoted by $(\cdot, \cdot)$ and the norm on $\Omega$ is denoted by $\| \cdot \|_{m,\Omega}$ [1]. Similar notations apply for the element $E$ and face/edge $\gamma$. We simply write $\| \cdot \|$ for $\| \cdot \|_{0,\Omega}$.

We give the approximation spaces for the flow equation first. As the solution $p$ of (1.1) is determined only up to an additive constant, therefore we will use a closed subspace of $L^2(\Omega)$ consisting of functions with vanishing mean value. Thus, we define the following spaces:

$$V = H(\text{div}; \Omega) = \{u | (L^2(\Omega))^d : \text{div } u \in L^2(\Omega)\},$$

$$W = L^2(\Omega).$$
Let the approximating subspace \( V_k(\mathcal{T}_h) \times W_k(\mathcal{T}_h) \) of \( V \times W \) be the \( k \)th \((k \geq 0)\) order Raviart–Thomas–Nédélec spaces [25,23] of the partition \( \mathcal{T}_h \). We know that \( u \) and \( p \) can be approximated to the same order of accuracy in \( L^2 \) in this space. Denoted by \( V^0 \) the subspace of \( V \) consisting of functions with normal trace on \( \partial \Omega \) (in the sense defined by, for example, [7]) equal to 0, i.e.,

\[
V^0 = H_0(\text{div}; \Omega) = \{ u \in H(\text{div}; \Omega), \langle u \cdot v, v \rangle = 0, \forall v \in H^1(\Omega) \},
\]

where the bracket \( \langle \cdot, \cdot \rangle \) denotes duality between \( H^{-1/2}(\partial \Omega) \) and \( H^{1/2}(\partial \Omega) \). Then, we can define the following subspace:

\[
V_k^0(\mathcal{T}_h) = V_k(\mathcal{T}_h) \cap V^0
\]

which will be used to solve the velocity \( u \).

Now we give the discrete approximation space for the transport equation. Note that different from the flow equation, local spaces of different polynomial orders for the transport equation will be used. The discontinuous finite element space is given as follows [19]. To each \( E \in \mathcal{T}_h \) we assign a nonnegative integer \( r_E \) (local polynomial degree) and a nonnegative integer \( s_E \) (local Sobolev index), collect the \( r_E, s_E \) and \( F_E \) in the vectors \( \mathbf{r} = \{ r_E : E \in \mathcal{T}_h \} \), \( \mathbf{s} = \{ s_E : E \in \mathcal{T}_h \} \), and consider the finite element space

\[
S^\mathbf{r}(\Omega, \mathcal{T}_h, \mathbf{F}) = \{ v \in L^2(\Omega) : v|_E \circ F_E \in R_{pE} \},
\]

where \( R_{pE} \) is either \( P_{pE}(\hat{E}) \) (for triangular partition) or \( Q_{pE}(\hat{E}) \) (for quadrilateral partition), and

\[
P_n(\hat{E}) = \text{span}\{ \hat{x}^a : 0 \leq |a| \leq n \}, \quad Q_n(\hat{E}) = \text{span}\{ \hat{x}^a : 0 \leq a_i \leq n, 1 \leq i \leq d \}
\]

with \( a = (a_1, \ldots, a_d) \) being the multi-index notation [10]. Further, associated with \( \mathcal{T}_h \), the broken Sobolev space of composite order \( \mathbf{s} \) is defined by

\[
H^\mathbf{s}(\mathcal{T}_h) = \{ v \in L^2(\Omega) : v|_E \in H^{s_E}(E), E \in \mathcal{T}_h \}
\]

and it is equipped with the broken Sobolev norm

\[
\| v \|_s = \left( \sum_{E \in \mathcal{T}_h} \| v \|_{H^{s_E}(E)}^2 \right)^{1/2}
\]

with respect to the subdivision \( \mathcal{T}_h \).

For the simplicity of presentation, we put \( \| v \| = \| v \|_0 \). If \( v \) is a \( d \)-dimensional vector function, then \( \| v \|_s \) stands for \( (\sum_{E \in \mathcal{T}_h} \| v \|_{H^{s_E}(E)}^2)^{1/2} \). The inner product \( \sum_{E \in \mathcal{T}_h} \int_E \phi \psi \) is also denoted by \( \langle \phi, \psi \rangle \) for \( \phi, \psi \in S^\mathbf{r}(\Omega, \mathcal{T}_h, \mathbf{F}) \).

Since piecewise polynomial functions are used in discontinuous Galerkin methods, consequently, there is no continuity constraint across element interfaces. As a result, jump terms across interfaces must be included in the variational formulations [38]. Therefore, we adopt the following notations.

Let \( E_i, E_j \in \mathcal{T}_h \) and \( \gamma = \partial E_i \cap \partial E_j \in \Gamma_h \) with \( n_\gamma \) exterior to \( E_i \). For \( v \in H^s(\mathcal{T}_h) \), \( s > \frac{1}{2} \), we define the mean value of \( v \) on \( \gamma \) and the jump of \( v \) across \( \gamma \), respectively, by

\[
\{ v \} = \frac{1}{2} (v|_{E_i})|_{\gamma} + (v|_{E_j})|_{\gamma}, \quad [v] = (v|_{E_i})|_{\gamma} - (v|_{E_j})|_{\gamma}.
\]

Now we can define

\[
b(u; c, w) = \sum_{E \in \mathcal{T}_h} \int_E D \nabla c \cdot \nabla w - \sum_{\gamma \in \Gamma_h} \int_\gamma \{ D \nabla c \cdot n_\gamma \}[w] - \sum_{\gamma \in \Gamma_h} \int_\gamma \{ D \nabla w \cdot n_\gamma \}[c]
\]

\[
+ \sum_{E \in \mathcal{T}_h} \int_E (u \cdot \nabla c) w + J^R_T(c, w)
\]
The function $J_T^\beta(c, w)$ is the interior penalty term as there is no continuity constraint imposed across the element interfaces in the discontinuous Galerkin method. We give its definition now. For $x$ in some $\gamma = \partial E_i \cap \partial E_j \in \Gamma_h$, and assume that $\gamma \neq \phi$, we first define the functions $h_\gamma$ be the measure of $\gamma$ and $r(x) \in L^\infty(\Gamma_h)$ by (cf. [33])

$$r_\gamma(x) = r(x)|_\gamma = \max\{r_{E_i}, r_{E_j}\}, \quad \gamma \in \Gamma_h.$$  

The reasons are that the partition is conforming, thus no hanging node is allowed, and different order of polynomials can be used in adjacent elements. Then we set

$$J_T^\beta(c, w) = \sum_{\gamma \in \Gamma_h} \int_{\gamma} \beta_\gamma r_\gamma^2 h_\gamma^{-1}[c][w].$$

Here $\beta_\gamma$ is a positive constant on each edge of face $\gamma$, and we assume that $r_{E_i}/r_{E_j}$ is bounded from below and above, and $0 < \beta_{\gamma, \min} \leq \beta_\gamma \leq \beta_{\gamma, \max}$. The parameter $\beta_\gamma$ is independent of $h$ and $r$, and it should be taken large enough [13].

