JOURNAL OF ALGEBRA 33, 435–462 (1975)

P. I. Degrees and Prime Ideals

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Communicated by I. N. Herstein

Received June 1, 1973

If R is a prime ring with polynomial identity, let p.i.deg R denote the least integer n such that R satisfies all identities of the ring of $n \times n$ matrices over the integers. (The square of this number is the dimension of the ring of quotients S of R, over its center.)

We shall find below that if R is (quasi) local, with maximal ideal P, then p.i.deg R is a multiple of p.i.deg R/P. More generally (but not quite most generally) we show that if R is any prime p.i. ring, then p.i.deg R can be written as a linear combination, with nonnegative integer coefficients, of the values p.i.deg R/P, as P ranges over the maximal ideals of R; and for any specified maximal ideal P_1 this expression for p.i.deg R can be chosen to involve p.i.deg R/P_1 with positive coefficient.

If R' is a prime p.i. ring, and $R \subsetneq R'$ a not necessarily prime subring, we also obtain analogous relations between p.i.deg R' and the numbers p.i.deg R/P. These are applied to give criteria for a semiprime ring R to be embeddable in $n \times n$ matrices over a commutative ring, or to be an Azumaya algebra.

The results of this paper will be applied in Bergman [4] to obtain strong new results on rational identities holding in division algebras.

NOTE. The development of our main results in Sections 1–6 is rather lengthy, and we have included many remarks and examples on ways that our Lemmas can or cannot be extended, open questions, etc. The reader may prefer to skip over this material, especially on first reading, to stay with the main line of the proof. In particular, the parenthetical assertion at the end of the statement of Proposition 6.2 is not used thereafter, and so its proof may be skipped.

* While this work was done the second author, and for part of the time, the first author, were guests at the Ring Theory Year at Leeds University, England. The first author was supported by an Alfred P. Sloan fellowship.

Also, the results in Sections 4 and 5 are used solely to go from the case where the center of our ring is a valuation ring of finite rank to that of an arbitrary valuation ring. But any valuation on a field of *finite transcendence degree* has finite rank, so if the center K of the ring of quotients of R has finite transcendence degree (over some subfield $\subseteq R$), our proof can be completed without this reduction. Since many of the p.i. rings R commonly dealt with have this property, one may feel justified in skipping (or skimming) Sections 4–5 on first reading, and returning to them afterwards.

Altogether, these exclusions would shorten Sections 1–6 by about one half.

For basic results on rings with polynomial identity, see Herstein [11, Chap. 6, 7] (or Jacobson [13, Chap. 10 Section 3 and Appendix B]).

1. Finite-Dimensional Algebras over Valuation Rings

All rings will be associative and unital. If R is a ring, $J(R)$ will denote the Jacobson radical of R, and $M_n(R)$ the ring of $n \times n$ matrices over R.

Let C be an integral domain with field of fractions K, and M a C-module. Then by $\dim_C M$ we shall mean $\dim_K M \otimes_C K$; equivalently, the maximum cardinality of a C-linearly independent subset of M.

Lemma 1.1. *Suppose C is a commutative valuation ring, M a torsion-free C-module, and $U \subsetneq V$ prime ideals of C. Then*

$$\dim_{C/U} M/MU \geqslant \dim_{C/V} M/MV.$$

Proof. It is easy to show using the fact that C is a valuation ring that if M is torsion-free over C, then M/MU is torsion-free over C/U, and C/U is again a valuation ring. Hence, dividing out by U, we can assume without loss of generality that $U = \{0\}$. We claim that any set of elements of M linearly dependent over C has image in M/MV linearly dependent over C/V. Indeed, in any dependence relation among elements $m_1, \ldots, m_r \in M$, we can divide by the g.c.d. of the coefficients in C to get a dependence relation such that, in particular, not all coefficients lie in V. This yields a dependence relation in M/MV.

The asserted inequality of dimensions follows. ∎

(We remark that the most general class of commutative integral domains C for which the above Lemma holds are the Prüfer domains. One can obtain this result for Prüfer domains as a corollary of the above Lemma by using the characterization of Prüfer domains as integral domains all of whose localizations at prime ideals are valuation rings. (Jaffard [12, p. 7, Theorem 1], Bourbaki, [8, Section 2, Example 12, p. 93].) Conversely, if a domain C

has the property of the above lemma, all localizations C' of C must have it also, and if we apply this property to the local ring C' in the case where M is an ideal of C' generated by two elements a and b, $U = \{0\}$, and $V =$ the maximal ideal of C', we find that $aC' \subseteq bC'$ or $bC' \subseteq aC'$. Hence C' is a valuation ring; hence C is Prüfer. Lemmas 1.2 and 3.2 below likewise go over without change to the case of C a Prüfer domain.)

(It would be interesting to know for what class of rings C Lemma 1.1 is true with the weaker conclusion,

$$\dim_{C/U} M/MU < \infty \Rightarrow \dim_{C/V} M/MV < \infty.)$$

An ideal P of a ring R is called *prime* if $P \neq R$ and $xRy \subseteq P \Rightarrow x \in P$ or $y \in P$, and R is called a *prime ring* if $\{0\}$ is prime in R. If R is an algebra over a commutative ring C, and P a prime ideal of R, then $U = C \cap P$ will be a prime ideal of C; we shall say that P *belongs to* the prime ideal U of C. (If R is not a faithful C-algebra, $C \cap P$ is shorthand for the inverse image of P under the map $C \to R$.)

If U is a prime ideal of a commutative integral domain C, K_U will always denote the field of fractions of C/U. Then the partially ordered set of prime ideals belonging to U in a C-algebra R can be naturally identified with the partially ordered set of all prime ideals of the K_U-algebra $R \otimes K_U$. (Proof straightforward!) In particular, $R \otimes K_U = \{0\} \Leftrightarrow R$ has no primes belonging to $U \Leftrightarrow UR \cap C \neq U$.

By a *finite-dimensional torsion-free* C-algebra, we shall mean a C-algebra which has these properties *as a C-module*.

LEMMA 1.2. *Let C be a commutative valuation ring, R a finite-dimensional torsion-free C-algebra, and U a prime ideal of C. Then there are no inclusions among the prime ideals of R belonging to U; these are finite in number; and every prime ideal P of R which contains U contains a prime ideal P' belonging to U.*

Proof. The first two statements follow immediately from the corresponding facts about the K_U-algebra $R \otimes K_U$, which is finite-dimensional by Lemma 1.1. Let us in fact record a stronger result than the finiteness of the number of primes belonging to U. Let $n = \dim_C R$. Then

$$\begin{aligned} \sum_{Q \subsetneq R \otimes K_U \text{ prime}} \dim_{K_U} R \otimes K_U/Q &= \dim_{K_U} R \otimes K_U/J(R \otimes K_U) \\ &\leqslant \dim_{K_U} R \otimes K_U \\ &\leqslant n, \qquad \text{by Lemma 1.1.} \end{aligned}$$

This translates to

$$\sum_{P \subsetneq R \text{ prime},\ C \cap P = U} \dim_{C/U} R/P \leqslant n. \tag{1}$$

Note that if L is any prime ideal of R such that $C \cap L \subseteq U$, then formula (1) applied to R/L gives

$$\sum_{L \subseteq P \subseteq R,\, P \text{ prime},\ C \cap P = U} \dim_{C/U} R/P \leqslant \dim_{C/C \cap L} R/L. \tag{2}$$

(We shall need these formulas in Section 4.)

To obtain the final assertion of our lemma, let $Q_1, ..., Q_r$ be the distinct primes of $R \otimes K_U$. Then the product ideal $Q_1 \cdots Q_r$ is contained in the intersection of these ideals, the Jacobson radical $J(R \otimes K_U)$, which is nilpotent. Hence some finite product of $Q_1, ..., Q_r$ with sufficiently many repetitions is zero. If we let $P_1, ..., P_r$ denote the corresponding prime ideals of R belonging to U, this says that the product of these ideals is contained in RU. Since $RU \subseteq P$ and P is prime, one of the P_i must lie in P, as claimed. ∎

Remark. For R as in the above lemma, the final assertion tells us that if $U \subseteq V$ are primes of C, then every prime of R belonging to V contains a prime belonging to U. It is not true, however, that every prime belonging to U is contained in a prime belonging to V. The simplest counterexample is to take for R the localization of C at U. For an example where C is the center of R, let C be a rank 2 valuation ring, say for simplicity with valuation group $\mathbf{Z} \times \mathbf{Z}$ ordered lexicographically, and let K be the field of fractions of C. Then C will have elements s and t such that the principal ideals of C are all of the form $s^m C$ ($m \geqslant 0$) or $s^m t^n C$ (m arbitrary, $n > 0$). In particular, $tC[s^{-1}] \subsetneq C$. Consider the C-subalgebra of $M_2(K)$ given by

$$R = \begin{pmatrix} C & C[s^{-1}] \\ tC[s^{-1}] & C[s^{-1}] \end{pmatrix}.$$

It is easy to see that the center of R is C. The partially ordered set of prime ideals of R is shown below:

primes $U \subseteq C$ primes $P \subseteq R$ belonging to U residue rings R/P

$$\begin{array}{ccccc}
sC & \begin{pmatrix} sC & * \\ * & * \end{pmatrix} & & C/sC & \\
\cup & \cup & & \uparrow & \\
tC[s^{-1}] & \begin{pmatrix} tC[s^{-1}] & * \\ * & * \end{pmatrix} & \begin{pmatrix} * & * \\ * & tC[s^{-1}] \end{pmatrix} & C/tC[s^{-1}] & C[s^{-1}]/tC[s^{-1}] \\
\cup & \cup & \cup & \nwarrow & \nearrow \\
\{0\} & \multicolumn{2}{c}{\left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}} & \multicolumn{2}{c}{R}
\end{array}$$

Thus not every prime of R belonging to $tC[s^{-1}]$ is contained in a prime belonging to sC.

Note that R is finitely generated as a C-algebra—by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 0 \\ t & 0 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 0 & 0 \\ 0 & s^{-1} \end{smallmatrix}\right)$.

2. Lifting Projective Modules

Let R be a prime ring with polynomial identity, C the center of R, and K the field of fractions of C. E. Formanek [10] has shown that R lies in a finitely generated C-submodule of the K-vector-space RK, which can (clearly) be taken to be free over C.

Now suppose C is a complete rank 1 (not necessarily discrete) valuation ring. It is clear that a free C-module F of finite rank will be separated and complete with respect to the chain of submodules Fa induced by the nonzero principal ideals Ca of C, and it is not hard to deduce that any submodule M of such an F will also be separated and complete with respect to the chain of submodules Ma. (Key steps: Without loss of generality, we can assume $MK = FK$, and then we see that for some $a \in C - \{0\}, Fa \subseteq M \subseteq F$.)

Hence in the context of the result of Formanek's quoted, if C is a complete rank 1 valuation ring, R will be complete with respect to the chain of ideals Ra $(a \in C - \{0\})$. This makes it possible to prove a lifting result for projective modules.

Proposition 2.1. *Let R be a prime ring with center C, such that R is finite-dimensional over C, and C is a complete rank 1 valuation ring. Then the operation $-\otimes_R(R/J(R))$ from finitely generated projective right R-modules to finitely generated projective right $R/J(R)$-modules induces a bijection of isomorphism classes of such modules.*

Proof. Injectivity is given by Bass [3, Proposition III.2.12 (a), p. 90]. Surjectivity (i.e., "lifting" of projectives, or lifting of idempotents) would be given by part (b) of that Proposition if $J(R)$ were nil, or R were separated and complete with respect to the chain of powers of $J(R)$. Neither of these may be true, but we can get the desired result by three successive applications of that Proposition.

Let U be the maximal ideal of C, and $k = C/U$ its residue field. It is not hard to show that $RU \subseteq J(R)$. Now R/RU is a finite-dimensional k-algebra by Lemma 1.1, hence has nilpotent Jacobson radical, so we can lift projectives from $(R/RU)/J(R/RU) \cong R/J(R)$ to R/RU.

Next, choose any $b \in U - \{0\}$. In C/Cb, the image of U is nil (because C is a rank 1 valuation ring), so in R/Rb the image of RU is nil, so we can lift projectives from R/RU to R/Rb.

Finally, by our above observations on completeness, R is complete with respect to the chain of powers of Rb (since this is cofinal with the chain of all ideals Ra $(a \in C - \{0\})$), so we can lift projectives from R/Rb to R. ∎

Remarks. The result of Formanek's that we have used is proved from his famous "central polynomials" result [9]. Since the latter may, conversely be proved from the former, one cannot hope to get this embeddability result by any elementary argument. But we wonder whether this embeddability can be shown without using Formanek's deep theorem in the special case needed for the above proposition, where R is already known to be *finite-dimensional* over C. We shall not need the central polynomials theorem again below, though we shall at times acknowledge in passing that the center K of the ring of fractions S of a p.i. ring R is just the field of fractions of the center C of R.

(The result on embedding in a finite-dimensional module is quite delicate, as the following two counterexamples show. (1) The hypothesis that C is the center of R cannot be weakened to, "R is an order in a finite-dimensional simple algebra over the field of fractions, K, of C, and $C = R \cap K$." For instance, let C be a Dedekind domain (e.g., $\mathbf{Z}$), K its field of fractions, U a prime of C (e.g., $(5) \subseteq \mathbf{Z}$) and A the ring of algebraic integers in a commutative extension S of K (e.g., $\mathbf{Q}(i)$) in which U has more than one distinct prime factors, $U_1, \ldots, U_r$ (e.g., $5 = (2+i)(2-i)$). Put $R = A[U_1^{-1}]$. Then $C = R \cap K$, but R is not contained in a finitely generated C-module. (2) Under the hypothesis of the embedding result, R will lie in a finitely generated C-submodule of its ring of fractions S. But such a submodule cannot in general be taken to be a C-subalgebra of S. E.g., let C and R be as in the matrix example of the preceeding section. Let $e = \left(\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right) \in R$, and suppose R lay in a torsion-free C-algebra R' that was finitely generated as a C-module. Then $R'e$ is also a finitely generated torsion-free C-module. But $es^{-1} \in R \subseteq R'$, so $R'es \supseteq (R'es^{-1})\, es = R'es$ which is impossible by Nakayama's Lemma.)

Let us now recall some basic facts about prime rings with polynomial identity.

Any prime p.i. ring R has a ring of quotients S, which is simple Artinian. (Goldie's Theorem plus Posner's Theorem. See Herstein [11, Theorem 7.3.2.[1]]) S will satisfy precisely the same identities as R, and will be spanned by R over its center K. The dimension of S over K will be a square integer, $\dim_K S = n^2$, and in fact, on tensoring with an appropriate algebraic extension K' of K, S becomes the matrix ring $M_n(K')$. R and S satisfy all the polynomial identities of $n \times n$ matrices over a commutative ring, but not those of smaller matrices. In particular (Amitsur and Levitzki [1]) n can be characterized as the least integer such that R satisfies the "standard identity" $S_{2n} = 0$, where $S_{2n}(X_1, \ldots, X_{2n}) = \sum_\pi \pm X_{\pi(1)} \cdots X_{\pi(2n)}$, π ranging over all permutations of $\{1, \ldots, 2n\}$, and $\pm$ referring to the sign of the permutation π. Hence let us make the following definition.

[1] The "centroid" referred to there becomes the center in the case of rings with 1.

DEFINITION 2.2. If R is a *prime* ring with polynomial identity, then the least positive integer n such that R satisfies the standard identity $S_{2n} = 0$ will be called the p.i. *degree* of R, written p.i.deg R.

(There are several other "degrees" in the literature of not necessarily prime p.i. rings. The least degree d of a homogeneous polynomial identity satisfied by R, and the least degree d' of a standard identity satisfied by R (which may not exist) agree in the case of prime p.i. rings, and are, unfortunately, twice our "p.i. degree." On the other hand, a division ring of dimension n^2 over its center is called a division ring of degree n, and our p.i. degree is an extension of this usage.)