Let $a(c) = 1/a(c)$, then we can give the numerical scheme which solves the flow equation by the mixed finite element method and the concentration equation by discontinuous Galerkin method. The continuous in time scheme for solving problem (1.1) is given as follows.

When $t = 0$, we set $C(x, 0) = c_0$ and $U(x, 0) = U_0$ is determined by [37]

$$(\sigma(c_0) U_0, v) - (\nabla \cdot v, \tilde{u}_0) = 0, \quad \forall v \in V_k^0(\mathcal{T}_h). \tag{2.1a}$$

Then for $t > 0$, the unknowns $U(\cdot, t) : J \rightarrow L^\infty(V_k^0(\mathcal{T}_h)), P(\cdot, t) : J \rightarrow L^\infty(W_k(\mathcal{T}_h))$ and $C(\cdot, t) : J \rightarrow S^r(\Omega, \mathcal{T}_h, \mathbf{F})$ are given by

$$\begin{cases}
(d(c) \frac{\partial P}{\partial t}, w) + (\nabla \cdot U, w) = (q, w), & \forall w \in W_k(\mathcal{T}_h), \\
(\sigma(C) U, v) - (\nabla \cdot v, P) = 0, & \forall v \in V_k^0(\mathcal{T}_h), \\
(d[C] \frac{\partial C}{\partial t}, \psi) + \left( b(C) \frac{\partial P}{\partial t}, \psi \right) + b(U; C, \psi) = L_T(C, \psi), & \forall \psi \in S^r(\Omega, \mathcal{T}_h, \mathbf{F}).
\end{cases} \tag{2.1b}$$

Here, $c_0$ (resp. $\tilde{u}_0$) denotes the interpolant of $c_0$ (resp. $p_0$) to be defined below.

The nonlinear system (2.1) has one unique solution, because we can change it into an initial value problem of coupled nonlinear first-order ordinary differential equations after representing the solution by the basis functions, as in paper [39]. By the theory of ordinary differential equations, the existence and uniqueness of the solution of (2.1) follows from the assumptions on the coefficients. Therefore, (2.1) is solvable.

3. Error analysis of the numerical scheme

We now introduce a projection which will play an important role in establishing the error estimates [16]. The approximation results can be found in [20, 22]. Let $(\tilde{u}, \tilde{p})$, the projection of the Darcy velocity and the pressure be given as the solution of the following elliptic mixed method equations:

$$\begin{cases}
(d(c) \frac{\partial \tilde{p}}{\partial t}, w) + (\nabla \cdot \tilde{u}, w) = (q, w), & \forall w \in W_k(\mathcal{T}_h), \\
(\sigma(c) \tilde{u}, v) - (\nabla \cdot v, \tilde{p}) = 0, & \forall v \in V_k^0(\mathcal{T}_h), \\
(\tilde{p}, 1) = (p, 1).
\end{cases} \tag{3.1}$$
Separate the errors for the pressure and velocity as follows:

\[ p - P = (p - \bar{p}) + (\bar{p} - P) \equiv \eta + \pi, \]
\[ u - U = (u - \bar{u}) + (\bar{u} - U) \equiv \rho + \sigma, \]

and initialize the pressure by taking

\[ P(\cdot, 0) = \bar{p}(0) = \bar{p}_0, \]

then we get \( \pi(0) = 0 \).

Following the method given in [16], it is easy to see that the projection error satisfies the equations

\[
\begin{align*}
\rho_0 + \eta_0 & \leq K \sum_{E \in \mathcal{T}_h} h^{\min(k+1, \omega_E - 1)} \left( \| \rho \|_{\omega_E, E} + \left\| \frac{\partial \rho}{\partial t} \right\|_{\omega_E, E} \right), \\
\frac{\partial \rho}{\partial t} + \frac{\partial \eta}{\partial t} & \leq K \sum_{E \in \mathcal{T}_h} h^{\min(k+1, \omega_E - 1)} \left( \| \rho \|_{\omega_E, E} + \left\| \frac{\partial \rho}{\partial t} \right\|_{\omega_E, E} \right). (3.2)
\end{align*}
\]

Here, \( k \) is the order of the Raviart–Thomas–Nédélec spaces. As \( \| \rho \|_{H(\operatorname{div}; \Omega)} \) does not appear in the following estimates, therefore, instead of the \( H(\operatorname{div}; \Omega) \) norm of \( \rho \) in paper [16], we need the \( L^2 \) norm, so we lower the regularity of \( p \) (from \( k + 3 \) to \( k + 2 \), i.e., \( \omega \) can be \( k + 2 \)).

For the estimation of the error in the concentration equation, we need to separate the error into two parts, where one part lies in the finite element space \( S^r(\Omega, \mathcal{T}_h, \mathbf{F}) \). We will use the following known hp approximation results [5,6].

Let \( E \in \mathcal{T}_h, w \in H^s(E) \), then there exists a constant \( K \) depending on \( s, E \) and \( \rho \) but independent of \( w, r \) and \( h \), and there is a sequence \( \zeta^k \in P_r(E) \) (or \( Q_r(E) \)), \( r = 1, 2, \ldots \), such that for any \( 0 \leq q \leq s \),

\[
\begin{align*}
\| w - z^k_r \|_{q, E} & \leq K \frac{h^{k-q}}{r^{s-q}} \| w \|_{s, E}, \quad s \geq 0, \\
\| w - z^k_r \|_{0, E} & \leq K \frac{h^{k-1/2}}{r^{s-1/2}} \| w \|_{s, E}, \quad s > \frac{1}{2}, \\
\| w - z^k_r \|_{1, E} & \leq K \frac{h^{k-3/2}}{r^{s-3/2}} \| w \|_{s, E}, \quad s > \frac{3}{2},
\end{align*}
\]

where \( \mu = \min(r + 1, s) \) and \( \gamma \) is an edge or a face on \( E \). We will often use the results for \( q = 1 \) and \( q = 2 \) in this paper, and we will use the above approximation results for the function \( w \) being the function which is the derivative with respect to the time variable \( t \).

Let \( \tilde{c} \in S^r(\Omega, \mathcal{T}_h, \mathbf{F}) \) be the interpolant of \( c \), having the above optimal hp approximation errors. The finite element solution error for the concentration is separated as follows:

\[ c - C = (c - \tilde{c}) + (\tilde{c} - C) \equiv \zeta + \tilde{\zeta}, \]

then \( \zeta \in S^r(\Omega, \mathcal{T}_h, \mathbf{F}) \) and we take \( C(x, 0) = \tilde{c}_0 \), as declared before.
Note that the estimates for \( \eta, \zeta, \partial \eta / \partial t \) and \( \partial \zeta / \partial t \) have been given above, we will give the estimates for \( \pi, \sigma \) and \( \xi \) below. Then we can obtain the error estimates by using triangle inequality.

In order to drive the error estimates, the following two trace inequalities provided in \([30,3,31]\) will be used frequently.

**Lemma 3.1.** The following trace inequalities hold for all \( v \in H^1(E_j) \).

\[
\|v\|_{L^2(\gamma_j)}^2 \leq K (h_j^{-1} \|v\|_{L^2(E_j)}^2 + h_j \|v\|_{L^1(E_j)}),
\]

\[
\|\nabla v \cdot n_{\gamma_j}\|_{L^2(\gamma_j)}^2 \leq K (h_j^{-1} \|\nabla v\|_{L^2(E_j)}^2 + \|\nabla v\|_{L^2(E_j)} \|\nabla^2 v\|_{L^2(E_j)}),
\]

where \( \gamma_j \) is an edge or a face on \( E_j \), \( n_{\gamma_j} \) is the unit normal vector on it, \( h_j \) is the diameter of \( E_j \).

We will use the following inverse inequalities true for functions in finite dimensional spaces from paper \([30]\) as well.

**Lemma 3.2.** Let \( E \) be an element in \( \mathbb{R}^d \) (\( n = 2, 3 \)) of diameter \( h_E \), let \( e_k \) be an edge or a face of \( E \), and let \( v_k \) be a unit vector normal to \( e_k \). Then, if \( \chi \) is a polynomial of degree \( r \) on \( E \), then there exists a constant \( K \) independent of \( E \) and \( r \) such that

\[
\|
\|
\|
\leq K r h_E^{-1/2} \|\chi\|_{L^2(E)},
\]

\[
\|
\|
\|
\leq K r h_E^{-1/2} \|\nabla \chi\|_{L^2(E)}.
\]

Now we can give the following theorem describing the property for the convergence of the mixed finite element and discontinuous Galerkin scheme (2.1).

**Theorem 3.1.** We assume that the hypotheses \((H)\) in Section 1 holds, and the regularity of the true solution of (1.1) is given as follows.

1. \( p \in L^2(J; H^0(\mathcal{T}_h)), \partial p / \partial t \in L^2(J; H^0(\mathcal{T}_h)); \)
2. \( c \in L^2(J; H^2(\mathcal{T}_h)), \partial c / \partial t \in L^2(J; H^1(\mathcal{T}_h)); \)
3. \( p, \nabla p, c \) and \( \nabla c \) are essentially bounded.

Let \( h = \max_{E \in \mathcal{T}_h} h_E \), we assume that the parameters satisfy

\[
h_E^{\min(r_E, \lambda E - 1, \mu E - 1, \omega E - 1, k + 1)} = o(h^{d/2}), \quad \forall E \in \mathcal{T}_h.
\]

Then, there exists a constant \( K > 0 \) independent of \( h \) and \( r \), and a constant \( h_0 > 0 \) such that

\[
\sup_{t \in J} \left\{ \left( \int_0^t \left\| \frac{\partial}{\partial t} (c - C) \right\|^2 dt \right)^{1/2} + \left( \int_0^t \left\| \frac{\partial}{\partial t} (p - P) \right\|^2 dt \right)^{1/2} + \|c - C\|_c(t) \right. \\
+ \left. \left\| \nabla (c - C) \right\|_c(t) + \|p - P\|_p(t) + \|u - U\|_u(t) + (J_E^p(c - C, c - C)(t))^{1/2} \right\} \right.

\[
\leq K \sum_{E \in \mathcal{T}_h} \left[ \frac{h_E^{\min(r_E, \lambda E - 1)}}{r_E^{3/2}} \|c\|_{L^2(E)} + \frac{h_E^{\min(r_E, \mu E - 1)}}{r_E^{3/2}} \|\partial c / \partial t\|_{L^2(E)} \\
+ \frac{h_E^{\min(k + 1, \omega E - 1)}}{k^{3/2}} \left( \|p\|_{L^2(E)} + \|\partial p / \partial t\|_{L^2(E)} \right) \right],
\]

holds for any \( 0 < h \leq h_0 \) and \( 0 \leq t \leq T \). Here, the integers \( \lambda, \mu \) and \( \omega \) describe the regularity orders of functions \( c, \partial c / \partial t \) and \( p \), integer \( r_E \) is the order of discontinuous space for the concentration on element \( E \), and \( k \) is the order of Raviart–Thomas–Nédélec space approximating \( p \) and \( u \).
Proof. We will present the proof in two steps. We consider the flow equation first. Subtract the first two equations in (3.1) from (2.1b), we have

\[
\left( d(c) \frac{\partial p}{\partial t} - d(C) \frac{\partial P}{\partial t}, v \right) + (\nabla \cdot \sigma, w) = 0, \quad \forall w \in W_k(\mathcal{F}_h),
\]

which lead to

\[
(\sigma(C) \tilde{u} - \sigma(C) U, v) - (\nabla \cdot v, \pi) = 0, \quad \forall v \in V_k^0(\mathcal{F}_h),
\]