We can now prove the first case of our result on p.i. degrees and prime ideals. To emphasize the development, the results in this series will be numbered as belonging to Section 6. (No result so numbered will be used in the proof of a result of lower number.)

LEMMA 3.1. *Let C be a complete rank 1 valuation ring, whose field of fractions K is algebraically closed. Let R be a finite-dimensional torsion-free prime C-algebra, let $S = R \otimes_C K$, and assume $R \neq S$. Let $n =$ p.i.deg R. Let $P_1, \dots, P_r$ be the primes of R belonging to the maximal ideal, U, of C, and let $m_i =$ p.i.deg R/P_i. Then there exist positive integers $c_1, \dots, c_r$ such that*

$$n = \sum c_i m_i . \tag{3}$$

Proof. S will be a finite-dimensional prime (hence simple) K-algebra. Hence as K is algebraically closed, S will be a matrix ring over K, and having p.i. degree n, S must be precisely $M_n(K)$. In particular, note that the center of S is K, so the center of R is $R \cap K$, which is C because $R \neq S$ and C is of rank 1. So again, $UR \subseteq J(R)$; hence the maximal ideals of R are precisely the P_i.

Let $k = C/U$. Because K is algebraically closed, k will also be so, hence each of the k-algebras R/P_i will likewise have the form $M_{m_i}(k)$. $P_1, \dots, P_r$ are precisely the maximal ideals of R, hence $R/J(R) \cong (R/P_1) \times \cdots \times (R/P_r) \cong M_{m_1}(k) \times \cdots \times M_{m_r}(k)$.

By the theory of modules over semisimple artin rings, the free right $R/J(R)$-module of rank 1 can be decomposed as a direct sum,

$$R/J(R) \cong \bigoplus_i A_i^{m_i},$$

where $A_1, \dots, A_r$ are nonzero projective $R/J(R)$-modules; modules of "row vectors" over the matrix rings $M_{m_i}(k)$. By Proposition 2.1, these can be lifted to finitely generated projective right R-modules A_i', which will satisfy

$$R \cong \bigoplus_i A_i'^{m_i}, \tag{4}$$

because the right and left hand sides become isomorphic modulo $J(R)$.

We now apply the theory of modules over simple artin rings to $S \cong M_n(K)$. This has a right module B (row-vectors again) such that as S-modules, $S \cong B^n$, and every finitely generated right S-module has the form B^c for a unique nonnegative integer c. In particular, we can write each of the modules $A_i' \otimes_R S$ as B^{c_i} for some *positive* integer c_i. Hence, tensoring (4) with S, we get

$$B^n = \bigoplus (B^{c_i})^{m_i}.$$

Equating exponents on the two sides, we get (3). ∎

3. Change of Base Ring

To get from the above lemma results without such strong hypotheses on C, we must study the behavior of prime ideals under change of C.

For the sake of generality, in the next lemma we shall understand "p.i.degree" to be defined for *all* prime rings R, taking the value ∞ if R does not satisfy a polynomial identity.

LEMMA 3.1. (Cf. Procesi [16, Lemma 2.2]). *Let $f\colon R \to R'$ be a ring homomorphism, such that R' is generated over $f(R)$ by central elements, and let P be a prime ideal of R'. Then $f^{-1}(P) \subseteq R$ is prime, and* p.i.deg $R/f^{-1}(P) =$ p.i.deg R'/P.

Proof. Replacing R and R' by $R/f^{-1}(P)$ and R'/P, the statement we wish to prove becomes: if R' is a prime ring with center C', and R a subring such that $R' = RC'$, then R is also a prime ring, and p.i.deg $R =$ p.i.deg R'.

To show R prime, let $x, y \in R - \{0\}$. As R' is prime, we have $\{0\} \neq xR'y = xRC'y = xRyC'$, so $\{0\} \neq xRy$, as desired.

Since $R \subseteq R'$, R satisfies any polynomial identity satisfied by R'. The converse may not quite be true (it will fail if R is a finite ring), but it is easy to show R' will satisfy any multilinear identity satisfied by R, for it suffices to check such an identity on elements $r_i c_i$ ($r_i \in R$, $c_i \in C'$), and we can factor out the c_i's and reduce to the same identity on R. Since p.i. degree is defined in terms of the standard identities, which are multilinear, the desired equality of p.i. degrees follows. ∎

(*Note.* Even the conclusion that $f^{-1}(P)$ is prime fails if R' is not assumed centrally generated over $f(R)$. E.g., let R' be the 2×2 matrix ring over a field K, $P = \{0\}$, and R the diagonal subring, isomorphic to $K \times K$.)

COROLLARY 3.2. *Let R be a finite-dimensional torsion-free algebra over a commutative valuation-ring C. Let f be a homomorphism of C into an arbitrary* *commutative ring* $\bar{C}$, *and* g *the induced homomorphism* $R \to \bar{R} = R \otimes_C \bar{C}$. *Then for every prime ideal* U *of* $\bar{C}$, "g^{-1}" *induces a surjective,* p.i. *degree-preserving map from the prime ideals of* $\bar{R}$ *belonging to* U *to the prime ideals of* R *belonging to* $f^{-1}(U)$. *In particular, if* f^{-1} *is surjective as a map of primes of* $\bar{C}$ *to primes of* C, *then* g^{-1} *is surjective from primes of* $\bar{R}$ *to primes of* R.

Proof. Everything follows from the preceeding lemma except the surjectivity of the map from primes of $\bar{R}$ belonging to U to primes of R belonging to $f^{-1}(U)$; it is here that we need the conditions that C is a valuation ring and R is finite-dimensional torsion-free. Given a prime ideal $P \subseteq R$ belonging to $f^{-1}(U)$, we want to find a $P' \subseteq \bar{R}$ such that $P = g^{-1}(P')$. By replacing R, C and C' by R/P, $C/f^{-1}(U)$, and C'/U, we may reduce to the case $P = \{0\}$, $U = \{0\}$, f injective, and by tensoring with the fields of fractions K and K' of C and C', we can further assume that R is a prime, finite dimensional algebra over a field, hence simple.

Since R embeds in R', the latter is nonzero, and so has some prime ideal P'. Since R is simple, $g^{-1}(P') = \{0\} = P$, as desired. ∎

In generalizing the hypothesis of Lemma 6.1, we must, in general, slightly weaken the conclusion.

PROPOSITION 6.2. *Let* C *be a rank* 1 *valuation ring, with field of fractions* K, R *a finite-dimensional torsion-free prime* C*-algebra such that* $R \neq S = R \otimes K$, *and* $n =$ p.i.deg R. *Let* $P_1, \ldots, P_r$ *be the prime ideals of* R *belonging to the maximal ideal* U *of* C, *and let* $m_i =$ p.i.deg R/P_i. *Then there exist nonnegative integers* $c_1, \ldots, c_r$ *such that*

$$n = \sum c_i m_i . \tag{5}$$

For any specified $i_0 \in \{1, \ldots, r\}$, *the* c_i*'s in* (5) *can be chosen so that* $c_{i_0} > 0$. (*If* C *is the center of* R, *or* K *is algebraically closed, or* C *is complete, or most generally, if the valuation on* K *has a unique extension to the center* K' *of* S, *then all* c_i *can be taken positive simultaneously.*)

Proof. Form the algebraic closure $\tilde{K}$ of K, extend the valuation on K to this field, and complete this field with respect to the extended valuation. The resulting field $\bar{K}$ will be complete and algebraically closed.[2] Let $\bar{C}$ denote the valuation subring of $\bar{K}$, $\bar{U}$ its maximal ideal, and $\bar{k}$ its residue field. Let $\bar{S} = S \otimes_K \bar{K}$, and $\bar{R} = R \otimes_C \bar{C}$, which may be identified with the $\bar{C}$-subalgebra of $\bar{S}$ generated by R (because $\bar{C}$ is torsion-free over C, hence flat). Note that $\bar{R}/\bar{R}\bar{U} \cong (R/RU) \otimes_k \bar{k}$.