Now we can put (3.7) and (3.9) together to get

\[
\left( d(C) \frac{\partial \pi}{\partial t}, \frac{\partial \pi}{\partial t} \right) + (\nabla \cdot \sigma, w) = \left( \left( d(C) - d(c) \right) \frac{\partial \tilde{p}}{\partial t}, \frac{\partial \pi}{\partial t} \right) - \left( d(c) \frac{\partial \eta}{\partial t}, \frac{\partial \pi}{\partial t} \right), \quad \forall w \in W_k(\mathcal{F}_h),
\]

Choosing the test functions \( w = \partial \pi / \partial t \) in the first equation, and differentiate the second equation of (3.6) with respect to time, and then choose \( v = \sigma \), we can have

\[
\left( d(C) \frac{\partial \pi}{\partial t}, \frac{\partial \pi}{\partial t} \right) + \left( \nabla \cdot \sigma, \frac{\partial \pi}{\partial t} \right) = \left( \left( d(C) - d(c) \right) \frac{\partial \tilde{p}}{\partial t}, \frac{\partial \pi}{\partial t} \right) - \left( d(c) \frac{\partial \eta}{\partial t}, \frac{\partial \pi}{\partial t} \right),
\]

and

\[
\left( \frac{\partial}{\partial t} (\sigma(C) \sigma), \sigma \right) - \left( \nabla \cdot \sigma, \frac{\partial \pi}{\partial t} \right) = \left( \frac{\partial}{\partial t} [(\sigma(C) - \sigma(c)) \tilde{u}], \sigma \right).
\]

Note that

\[
\frac{d}{dt} (\sigma(C) \sigma) = 2 \left( \frac{\partial}{\partial t} (\sigma(C) \sigma), \sigma \right) - \left( \frac{\partial}{\partial t} (\sigma(C) \sigma), \sigma \right),
\]

we see that

\[
\frac{1}{2} \frac{d}{dt} (\sigma(C) \sigma) - \left( \nabla \cdot \sigma, \frac{\partial \pi}{\partial t} \right) = \left( \frac{\partial}{\partial t} [(\sigma(C) - \sigma(c)) \tilde{u}], \sigma \right) - \frac{1}{2} \left( \frac{\partial}{\partial t} (\sigma(C) \sigma), \sigma \right).
\]

Now we can put (3.7) and (3.9) together to get

\[
\left( d(C) \frac{\partial \pi}{\partial t}, \frac{\partial \pi}{\partial t} \right) + \frac{1}{2} \frac{d}{dt} (\sigma(C) \sigma)
\]

\[
= \left( \left( d(C) - d(c) \right) \frac{\partial \tilde{p}}{\partial t}, \frac{\partial \pi}{\partial t} \right) - \left( d(c) \frac{\partial \eta}{\partial t}, \frac{\partial \pi}{\partial t} \right) + \left( \frac{\partial}{\partial t} [(\sigma(C) - \sigma(c)) \tilde{u}], \sigma \right)
\]

\[
- \frac{1}{2} \left( \frac{\partial}{\partial t} (\sigma(C) \sigma), \sigma \right).
\]

As to the right-hand side, using Cauchy–Schwarz inequality, we can get

\[
\left| \left( d(C) - d(c) \right) \frac{\partial \tilde{p}}{\partial t}, \frac{\partial \pi}{\partial t} \right| \leq K (\| \xi \| + \| \zeta \|) \left\| \frac{\partial \pi}{\partial t} \right\|^2 + K (\| \xi \|^2 + \| \zeta \|^2),
\]

\[
\left| \left( d(C) \frac{\partial \eta}{\partial t}, \frac{\partial \pi}{\partial t} \right) \right| \leq \epsilon \left\| \frac{\partial \pi}{\partial t} \right\|^2 + K \left\| \frac{\partial \eta}{\partial t} \right\|^2,
\]

\[
\left| \frac{\partial}{\partial t} [(\sigma(C) - \sigma(c)) \tilde{u}], \sigma \right| \leq \epsilon \left[ \left\| \frac{\partial \tilde{u}}{\partial t} \right\| + \left\| \frac{\partial \sigma}{\partial t} \right\| \right] \leq \epsilon \left\| \frac{\partial \tilde{u}}{\partial t} \right\|^2 + K \left[ \| \xi \|^2 + \| \zeta \|^2 \right].
\]

\[
\leq K \| \sigma \| \left[ \left\| \frac{\partial \xi}{\partial t} \right\| + \left\| \frac{\partial \zeta}{\partial t} \right\| + \| \xi \| + \| \zeta \| \right] \leq \epsilon \left\| \frac{\partial \xi}{\partial t} \right\|^2 + K \left[ \| \xi \|^2 + \| \zeta \|^2 \right].
\]
In order to give estimates for the last term on the right-hand side of (3.10), i.e., the integral of a multiply of three functions, we need to bound one function by its $L^\infty$ norm, then we can use Hölder inequality to bound the integral. Therefore, we need an induction hypothesis

$$\|\sigma\|_{L^\infty([0,T]; L^\infty(\Omega))} \leq L$$

(3.11)

holds true for some positive constant $L$. It is easy to see that under this induction hypothesis, $\|U\|_{L^\infty([0,T]; L^\infty(\Omega))}$ is bounded, too. The induction hypothesis technique has been used in paper [16].

With the induction hypothesis (3.11), it is easy to see that

$$\left| \left( \frac{\partial \sigma}{\partial c}(C) \frac{\partial c}{\partial t} \sigma, \sigma \right) \right| = \left| \left( \frac{\partial \sigma}{\partial c}(C) \frac{\partial \sigma}{\partial t}, \sigma \right) + \left( \frac{\partial \sigma}{\partial c}(C) \frac{\partial \sigma}{\partial t} \sigma, \sigma \right) \right| \leq K \|\sigma\|^2$$

$$+ K \left\| \frac{\partial \sigma}{\partial t} \right\| \|\sigma\|_{L^\infty([0,T]; L^\infty(\Omega))} \|\sigma\|$$

$$\leq \epsilon \left\| \frac{\partial \sigma}{\partial t} \right\|^2 + K \|\sigma\|^2.$$ 

Combining the above bounds, after integration with respect to $t$, and note that $\sigma(0) = 0$, we get

$$d_s \int_0^t \left\| \frac{\partial \sigma}{\partial t} \right\|^2 + (\alpha(C) \sigma, \sigma)(t)$$

$$\leq \epsilon \int_0^t \left\| \frac{\partial \sigma}{\partial t} \right\|^2 + K \int_0^t \left[ \|\sigma\|^2 + \|\sigma\|^2 + \left\| \frac{\partial \sigma}{\partial t} \right\|^2 + \right.\left\| \frac{\partial \sigma}{\partial t} \right\|^2 \right].$$