[2] We have not found a reference in the literature for this fact, but it is easy to prove. Given a polynomial $f \in \bar{K}[x]$, of degree n, write it as the limit of a Cauchy sequence of polynomials $f_i \in K[x]$ of the same degree. Then one shows that one can find a Cauchy sequence (r_i) of roots of the f_i, and its limit in $\bar{K}$ will be a root of f.

Let P_{i_0} be any one of the prime ideals of R belonging to U. By Corollary 3.2 this lifts to to an ideal P of $\bar{R}$ belonging to $\bar{U}$, and p.i.deg $\bar{R}/P$ will equal p.i.deg $R/P_{i_0} = m_{i_0}$. P contains an ideal Q of $\bar{R}$ belonging to the zero ideal of $\bar{C}$, by Lemma 1.2. $Q \cap R$ must be $\{0\}$, the unique prime of R belonging to $\{0\} \subseteq C$, so by Corollary 3.2, p.i.deg $\bar{R}/Q =$ p.i.deg $R/\{0\} = n$.

We now apply Lemma 6.1 to the finite-dimensional $\bar{C}$-algebra $\bar{R}/Q$, and conclude that n is a positive linear combination of the p.i. degrees of the quotients of $\bar{R}$ by its prime ideals belonging to $\bar{U}$ and containing Q. By Corollary 3.2 again, these degrees are equal to the p.i. degrees of the quotients of R by appropriate primes belonging to U; i.e., n is a positive linear combination of *some* of the m_i's. In particular, m_{i_0} will be among these, because Q was chosen to be contained in P. Throwing in zero coefficients for the nonoccurring m_i's, we get (5).

Before proving that under certain conditions all the c_i's can be taken positive, let us show that this is not so in general.

Let C be a rank 1 valuation ring, and K' a finite algebraic extension of its field of fractions K, such that the maximal ideal U of C splits into more than one prime $U_1, \ldots, U_r$ in the integral closure C' of C in K'. Then if we simply take $R = C'$, in the above proposition, we find $n = 1$, $m_1 = \cdots = m_r = 1$, so n cannot be written as a combination in which more than one of the m_i appears with positive coefficient/

For an example where the m_i's are distinct, let C, K, C', K', and $U_1, \ldots, U_r$ be as above, let n be any positive integer, and choose two expressions for it, $n = m_1 + m_2 = m_3 + m_4$. Take for R the subring of $M_n(C')$ given by the intersection of the two subrings

$$\begin{array}{cc} & \begin{array}{cc} \overbrace{\hphantom{C'\ \cdots}}^{m_1} & \overbrace{\hphantom{C'\ \cdots}}^{m_2} \end{array} \\ \begin{array}{c} m_1\Big\{ \\ m_2\Big\{ \end{array} & \left(\begin{array}{cc|cc} C' & \cdots & C' & \cdots \\ \vdots & & \vdots & \\ \hline U_1 & \cdots & C' & \cdots \\ \vdots & & \vdots & \cdots \end{array}\right) \end{array} \qquad \begin{array}{cc} & \begin{array}{cc} \overbrace{\hphantom{C'\ \cdots}}^{m_3} & \overbrace{\hphantom{C'\ \cdots}}^{m_4} \end{array} \\ \begin{array}{c} m_3\Big\{ \\ m_4\Big\{ \end{array} & \left(\begin{array}{cc|cc} C' & \cdots & C' & \cdots \\ \vdots & & \vdots & \\ \hline U_2 & \cdots & C' & \cdots \\ \vdots & & \vdots & \cdots \end{array}\right) \end{array}$$

Then R will have $r + 2$ maximal ideals, the quotients by which will be isomorphic respectively to $M_{m_1}(C'/U_1)$, $M_{m_2}(C'/U_1)$, $M_{m_3}(C'/U_2)$, $M_{m_4}(C'/U_2)$ and $M_n(C'/U_i)$ $(i > 2)$. (Considering R as an algebra over the nonvaluation ring C', the first two of these primes will belong to the ideal U_1, the next two to U_2, and the remainder to the U_i for $i > 2$.) Certainly, n cannot be written as a linear combination of these p.i. degrees, $m_1, m_2, m_3, m_4, n, \ldots, n$ so as to involve even the first four with positive coefficients.

We now return to the proof of the final parenthetical assertions of

Proposition 6.2. These will not be used in the remainder of this paper.

If C is the center of R, then K is the center of S, from which it follows that $\bar{S} = S \otimes_K \bar{K}$ is again a simple ring, and $\bar{R}$ again a prime ring. Thus $\{0\}$ is the unique prime of $\bar{R}$ belonging to $\{0\} \subseteq C$, and we can apply Lemma 6.1 directly to $\bar{R}$. We get $n = \text{p.i.deg}\,\bar{R} =$ a linear combination with positive coefficients of the p.i. degrees of the quotients of $\bar{R}$ by prime ideals belonging to $\bar{U}$. By Corollary 3.2 these will equal the p.i. degrees of quotients of R by various of the P_i, each of which will occur at least onece, giving (6) with all coefficients positive.

If K is algebraically closed, then as in the proof of Lemma 6.1, C will be the center of R, so we can apply the preceeding argument.

If C is complete, equivalently, if K is complete with respect to its valuation, then the valuation on C will extend to a unique valuation on the algebraic extension field K' (see Bourbaki [7, Section 8, no. 3, Cor. 1, p. 142]), so it will now suffice to prove the result under this last hypothesis.

Let $U' \subseteq C' \subseteq K'$ be the valuation subring and valuation ideal of the extended valuation. C' may fail to lie entirely in R, so write $R' = RC'$. Then C' will be the center of R', and we can apply the case already proved to conclude that n is a positive linear combination of the p.i. degrees of the residue rings of the maximal ideals P' of R'. By Lemma 3.1, these will equal the p.i. degrees of the residue rings of the ideals $P' \cap R$ of R, so it will suffice to show now that the map $P' \mapsto P' \cap R$, from primes of R' to primes of R is surjective. Equivalently: that every maximal ideal P of R lies in a maximal ideal P' of R'.

From this point on we sketch. By the nonsplitting of the valuation, it follows that the center of R contains a nonzero ideal I of C'. U', being the valuation ideal of a rank 1 valuation ring, is nil modulo the nonzero ideal UI of C'. Hence $U'R = U'R'$ is nil modulo $UIR \subseteq UR$, hence $U'R' \cap R$ is nil modulo UR. As $J(R')$ is nil modulo $U'R'$, it follows that $J(R') \cap R$ is nil modulo $UR \subseteq J(R)$, so $J(R') \cap R \subseteq J(R)$. The reverse inclusion is immediate (because $U \subseteq U'$), so we get an embedding $R/J(R) \hookrightarrow R'/J(R')$. Since both these rings are finite-dimensional k-algebras, and the latter is a central extension of the former, every proper ideal of the former generates a proper ideal of the latter. Hence every maximal ideal P of R generates a proper ideal of R', and so lies in a maximal ideal P', as desired. ∎

COROLLARY 6.3. *Let C be a commutative valuation ring of* finite *rank, R a finite-dimensional torsion-free C-algebra, and $P_0 \subseteq P_1$ prime ideals of R. Then* p.i.deg R/P_0 *can be written as a linear combination, with nonnegative integer coefficients, of the* p.i. *degrees of the residue rings R/P, as P ranges over the primes of R containing P_0, and such that the coefficient of* p.i.deg R/P_1, *in particular, is positive.*

Proof. Let the prime ideals of C be $\{0\} = U_0 \subset U_1 \subset \cdots \subset U_n$, and say the primes $P_0 \subseteq P_1$ of R belong to $U_i \subseteq U_j$ of C $(i \leqslant j)$ respectively.

If $j = i$, then by Lemma 1.2, $P_0 = P_1$ and the result is trivial.

If $j = i + 1$, let C' denote the ring obtained by localizing C/U_i at the image of U_{i+1}, and let $R' = R \otimes_C C'$. C' is a rank 1 valuation ring whose maximal ideal U' is induced by U_j. The ideals of R' belonging to $\{0\}$, respectively U', correspond to ideals of R belonging to U_i and U_j. Let $P_0' \subseteq P_1'$ denote the ideals of R' corresponding to P_0 and P_1. Then the preceding proposition, applied to R'/P_0', with P_1 for P_{i_0}, immediately leads to the desired result about primes of R.

The general case $j > i$ can now be obtained by induction on $j - i$. Given P_0 and P_1, choose an ideal P_2 belonging to U_{i+1}, and such that $P_0 \subseteq P_2 \subseteq P_1$. Such a P_2 exists by Lemma 1.2 applied to the algebra R/P_0 over the valuation ring C/U_i. By the preceding paragraph, p.i.deg R/P_0 can be written as a linear combination of p.i. degrees of rings R/P for primes P containing P_0, such that p.i.deg R/P_2 occurs with positive coefficient. By induction on $j - i$, p.i.deg R/P_2 can in turn be written in terms of p.i. degrees of residue rings at primes containing P_2, so that R/P_1 has positive coefficient. Substituting the latter expression into the former, we get the desired result. ▮

Note that in the above proposition, we cannot assert that p.i.deg R/P_0 will be a linear combination of the p.i. degrees of residue rings at primes belonging to the particular prime U_j to which P_1 belongs. For example, by slightly modifying the matrix ring constructed at the end of Section 1, we can get an algebra R over a rank 2 valuation ring C, yielding the following situation:

ideals U of C	primes P of R belonging to U		p.i.deg R/P	
$U_j = U_2$	P_1		2	
$\cup$	$\cup$			
U_1	P_2	P_3	2	1
$\cup$	$\cup$	$\cup$		
$U_i = U_0 = \{0\}$	P_0		3	

Here p.i.deg R/P_0 is not a linear combination of the p.i. degrees at the primes (that is, the one prime, P_1) belonging to U_j. The point is that Proposition 6.2 needs the hypothesis that $R \neq S$, i.e., that the zero ideal of R is not maximal; so we can use this proposition on a ring R/P only when we know that P is not maximal in R. But clearly we *can* easily apply the above Corollary inductively to get the following.

COROLLARY 6.4. *Let C be a commutative valuation ring of finite rank, R a finite-dimensional torsion-free C-algebra, P_0 a prime ideal of R, and P_1 a* *maximal ideal of R containing* P_0. *Then* p.i.deg R/P_0 *may be written as a linear combination with nonnegative integral coefficients of the integers* p.i.deg R/P, *where P ranges over the maximal ideals of R containing* P_0, *and where the coefficient of* p.i.deg R/P_1 *is positive.* ∎

4. Dimensions of Residue Rings

We next wish to eliminate the finite-rank hypothesis on C. We shall develop the tools needed in this and the next section. The idea will be to show that we can reduce our considerations to a subring of R which is "rationally finitely generated." The first step is to "tame" finite-dimensional algebras over arbitrary valuation rings, by proving an analog of Lemma 6.1, for dimensions rather than p.i. degrees, with inequalities instead of linear relations, and without any conditions of rank 1, completeness, or algebraic closure. The result is a generalization of formula (1) of Section 1.

Proposition 4.1. *Let C be a commutative valuation ring, and R a finite-dimensional torsion-free C-algebra say of dimension n. Let X be a set of prime ideals of R none of which is contained in another. Then X is finite; in fact*

$$\sum_{P\in X} \dim_{C/C\cap P} R/P \leqslant n. \tag{6}$$

Proof. It will suffice to establish (6) for X a finite set of primes. To do this, we shall induce on the number of distinct primes of C to which the prime ideals in X belong. For convenience, let us abbreviate $\dim_{C/C\cap P} R/P$ to $d(P)$.

When there is only one prime to which members of X belong, (6) follows from (1)(p. 3).

If there are more than one such prime, let V be the largest of these, and let $X = Y \cup Z$, where Z is the class of primes in X belonging to V, and Y consists of the remaining primes. Now let U denote the second-largest prime to which members of X belong. Using Lemma 1.2, let us choose for each $P \in Z$ a prime $\alpha(P) \subsetneq P$ belonging to U. (The map $\alpha: Z \to \{$primes of R belonging to $U\}$ need not be $1-1$.)

For each $L \in \alpha(Z)$, Eq. (2) (see Section 1) tells us that $\sum_{\alpha(P)=L} d(P) \leqslant d(L)$. Hence $\sum_{P\in X} d(P) \leqslant \sum_{P\in Y} d(P) + \sum_{L\in\alpha(Z)} d(L)$. Further, Y and $\alpha(Z)$ will be disjoint, and it is easy to see that no member of $Y \cup \alpha(Z)$ is contained in another. The number of primes to which members of $Y \cup \alpha(Z)$ belong is one fewer than for X; hence by induction, we may assume (6) is valid for this set. So we have

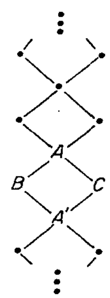
$$\sum_{P\in X} d(P) \leqslant \sum_{P\in Y\cup\alpha(Z)} d(P) \leqslant n,$$

establishing (6) for the set X. ∎

All we shall actually need of this proposition is

COROLLARY 4.2. *If R is a finite-dimensional torsion-free algebra over a commutative valuation ring C, then R has only finitely many maximal ideals. Thus, R is semilocal.* ∎

The above techniques can be carried further than we have done here. For example, though the prime ideals of a commutative valuation ring C can form an infinite chain, we claim that those of a finite- dimensional torsion-free C-algebra R cannot include an infinite array as shown below! Indeed, in that diagram, we see that $d(A') \geqslant d(B) + d(C) \geqslant 2d(A)$. By induction, if



our diagram continues indefinitely upward or downward, the values of $d(P)$ will be unbounded, contradicting (1). One can use this approach to deduce that the partially ordered set of prime ideals of R is the union of a finite number of nonoverlapping chains, and that formula (6) can be improved to

$$\sum_{X} d(P)\, e(P) \leqslant n,$$

where $e(P)$ is the total number of paths in this partially ordered set from P down to minimal primes of R.

5. RATIONALLY CLOSED SUBRINGS OF SEMILOCAL RINGS

In this section, we complete the taming of finite-dimensional algebras over valuation rings.

Let us call a subring R' of a ring R *rationally closed* in R if for every $x \in R'$

that is invertible in R, $x^{-1} \in R'$. The well-known fact that one can lift inverses modulo the Jacobson radical gives the following.

LEMMA 5.1. *If R' is a rationally closed subring of R, then the image of R' in $R/J(R)$ is rationally closed in $R/J(R)$.* ▮

Clearly, a rationally closed subring of a division ring is a division ring. It is easy to deduce.

LEMMA 5.2. *Let D be a division ring, $R = M_n(D)$, and R' a rationally closed subring of R containing the n^2 matrix units e_{ij}. Then R' has the form $M_n(D')$, for some sub-division-ring D' of D.* ▮

COROLLARY 5.3. *Let $R = M_{m_1}(D_1) \times \cdots \times M_{m_r}(D_r)$ be a semisimple artin ring. Then any rationally closed subring $R' \subseteq R$ containing all the elements $(0,..., 0, e_{ij}, 0,..., 0)$ will have the form $M_{m_1}(D_1') \times \cdots \times M_{m_r}(D_r')$ for a family of sub-division-rings $D_i' \subseteq D_i$.* ▮

Recall that a ring R is said to be *semilocal* if $R/J(R)$ is semisimple artin.

PROPOSITION 5.4. *Let R be a semilocal ring. Then there exists a finite set of elements $E \subseteq R$ such that any rationally closed subring R' of R containing E will have the property that the maximal ideals of R' are precisely the intersections with R' of the maximal ideals of R, and these are all distinct. (In particular, R' must again be semilocal.)*

Proof. Write $R/J(R)$ in the form $M_{m_1}(D_1) \times \cdots \times M_{m_r}(D_r)$, and let E be a finite subset of R whose image in $R/J(R)$ gives the elements specified in Corollary 5.3.