(3.12)

We now come to the second step, to deal with the transport equation. If $(p, u, c)$ is the true solution of (1.1), on each element $E_i$, we multiply the second equation of (1.1) by a test function $\psi_{E_i}$, then integrate over the element $E_i$. After applying Green’s formula and summing over $i$ (extend each $\psi_{E_i}$ to zero outside of $E_i$), we can put $\psi = \sum_{E_i \in \mathcal{F}_h} \psi_{E_i}$ to obtain

$$\sum_{E \in \mathcal{F}_h} \int_E \phi \frac{\partial c}{\partial t} \psi + \sum_{E \in \mathcal{F}_h} \int_E b(c) \frac{\partial p}{\partial t} \psi + \sum_{E \in \mathcal{F}_h} \int_E (u \cdot \nabla c) \psi + \sum_{E \in \mathcal{F}_h} \int_E D\nabla c \cdot \nabla \psi$$

$$- \sum_{\gamma \in \mathcal{G}_h} \int_{\gamma} (D\nabla c \cdot n) \gamma \psi = \int_{\Omega} (\hat{c} - c) q \psi.$$ 

With the boundary condition it comes to

$$\left( \frac{\partial c}{\partial t}, \psi \right) + \left( b(c) \frac{\partial p}{\partial t}, \psi \right) + \sum_{E \in \mathcal{F}_h} \int_E D\nabla c \cdot \nabla \psi + \sum_{E \in \mathcal{F}_h} \int_E (u \cdot \nabla c) \psi - \sum_{\gamma \in \mathcal{G}_h} \int_{\gamma} (D\nabla c \cdot n) \gamma \psi$$

$$- \sum_{\gamma \in \mathcal{G}_h} \int_{\gamma} (D\nabla \psi \cdot n) \gamma \psi + J^B_{\mathbb{F}}(c, \psi) = \int_{\Omega} (\hat{c} - c) q \psi.$$ 

Therefore, as $[c] = 0$, the following weak formulation in the broken Sobolev space $S^r(\Omega, \mathcal{F}_h, \mathbb{F})$ holds true for the true solution of (1.1).

$$\left( \frac{\partial c}{\partial t}, \psi \right) + \left( b(c) \frac{\partial p}{\partial t}, \psi \right) + b(u; c, \psi) = \mathcal{L}_T(c, \psi), \quad \forall \psi \in S^r(\Omega, \mathcal{F}_h, \mathbb{F}), \quad t \in J.$$
Subtract the above equation from the third equation in (2.1b), and note that [16]

\[ \hat{c} - \hat{C} = \begin{cases} 0, & q > 0, \\ \zeta + \xi, & q < 0, \end{cases} \]

we get

\[
\left( \phi \frac{\partial \zeta}{\partial t}, \psi \right) + \left( b(C) \frac{\partial \pi}{\partial t} \right) + \sum_{E \in \mathcal{F}_h} \int_E D \nabla \zeta \cdot \nabla \psi - \int_\Omega \zeta q \psi + J^0_T (\zeta, \psi) = - \left( \phi \frac{\partial \zeta}{\partial t}, \psi \right) - \left( b(C) - b(c) \right) \frac{\partial p}{\partial t}, \psi \right) - \sum_{E \in \mathcal{F}_h} \int_E D \nabla \zeta \cdot \nabla \psi \\
- \sum_{E \in \mathcal{F}_h} \int_E (u \cdot \nabla c - U \cdot \nabla C) \psi + \sum_{\gamma \in \Gamma_h} \int_{\gamma} \{ D \nabla (\zeta + \xi) \cdot n_\gamma \} \psi] + \sum_{\gamma \in \Gamma_h} \int_{\gamma} \{ D \nabla \psi \cdot n_\gamma \} [\zeta + \xi] \\
- \int_\Omega \zeta q \psi - J^0_T (\zeta, \psi), \quad \forall \psi \in \mathcal{S}^r (\Omega, \mathcal{F}_h, \mathcal{F}), \quad t \in J.
\]

Choose the test function \( \psi = \partial \zeta / \partial t \), and make use of the following identity:

\[
u \cdot \nabla c - U \cdot \nabla C = (u - U) \cdot \nabla c + U \cdot \nabla (c - C) = (\rho + \sigma) \cdot \nabla c + U \cdot \nabla (\zeta + \xi),
\]

leads to

\[
\left( \phi \frac{\partial \zeta}{\partial t}, \frac{\partial \zeta}{\partial t} \right) + \frac{1}{2} \frac{d}{dt} \left\{ \sum_{E \in \mathcal{F}_h} \int_E D |\nabla \zeta|^2 - \int_\Omega q - \zeta^2 + J^0_T (\zeta, \xi) \right\} \\
= - \left( \phi \frac{\partial \zeta}{\partial t}, \frac{\partial \zeta}{\partial t} \right) - \left( b(C) \frac{\partial \pi}{\partial t}, \frac{\partial \zeta}{\partial t} \right) + \left( b(C) - b(c) \right) \frac{\partial p}{\partial t}, \frac{\partial \zeta}{\partial t} \right) - \left( b(C) \frac{\partial \eta}{\partial t}, \frac{\partial \zeta}{\partial t} \right) \\
+ \sum_{E \in \mathcal{F}_h} \int_E (\rho + \sigma) \cdot \nabla c + \sum_{E \in \mathcal{F}_h} \int_E U \cdot \nabla (\zeta + \xi) \frac{\partial \zeta}{\partial t} \\
- \sum_{E \in \mathcal{F}_h} \int_E D \nabla \zeta \cdot \nabla \frac{\partial \zeta}{\partial t} + \sum_{\gamma \in \Gamma_h} \int_{\gamma} \{ D \nabla (\zeta + \xi) \cdot n_\gamma \} \left[ \frac{\partial \zeta}{\partial t} \right] + \sum_{\gamma \in \Gamma_h} \int_{\gamma} \{ D \nabla \zeta \cdot n_\gamma \} [\zeta + \xi] \\
- \frac{1}{2} \int_\Omega q - \zeta^2 + \int_\Omega \zeta q - \frac{\partial \zeta}{\partial t} - J^0_T (\zeta, \xi) \\
\equiv \sum_{i=1}^{12} T_i. \tag{3.13}
\]

Note that \( \| U (t) \|_{L^\infty ([0, T]; L^\infty (\Omega))} \) is bounded, we can easily get the following bounds by using Cauchy–Schwarz inequality:

\[
|T_1| + |T_2| + |T_3| + |T_4| + |T_5| + |T_6| + |T_{10}| + |T_{11}| \\
\leq e \left\| \frac{\partial \zeta}{\partial t} \right\|^2 + K \left\| \frac{\partial \zeta}{\partial t} \right\|^2 + K_3 \left\| \frac{\partial \pi}{\partial t} \right\|^2 + K (\| \zeta \|^2 + \| \xi \|^2) + K \left\| \frac{\partial \eta}{\partial t} \right\|^2 + K (\| \rho \|^2 + \| \sigma \|^2) \\
+ K (\| \nabla \zeta \|^2 + \| \nabla \xi \|^2).
\]
The terms \( T_i, \ i = 7, 8, 9, 12 \) have been carried momentarily. Now, integrate in time and with the integration by part formula, we obtain

\[
\phi_s \int_0^t \left\| \frac{\partial \xi}{\partial t} \right\|^2 dt + \sum_{E \in \mathcal{F}_h} \int_E D|\nabla \xi|^2(t) - \int_\Omega q^- \xi^2(t) + J^B_T(\xi, \xi)(t) \leq e \int_0^t \left\| \frac{\partial \xi}{\partial t} \right\|^2 dt + K_3 \int_0^t \left\| \frac{\partial \pi}{\partial t} \right\|^2 + K \int_0^t \left[ \|\sigma\|^2 + \|\xi\|^2 + \|\nabla \xi\|^2 \right] \\
+ K \int_0^t \left[ \|\rho\|^2 + \|\xi\|^2 + \|\nabla \xi\|^2 + \left\| \frac{\partial \xi}{\partial t} \right\|^2 + \left\| \frac{\partial \eta}{\partial t} \right\|^2 \right] \\
- \sum_{E \in \mathcal{F}_h} \int_0^t \int_E D\nabla \xi \cdot \nabla \frac{\partial \xi}{\partial t} + \sum_{\gamma \in \mathcal{I}_h} \int_\gamma \int_0^t \left\{ D\nabla(\xi + \xi) \cdot n_\gamma \right\} \left[ \frac{\partial \xi}{\partial t} \right] \\
+ \sum_{\gamma \in \mathcal{I}_h} \int_\gamma \int_0^t \left\{ D\nabla \frac{\partial \xi}{\partial t} \cdot n_\gamma \right\} [\xi + \xi] - \int_0^t J^B_T(\xi, \frac{\partial \xi}{\partial t}). \tag{3.14}
\]

For the remaining terms in (3.14), as we have assigned \( C(0) = \tilde{c}(0) \), consequently, \( \xi(0) = 0 \). By integrating by part in time, we have

\[
\left| \int_0^t \sum_{E \in \mathcal{F}_h} \int_E D\nabla \xi \cdot \nabla \frac{\partial \xi}{\partial t} \right| = \left| \sum_{E \in \mathcal{F}_h} \int_E D\nabla \xi \cdot \nabla \xi(t) - \int_0^t \sum_{E \in \mathcal{F}_h} \int_E D\nabla \frac{\partial \xi}{\partial t} \cdot \nabla \xi \right| \\
\leq \frac{1}{4} \sum_{E \in \mathcal{F}_h} \int_E D|\nabla \xi|^2(t) + \sum_{E \in \mathcal{F}_h} \int_E D|\nabla \xi|^2(t) \\
+ \frac{1}{2} \int_0^t \sum_{E \in \mathcal{F}_h} \int_E D|\nabla \xi|^2 + \frac{1}{2} \int_0^t \sum_{E \in \mathcal{F}_h} \int_E D \left| \nabla \frac{\partial \xi}{\partial t} \right|^2.
\]

\[
\int_0^t \sum_{\gamma \in \mathcal{I}_h} \int_\gamma \left\{ D\nabla(\xi + \xi) \cdot n_\gamma \right\} \left[ \frac{\partial \xi}{\partial t} \right] + \int_0^t \sum_{\gamma \in \mathcal{I}_h} \int_\gamma \left\{ D\nabla \frac{\partial \xi}{\partial t} \cdot n_\gamma \right\} [\xi + \xi] \\
= \int_0^t \sum_{\gamma \in \mathcal{I}_h} \int_\gamma \left\{ D\nabla \xi \cdot n_\gamma \right\} \left[ \frac{\partial \xi}{\partial t} \right] + \int_0^t \sum_{\gamma \in \mathcal{I}_h} \int_\gamma \left\{ D\nabla \xi \cdot n_\gamma \right\} \left[ \frac{\partial \xi}{\partial t} \right] + \int_0^t \sum_{\gamma \in \mathcal{I}_h} \int_\gamma \left\{ D\nabla \frac{\partial \xi}{\partial t} \cdot n_\gamma \right\} [\xi] \\
+ \int_0^t \sum_{\gamma \in \mathcal{I}_h} \int_\gamma \left\{ D\nabla \frac{\partial \xi}{\partial t} \cdot n_\gamma \right\} [\xi] \\
= \int_0^t \sum_{\gamma \in \mathcal{I}_h} \int_\gamma \left\{ D\nabla \xi \cdot n_\gamma \right\} \left[ \frac{\partial \xi}{\partial t} \right] + \sum_{\gamma \in \mathcal{I}_h} \int_\gamma \left\{ D\nabla \xi \cdot n_\gamma \right\} [\xi](t) + \int_0^t \sum_{\gamma \in \mathcal{I}_h} \int_\gamma \left\{ D\nabla \frac{\partial \xi}{\partial t} \cdot n_\gamma \right\} [\xi] \\
\equiv \sum_{i=1}^3 R_i.
\]
The terms $R_i$ ($1 \leq i \leq 3$) can be treated similarly

$|R_1| = \left| \sum_{\gamma \in \Gamma_h} \int_\gamma \left( D\nabla \zeta \cdot n_\gamma \right)[\zeta](t) - \int_0^l \sum_{\gamma \in \Gamma_h} \int_\gamma \left( D\nabla \frac{\partial \zeta}{\partial t} \cdot n_\gamma \right)[\zeta] \right|