If R' is a rationally closed subring of R containing E, then by Lemma 5.1, the image of R' in $R/J(R)$ will be rationally closed, and so will have the form described in the conclusion of Corollary 5.3. In particular, the factor ring of R' by its intersection with each maximal ideal of R will have the form $M_{m_i}(D_i')$, which is a simple ring, so these intersections are again maximal ideals; and clearly they are likewise distinct. To see that they are the only maximal ideals of R' it suffices (since they are finite in number) to verify that an element of R' invertible modulo each of them is invertible in R'. But such an element will be invertible in R, hence invertible in R' by rational closure. ▮

PROPOSITION 5.5. *Let R be a finite-dimensional torsion-free algebra over a commutative valuation ring C. Then R has a subring R' such that* (a) *the maximal ideals of R' are precisely the intersections of R' with the mnximal ideals of R, and these intersections are distinct,* (b) *for each maximal* *ideal P of R,* p.i.deg$(R'/R' \cap P) =$ p.i.deg R/P, (c) $C' = C \cap R'$ *is a valuation ring of finite rank, and* (d) *if K is the field of fractions of C, then* $R' \otimes_{C'} K \cong R \otimes_C K$. (*In particular, if R is prime, then R' is prime of the same* p.i.*-degree.*)

Proof. Let $b_1, \ldots, b_n \in R$ be a K-basis for $S = R \otimes_C K$, with $b_1 = 1$. There will exist elements (structure constants) $\alpha_{ijk} \in K$ $(i, j, k = 1, \ldots, n)$ such that $b_i b_j = \sum \alpha_{ijk} b_k$.

Let X be a finite set of elements of R, on which we shall put conditions later, and write each $x \in X$ as $\sum \beta_{xi} b_i$.

Let K' be the subfield of K generated by the finitely many elements α_{ijk} and β_{xi}. Let S' be the K'-subspace $\sum b_i K' \subseteq S$. (Note that $S = S' \otimes_{K'} K$.) Because K' contains the α_{ijk}, S' will be a subring of S, and because K' contains the β_{xi}, S' will contain X.

Let R' be the subring $R \cap S'$; then $C' = C \cap R' = C \cap K'$ will be a valuation ring with field of fractions K', and S' will equal $R' \otimes_{C'} K'$. Hence $S = S' \otimes_{K'} K = R' \otimes_{C'} K$, establishing (d). Condition (c) holds because a valuation on a field of finite transcendence degree has finite rank (Bourbaki [7, Section 10, no. 3]).

Now S' will be rationally closed in S. (Easy proof: S' is finite-dimensional over K', hence artinian. In an artinian ring, all nonzero-divisors are invertible, hence such a ring is rationally closed in any overring. Alternate approach: if A and B are algebras over a field k, one can show that both are rationally closed in $A \otimes_k B$.) Hence R' is rationally closed in R. Since $X \subseteq S'$ by construction, we also have $X \subseteq R'$. Now let E be a finite subset of R with the property described in Proposition 5.4. Then if $E \subseteq X$, condition (a) will be satisfied. Also, for each of the finitely many maximal ideals $P \subseteq R$, let the p.i. degree of R/P be n_P, and choose a $2(n_P - 1)$-tuple F_P of elements of R whose images in R/P do not satisfy the standard identity $S_{2(n_P-1)}$. Then if X contains F_P we will have p.i.deg$(R'/R' \cap P) =$ p.i.deg(R/P). Thus, taking $X = E \cup \bigcup F_P$, we get conclusions (a)–(d) as desired. ∎

We can now eliminate the finite-rank hypothesis from Corollary 6.4.

COROLLARY 6.5. *Let C be a commutative valuation ring, R a finite-dimensional torsion-free C-algebra, P_0 a prime ideal of R, and P_1 a maximal ideal of R containing P_1. Then* p.i.deg R/P_0 *can be written as a linear combination with nonnegative integral coefficients of the integers* p.i.deg R/P, *where P ranges over the maximal ideals of R containing P_0, and where the coefficient of* p.i.deg R/P_1 *is positive.*

Proof. Without loss of generality, we may take $P_0 = \{0\}$. Then Proposition 5.5 immediately reduces the problem to the case where C is a valuation ring of finite rank, which is taken care of by Corollary 6.4. (We took $P_0 = \{0\}$ so that condition (d) would assure us that the intersection of P_0 with R' would be a prime ideal giving the same p.i. degree.) ▮

6. Proof of the Main Theorem

We are now ready to free our results from valuation rings entirely, with the help of a theorem of C. Procesi: Let R be a prime p.i.-ring and P a prime ideal of R. Let S be the simple artinian ring of quotients of R, and K its center. Then there exists a valuation subring $C' \subseteq K$ such that the ring $R' = RC' \subseteq S$ has a prime ideal P' with $P = P' \cap R$. (Procesi [15, Prop. 2.10]). Note that since K is the field of fractions of C', R' will be a finite-dimensional torsion-free C'-algebra.

PROPOSITION 6.6. *Let R be a* p.i. *ring, and $P_0 \subseteq P_1$ prime ideals of R. Then* p.i. *degree R/P_0 can be written as a linear combination with nonnegative integral coefficients, of the integers* p.i.deg R/P, *where P ranges over the prime ideals of R containing P_0, and where the coefficient of* p.i.deg R/P_1 *is positive.*

Proof. Again, without loss of generality we take $P_0 = \{0\}$. Apply the result of Procesi's quoted above to the ring R, with P_1 for "P." Localizing C' further if necessary, we can assume that P_1' belongs to the maximal ideal of C', and hence is maximal by Lemma 1.2. We now apply Corollary 6.5, and conclude that p.i.deg R may be written as a linear combination of the p.i. degrees of the residue rings of R' at its maximal ideals, with p.i.deg R'/P_1' occurring with positive coefficient. But because R' is an extension of R by central elements, Lemma 3.1 tells us that these p.i. degrees will equal the degrees of the residue rings of R at corresponding prime ideals. ▮

PROPOSITION 6.7. *Let R be a* p.i.-ring *and P_0 a prime ideal of R. Then* p.i.deg R/P_0 *may be written as a linear combination with nonnegative integer coefficients of the integers* p.i.deg R/P *as P ranges over the* maximal *ideals of R containing P_0, and in this expression, the coefficient of any specified term* p.i.deg R/P_1 *can be taken positive.*

This will follow as a special case of the next statement, which also embraces Proposition 6.6.

THEOREM 6.8. *Let R be a* p.i. *ring and $P_0 \subseteq P_1$ prime ideals of R. Then* p.i.deg R/P_0–p.i.deg R/P_1 *can be written as a linear combination with non-negative integer coefficients of the integers* p.i.deg R/P, *as P ranges over the maximal ideals of R containing P_0.*

Proof. Let $n = \text{p.i.deg } R/P_0 - \text{p.i.deg } R/P_1$. By Proposition 6.6 we can write

$$n = \sum_{P_0 \subseteq P, P \text{ prime}} c_P \text{ p.i.deg } R/P \tag{7}$$

for some nonnegative integer coefficients c_P. Let us choose the expression (7) so as to maximize the subsum of the right-hand side taken over *maximal* P. Then we claim that *all* the nonzero coefficients c_P occur for maximal ideals P. For if P_2 were a nonmaximal ideal with $c_{P_2} > 0$, let P_3 be a maximal ideal containing P_2, and use Proposition 6.6 to write p.i.deg R/P_2 as a linear combination of the same sort, with a positive coefficient for p.i.deg R/P_3. Substituting this expression for p.i.deg R/P_2 into (7), we get a formula of the same form as (7) but with the subsum over maximal ideals increased, contradicting our maximality assumption.