\leq K \sum_{\gamma \in \Gamma_h} \int_\gamma \left( \beta_\gamma r_\gamma^2 h_\gamma^{-1} \right) \left\{ D\nabla \zeta \cdot n_\gamma \right\}^2(t) + \frac{1}{4} \sum_{\gamma \in \Gamma_h} \int_\gamma \beta_\gamma r_\gamma^2 h_\gamma^{-1} \left[ \zeta \right]^2(t)

+ K \int_0^l \sum_{\gamma \in \Gamma_h} \int_\gamma \left( \beta_\gamma r_\gamma^2 h_\gamma^{-1} \right) \left\{ D\nabla \frac{\partial \zeta}{\partial t} \cdot n_\gamma \right\}^2 + K \int_0^l \sum_{\gamma \in \Gamma_h} \int_\gamma \beta_\gamma r_\gamma^2 h_\gamma^{-1} \left[ \zeta \right]^2

\leq \frac{1}{4} J_T^\beta (\zeta, \zeta)(t) + K \int_0^l J_T^\beta (\zeta, \zeta) + K \sum_{\gamma \in \Gamma_h} \int_\gamma h_\gamma r_\gamma^2 \left\{ D\nabla \zeta \cdot n_\gamma \right\}^2(t)

+ K \int_0^l \sum_{\gamma \in \Gamma_h} \int_\gamma h_\gamma r_\gamma^{-2} \left\{ D\nabla \frac{\partial \zeta}{\partial t} \cdot n_\gamma \right\}^2 \right|

|R_2| \leq \sum_{\gamma \in \Gamma_h} \int_\gamma \left( \beta_\gamma r_\gamma^2 h_\gamma^{-1} \right) \left\{ D\nabla \zeta \cdot n_\gamma \right\}^2(t) + \frac{1}{4} \sum_{\gamma \in \Gamma_h} \int_\gamma \beta_\gamma r_\gamma^2 h_\gamma^{-1} \left[ \zeta \right]^2(t)

\leq K_4 \left( \min_{\gamma \in \Gamma_h} \beta_\gamma \right)^{-1} \sum_{E \in \mathcal{F}_h} \int_E D\left| \nabla \zeta \right|^2(t) + \frac{1}{4} J_T^\beta (\zeta, \zeta)(t),

|R_3| = \sum_{\gamma \in \Gamma_h} \int_\gamma \left( D\nabla \zeta \cdot n_\gamma \right)[\zeta](t) - \int_0^l \sum_{\gamma \in \Gamma_h} \int_\gamma \left( D\nabla \frac{\partial \zeta}{\partial t} \cdot n_\gamma \right) \left[ \frac{\partial \zeta}{\partial t} \right]

\leq K_4 \sum_{\gamma \in \Gamma_h} \int_\gamma \left( \beta_\gamma r_\gamma^2 h_\gamma^{-1} \right) \left\{ D\nabla \zeta \cdot n_\gamma \right\}^2(t) + \sum_{\gamma \in \Gamma_h} \int_\gamma \beta_\gamma r_\gamma^2 h_\gamma^{-1} \left[ \zeta \right]^2(t)

+ K \int_0^l \sum_{\gamma \in \Gamma_h} \int_\gamma \left( \beta_\gamma r_\gamma^2 h_\gamma^{-1} \right) \left\{ D\nabla \frac{\partial \zeta}{\partial t} \cdot n_\gamma \right\}^2 + K \int_0^l \sum_{\gamma \in \Gamma_h} \int_\gamma \beta_\gamma r_\gamma^2 h_\gamma^{-1} \left[ \frac{\partial \zeta}{\partial t} \right]^2

\leq K_4 \left( \min_{\gamma \in \Gamma_h} \beta_\gamma \right)^{-1} \sum_{E \in \mathcal{F}_h} \int_E D\left| \nabla \zeta \right|^2(t) + K \int_0^l \sum_{E \in \mathcal{F}_h} \int_E D\left| \nabla \zeta \right|^2

+ K \sum_{\gamma \in \Gamma_h} \frac{r_\gamma^2}{h_\gamma} \left\| \| \zeta \|_{\mathcal{B}_{0, \gamma}}^2 \right) + K \int_0^l \sum_{\gamma \in \Gamma_h} \frac{r_\gamma^2}{h_\gamma} \left( \| \zeta \|_{\mathcal{B}_{0, \gamma}}^2 + \left\| \frac{\partial \zeta}{\partial t} \right\|_{0, \gamma}^2 \right)

For the last term in (3.14), we have

\left| \int_0^l J_T^\beta (\zeta, \zeta)(t) \right| = \int_0^l \sum_{\gamma \in \Gamma_h} \int_\gamma \beta_\gamma r_\gamma^2 h_\gamma^{-1} \left[ \zeta \right] \left[ \frac{\partial \zeta}{\partial t} \right]

= \int_0^l \sum_{\gamma \in \Gamma_h} \int_\gamma \beta_\gamma r_\gamma^2 h_\gamma^{-1} \left[ \zeta \right] \left[ \frac{\partial \zeta}{\partial t} \right]

\leq \frac{1}{2} J_T^\beta (\zeta, \zeta)(t) + K \int_0^l J_T^\beta (\zeta, \zeta) + K \sum_{\gamma \in \Gamma_h} \int_\gamma r_\gamma^2 h_\gamma^{-1} \left[ \zeta \right]^2(t)

+ K \int_0^l \sum_{\gamma \in \Gamma_h} \int_\gamma r_\gamma^2 h_\gamma^{-1} \left[ \frac{\partial \zeta}{\partial t} \right]^2 \right|.
Let $\beta_\gamma$ be large enough, so that $4K_4 \leq \min_{\gamma \in \Gamma} \beta_\gamma$. Using the above estimates, and let $\varepsilon \leq \phi_\pi/4$ be sufficiently small, (3.14) now becomes

$$
\phi_\pi \int_0^t \left\| \frac{\partial \xi}{\partial t} \right\|^2 + \sum_{E \in \mathcal{F}} \int_E D|\nabla \xi|^2(t) - \int_\Omega q^{-2} \xi^2(t) + J_\pi^\beta(\xi, \xi)(t)
\leq K_3 \int_0^t \left\| \frac{\partial \pi}{\partial t} \right\|^2 + K \int_0^t \left[ ||\sigma||^2 + ||\xi||^2 + ||\nabla \xi||^2 + J_\pi^\beta(\xi, \xi) \right]
+ K \int_0^t \sum_{\gamma \in \Gamma} h_\gamma r^{-2}_\gamma \left( ||\nabla \xi \cdot n_\gamma||^2_{0, \gamma} + \left\| D\nabla \frac{\partial \xi}{\partial t} \cdot n_\gamma \right\|^2_{0, \gamma} \right)
+ K \sum_{\gamma \in \Gamma} \int_E D|\nabla \xi|^2(t)
+ K \sum_{\gamma \in \Gamma} \int_0^t \left( h_\gamma r^{-2}_\gamma (D\nabla \xi \cdot n_\gamma)^2(t) + r^{-2}_\gamma h^{-1}_\gamma [\xi]^2(t) \right).
\tag{3.15}
$$

Having obtained error estimates for both flow and transport equations, what we need to is to combine (3.12) and (3.15). Note that the following trivial inequality holds for $v(0) = 0$:

$$
||v(t)||^2 = \int_0^t \frac{d}{dt} ||v(t)||^2 \leq \varepsilon \int_0^t \left\| \frac{\partial v}{\partial t} \right\|^2 + K \int_0^t ||v||^2,
$$

and use the inequality for $\xi$ and $\pi$, multiply (3.12) by $(K_3 + 1)/d_\pi$ and add it to (3.15), we obtain

$$
\int_0^t \left\| \frac{\partial \xi}{\partial t} \right\|^2 + \int_0^t \left\| \frac{\partial \pi}{\partial t} \right\|^2 + ||\xi||^2(t) + ||\nabla \xi||^2(t) + ||\pi||^2(t) + ||\sigma||^2(t) + J_\pi^\beta(\xi, \xi)(t)
\leq K \int_0^t \left[ ||\xi||^2 + ||\nabla \xi||^2 + ||\pi||^2 + ||\sigma||^2 + J_\pi^\beta(\xi, \xi) \right]
+ K \sum_{E \in \mathcal{F}} \int_E D|\nabla \xi|^2(t) + K \sum_{\gamma \in \Gamma} \int_0^t \left( h_\gamma r^{-2}_\gamma (D\nabla \xi \cdot n_\gamma)^2(t) + r^{-2}_\gamma h^{-1}_\gamma [\xi]^2(t) \right)
+ K \int_0^t \left[ \left\| \frac{\partial \xi}{\partial t} \right\|^2 + \left\| \nabla \frac{\partial \xi}{\partial t} \right\|^2 + \left\| \frac{\partial \pi}{\partial t} \right\|^2 + \left\| \frac{\partial \sigma}{\partial t} \right\|^2 \right]
+ K \int_0^t \sum_{\gamma \in \Gamma} h_\gamma r^{-2}_\gamma \left( ||\nabla \xi \cdot n_\gamma||^2_{0, \gamma} + \left\| D\nabla \frac{\partial \xi}{\partial t} \cdot n_\gamma \right\|^2_{0, \gamma} \right)
+ K \int_0^t \sum_{\gamma \in \Gamma} \int_0^t \left( h_\gamma r^{-2}_\gamma (D\nabla \xi \cdot n_\gamma)^2(t) + r^{-2}_\gamma h^{-1}_\gamma [\xi]^2(t) \right).
\tag{3.16}
$$
Thus, with the estimates for $\eta, \rho$ and $\zeta$, it follows from Lemmas 3.1 and 3.2, the Gronwall lemma and the approximation properties that

$$
\left( \int_0^t \left\| \frac{\partial \zeta}{\partial t} \right\|^2 dt \right)^{1/2} + \left( \int_0^t \left\| \frac{\partial \pi}{\partial t} \right\|^2 dt \right)^{1/2} + \| \zeta \|\|(t) + \| \nabla \zeta \|\|(t) + \| \pi \|\|(t)
$$

$$
+ \| \sigma \|(t) + \left( J_T^B (\bar{\pi}, \bar{\zeta})(t) \right)^{1/2}
$$

$$
\leq K \sum_{E \in T_h} \left[ \frac{h_{\min}(r_E, \lambda_E-1)}{r_E} \| c \|_{\lambda_E,E} + \frac{h_{\min}(r_E, \mu_E-1)}{r_E^{\mu_E-3/2}} \left\| \frac{\partial c}{\partial t} \right\|_{\mu_E,E}
$$

$$
+ \frac{h_{\min}(k+1, \omega_E-1)}{k^{\omega_E-1/2}} \left( \| p \|_{\omega_E,E} + \left\| \frac{\partial p}{\partial t} \right\|_{\omega_E,E} \right) \right], \quad 0 \leq t \leq T. \tag{3.17}
$$

Using triangle inequality, we finally get (3.5).

It remains to justify the induction hypothesis (3.11). We can give a proof in the way provided in paper [9]. The initial conditions were chosen so that $\sigma(0) = 0$, hence, (3.11) holds for $t = 0$. Set

$$
F(t) = \| \sigma \|_{L^\infty([0,t], L^\infty(\Omega))}.
$$

Since $F(t)$ is continuous in $t$, there exists a $t^*$ such that

$$
F(t) < L, \quad 0 \leq t < t^*,$$

$$
F(t) = L, \quad t = t^*.
$$

We prove that $t^* = T$. Otherwise, if $t^* < T$, then using (3.17) and inverse inequality

$$
\| \chi \|_{W^{1,\infty}} \leq K h^{-d/2} \| \chi \|_{W^{1,2}},
$$

then bound (3.17) imply that

$$
F(t^*) \leq K h^{-d/2} \sum_{E \in T_h} \left[ \frac{h_{\min}(r_E, \lambda_E-1)}{r_E} \| c \|_{\lambda_E,E} + \frac{h_{\min}(r_E, \mu_E-1)}{r_E^{\mu_E-3/2}} \left\| \frac{\partial c}{\partial t} \right\|_{\mu_E,E}
$$

$$
+ \frac{h_{\min}(k+1, \omega_E-1)}{k^{\omega_E-1/2}} \left( \| p \|_{\omega_E,E} + \left\| \frac{\partial p}{\partial t} \right\|_{\omega_E,E} \right) \right].
$$

Choose $h$ to be sufficiently small and with (3.4) we can have $F(t^*) \leq \frac{1}{2} L$. This contradiction shows that $t^* = T$, which completes the whole proof. \[\Box\]

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