Hence, all nonzero terms on the right-hand-side of (7) occur for maximal P, and we have the desired expression for n. ▮

A ring R is called quasilocal if $R/J(R)$ is simple artinian. (The term "local" is also sometimes used for this condition, but we shall use it, more conventionally, for the condition that $R/J(R)$ be a division ring.) In this situation, $J(R)$ is the unique maximal 2-sided ideal $P \subsetneq R$. ▮

Corollary 6.9 (To Proposition 6.7). *Let R be a quasilocal prime* p.i. *ring, with maximal ideal P. Then* p.i.deg R/P *divides* p.i.deg R.

Note. In Proposition 6.2, we found that under special assumptions, *all* coefficients could be taken positive in our linear expressions. Let us sketch here to what extent this can be carried over to subsequent results:

In Corollary 6.3, if K is *algebraically closed,* we can deduce that if X is a family of primes containing P_0 and such that no member of X contains another, (cf. Section 4) then the coefficients associated with the members of X can be simultaneously taken positive. In particular, this gives Corollary 6.4 with positive coefficients at all maximal ideals containing P_0. (The hypothesis that C is the center of R, on the other hand, doesn't seem usable here because it need not go over to residue rings R/P. Completeness conditions on non-rank-1 valuation rings are messy, and we shall not look at them.) The same result can be gotten for Corollary 6.5: the point is that in Proposition 5.5 we can throw in the condition that K' is algebraically closed in K, since going to the algebraic closure in our construction won't interfere with finite transcendence degree. Then if K is algebraically closed, K' will also be so, and we can reduce to the version of Corollary 6.4 described above.

Whether any such generalizations can be proved for our results in which C is not a valuation ring we do not know. Certainly we cannot get all maximal ideals to have positive coefficient, for there may be infinitely many, even if $R = C$. A reasonable hope is that we can do this for a family X of prime ideals such that no member of X contains another but such that the ideals of of the center C of R to which members of X belong form a chain. What is needed is to study "Procesi localization" and determine how large a class of primes can be simultaneously "saved."

7. Subrings

In this section, we shall extend the preceeding results to deal with p.i. degrees of resdiue rings of *subrings* of a given p.i. ring R. In particular, this will yield information on rings representable by matrices over commutative rings, which will assist us in the study of conditions for a ring to be Azumaya in the next section.

Let us start by looking at a case which can be studied much more easily than the general one.

Suppose $D \subseteq D'$ are division rings with polynomial identity. Let K' denote the center of D'. Then DK' will be a subdivision ring of D', having the same p.i. degree as D by Lemma 3.1. Let $K \supseteq K'$ denote the center of DK'. Let $E \supseteq K$ be a maximal subfield of DK', and $E' \supseteq E$ a maximal subfield of D' containing E. Then we see

$$\text{p.i.deg}\, D = \text{p.i.deg}\, DK' = [E : K] \mid [E : K'] \mid [E' : K'] = \text{p.i.deg}\, D'.$$

Now suppose R' is a domain (ring without zero-divisors) with polynomial identity, and $R \subseteq R'$ any subring. Then the division ring of fractions D of R will embed in the division ring of fractions D' of R'. Hence $\text{p.i.deg}\, R = \text{p.i.deg}\, D$ divides $\text{p.i.deg}\, R' = \text{p.i.deg}\, D'$.

We shall see below (Cor. 7.2) that the observation $\text{p.i.deg}\, D \mid \text{p.i.deg}\, D'$ is true more generally if $D \subseteq D'$ are simple p.i. rings. But the result $\text{p.i.deg}\, R \mid \text{p.i.deg}\, R'$ does not similarly go over to arbitrary prime p.i. rings. The difficulty is that a regular element (nonzero divisor) of R may not remain regular in R', whence the simple ring of quotients of R may not embed in that of R'. Here is an example:

Let k be a field, and $n > m$ arbitrary positive integers. We form the polynomial ring $A = k[x_{ij}, y_{ij}\ (i, j \leqslant m)]$ in $2m^2$ indeterminates over k, and let $R' = M_n(A)$. This is a prime ring of p.i.-degree n. Now let R be the subalgebra of R' generated by the two matrices X and Y, each defined to have an $m \times m$ upper left-hand square block of indeterminates, x_{ij}, respectively y_{ij}, and zeroes everywhere else. Then R will be isomorphic to the k-algebra generated by two generic $m \times m$ matrices, which is a domain of p.i.-degree m, though m was an arbitrary integer $< n$.

Nevertheless, by taking into account the p.i. degrees of *factor* rings R/P, we shall obtain useful results. In the above case, note that every element of R has, as an $n \times n$ matrix, the form of an $m \times m$ block plus a k-valued scalar matrix. The map $((a_{ij})) \mapsto a_{nn}$ takes each element of R to its "scalar part," and so defines a homomorphism of R onto k. If we write P for the kernel of this map, then $R/P \cong k$, and we see that p.i.deg $R' = n$ is in fact a linear combination p.i.deg $R = m$ and p.i.deg $R/P = 1$. We shall now show that essentially the same phenomenon happens in general.

LEMMA 7.1. *Let S' be a simple* p.i. *ring with center K and S any K-subalgebra of S'. Then* p.i.deg S' *may be written as a linear combination with* positive *integer coefficients of the integers* p.i.deg S/P, *as P ranges over the prime ideals of S.*

Proof. Case 1. K algebraically closed. Then S' has the form $M_n(K)$, and for every prime ideal P of S, S/P has the form $M_{m_P}(K)$. S' will have p.i. degree n, and each S/P, p.i. degree m_P. S' has a unique simple right module V, which has dimension n over K, and each S/P has unique simple right module V_P of K-dimension m_P.

Note that $\{V_P\}$ as P ranges over the maximal ideals of S gives precisely all the nonisomorphic simple S-modules. Now consider a composition series for V, considered an S-module by restriction of scalars. Each composition factor will have the form V_P, and each V_P must occur at least once—for each V_P is a composition factor of the free right S-module of rank 1, which embeds as a right S-module in S', which is a direct sum as a module of n copies of V. Hence, adding up K-dimensions, we see that n will be a linear combination with positive integer coefficients of the integers $\dim_K V_P = m_P =$ p.i.deg S/P, as claimed.

In the general case, let K' denote the algebraic closure of K. Then $S' \otimes_K K'$ will be a simple p.i. ring of the same p.i. degree as S. We apply Case 1 to the subalgebra $S \otimes K' \subseteq S' \otimes K'$. By Corollary 3.2, the p.i. degrees of residue rings of $S \otimes K'$ at primes are the same as those of S, with possibly greater multiplicities. The result follows. ∎

COROLLARY 7.2. *Suppose $R \subseteq R'$ are prime* p.i. *rings, and every regular element of R remains regular in R'. (In particular, this holds if R is simple, or R' is a domain.) Then* p.i.deg R *divides* p.i.deg R'.

Proof. To see the parenthetical remark, note that if R is simple, every regular element of R is invertible, while if R' is a domain, all elements of $R' - \{0\}$ are regular.

The hypothesis on regular elements implies that the simple ring of fractions S_0 of R embeds in the simple ring of fractions S' of R'. Let K denote the center of S', and let $S = S_0K \subseteq S'$. By the preceeding lemma, p.i.deg S' is a linear combination of the p.i. degrees of the residue rings of the primes of S, and by Lemma 3.1, these must each equal the p.i. degree of the residue ring of S_0 at its only prime, $\{0\}$. The result follows. ∎

The above Corollary includes our earlier observations on p.i. degrees of subdomains.

We can now give a result on general inclusions of p.i. rings.

THEOREM 7.3. *Let R' be a prime* p.i. *ring, and R a subring of R'. Then* p.i.deg R *can be written as a linear combination with nonnegative integer coefficients of the integers* p.i.deg R/P, *as P ranges over the prime ideals of R.*

(In fact, this representation can be chosen either so that (a) *for* every minimal *prime $P \subseteq R$,* p.i.deg R/P *occurs with positive coefficient, or so that* (b) *for* any specified *prime $P_0 \subseteq R$,* p.i.deg R/P_0 *occurs with positive coefficient. In each of those cases, all primes occurring with positive coefficient other than those mentioned can be taken to be maximal.)*

Proof. Let S' denote the simple ring of fractions of R', K its center, and $S = RK$. Then p.i.deg R' = p.i.deg S', and by Lemma 7.1 this is a linear combination with positive integer coefficients of the integers p.i.deg S/Q, as Q ranges over the prime ideals of S. Because S is a central extension of R, for each prime $Q \subseteq S$, $Q \cap R$ will be a prime ideal $P \subseteq R$, with p.i.deg R/P = p.i.deg S/Q (Lemma 3.1). The main statement of the theorem follows.

Note that since S is a finite-dimensional K-algebra, it will have nilpotent Jacobson radical, and so some product of the (finitely many) primes of S will be zero: $Q_1 \cdots Q_n = \{0\}$. Hence in R, some product of the primes $P = Q \cap R$ is zero, hence the set of these primes includes all the minimal primes of R. Hence the expression for p.i.deg R' which we have obtained satisfies condition (a). To get condition (b), let P_1 be an arbitrary prime ideal of R, choose a minimal prime $P_0 \subseteq P_1$, and apply Proposition 6.6 to the term p.i.deg R/P_0 in the above-constructed expression for p.i.deg R'.

Finally, given an expression satisfying either condition (a) or (b), we may apply Proposition 6.7 to any term p.i.deg R/P_0 which is not required by that condition, and replace it by a linear combination of p.i. degrees at maximal primes. ∎

If we want to study a subring R of an *arbitrary* p.i. ring R', the following lemma is useful.

LEMMA 7.4. *Let $R \subseteq R'$ be two rings, P a prime ideal of R, and I a 2-sided ideal of R' maximal for the property $I \cap R \subseteq P$. Then I is prime.*

Proof. Suppose J_1, J_2 are proper overideals of I in R'. Then by maximality hypothesis, $P \not\supseteq R \cap J_i$ $(i = 1, 2)$, hence as P is prime,

$$P \not\supseteq (R \cap J_1)(R \cap J_2) \subseteq R \cap J_1 J_2 .$$

Hence $J_1 J_2 \not\subseteq I$, establishing the primality of I as desired. ∎

COROLLARY 7.5. *Let $R \subseteq R'$ be two* p.i. *rings and P_1 a prime ideal of R. Let P_0 be a prime ideal of R' such that $P_0 \cap R \subseteq P_1$. (Such an ideal exists by the preceding lemma.) Then* p.i.deg R'/P_0 − p.i.deg R/P_1 *is a linear combination with nonnegative integer coefficients of the integers* p.i.deg R/P *as P ranges over the maximal ideals of R.*

Proof. Apply Theorem 7.5 (with condition (b)) to the prime p.i. ring $\bar{R}' = R'/P_0$, its subring $\bar{R} = R/(P_0 \cap R)$, and the prime ideal $\bar{P}_1 = P_1/(P_0 \cap R)$ of $\bar{R}$, noting that every residue ring of $\bar{R}$ at a maximal ideal is naturally isomorphic to a residue ring of R at a maximal ideal. ∎

THEOREM 7.6. *Suppose a ring R can be embedded in $n \times n$ matrices over a commutative ring C, and let P_1 be a prime ideal of R. Then n −* p.i.deg R/P_1 *equals a linear combination with nonnegative integer coefficients of the integers* p.i.deg R/P, *as P ranges over the maximal ideals of R.*

In particular (taking P_1 maximal), n equals a linear combination with nonnegative integer coefficients of these integers p.i.deg R/P.

Proof. Apply the above Corollary to $R \subseteq R' = M_n(C)$, noting that $P_0 \subseteq M_n(C)$ will have the form $M_n(U)$, for U a prime ideal of C, hence p.i.deg $R'/P_0 = n$. ∎

For R a semiprime algebra over a field, the converse is also true:

COROLLARY 7.7. *Let k be a field, R a semiprime k-algebra with polynomial identity, and n a positive integer. Then R is embeddable (as a k-algebra or as a ring) in $n \times n$ matrices over a commutative k-algebra C* if and only if *for each prime $P_1 \subseteq R$, the integer n −* p.i.deg R/P_1 *is expressible as a linear combination, with nonnegative integer coefficients, of the integers* p.i.deg R/P, *as P ranges over the prime ideals of R. (This is also equivalent to the same condition with P_1 limited to minimal prime ideals of R, and/or P ranging only over maximal ideals.)*

Proof. "$\Rightarrow$" is given by the preceding theorem.

Conversely, suppose the prime ideals of R satisfy the stated conditions. To show R embeddable in $M_n(C)$ for some commutative k-algebra C, it suffices to display, for each prime $P \subseteq R$, a homomorphism $R \to M_n(C_P)$ (for some commutative k-algebra C_P) whose kernel is contained in P; for then the kernel of the product map $R \to M_n(\prod_P C_P)$ will lie in $\bigcap P = \{0\}$.

Now given any prime ideal P_1, we can by hypothesis write

$$n = \text{p.i.deg}\, R/P_1 + \cdots + \text{p.i.deg}\, R/P_r$$

for some primes $P_2, \dots, P_r$. Let $n_i = \text{p.i.deg}\, R/P_i$ ($i = 1, \dots, r$). For each i, R/P_i is a prime k-algebra of p.i.-degree n_i, hence can be embedded in $M_{n_i}(C_i)$ for some commutative k-algebra C_i. Further, all the C_i can be simultaneously embedded in a single commutative k-algebra,

$$C = C_1 \otimes_k \cdots \otimes_k C_r .$$

If we now consider the composite map,

$$R \to \prod_i R/P_i \hookrightarrow \prod M_{n_i}(C_i) \hookrightarrow \prod M_{n_i}(C) \hookrightarrow M_{n_1 + \cdots + n_r}(C) = M_n(C)$$

(the last embedding by diagonal blocks) we see that the kernel is $\bigcap P_i$, which is contained in the arbitrarily chosen prime ideal P_1, as required to complete the proof. ∎

The final parenthetical statements of equivalence are easily deduced using Theorem 6.8.

The hypothesis that R be an algebra over a *field* was needed above to conclude that any family of commutative k-algebras could be embedded in a common commutative k-algebra. The Corollary fails with k replaced by any commutative integral domain A which is not a field. Indeed, let I be a nonzero prime ideal of such an A, and let R be the ring $\left(\begin{smallmatrix} A & A \\ I & A \end{smallmatrix}\right) \subseteq M_2(A)$. The p.i. degrees of the factor-rings R/P are 1 and 2 (e.g., 1 when $P = \left(\begin{smallmatrix} I & A \\ I & A \end{smallmatrix}\right)$, 2 when $P = \{0\}$); hence for $n = 3$, the numerical condition of the preceding corollary is satisfied. But suppose we had an embedding $R \subseteq M_3(C)$ for some commutative A-algebra C. Tensoring with the field of fractions K of A, we would get an embedding of the simple ring $M_2(K)$ in $M_3(C \otimes K)$, which contradicts Theorem 7.6. (Taking $A = \mathbf{Z}$, we get an R satisfying our numerical conditions but not embeddable in 3×3 matrices over any commutative *ring*.)

The final Corollary to Theorem 7.6, which we shall need in the next section, eliminates reference to embeddings and subrings; it can be looked at as a generalization of Theorem 6.8, with the prime ideal P_0 replaced by an intersection I of prime ideals all associated with the same p.i. degree:

COROLLARY 7.8. *Let R be a ring, and $I \subseteq R$ an ideal which is the intersection of a family of prime ideals P whose residue rings R/P all have the same* p.i. *degree, $n < \infty$. Let P_1 be any prime ideal of R containing I. Then*

n — p.i.deg R/P_1 *equals a linear combination with nonnegative integer coefficients, of the integers* p.i.deg R/P, *as* P *ranges over the maximal prime ideals of* R.

Proof. For each ideal P in the given family, R/P has p.i. degree n, hence is embeddable in $n \times n$ matrices over a commutative ring C_P. Hence R/I embeds in $M_n(\prod C_P)$, and we apply Theorem 7.6 to R/I and get the desired conclusion. ∎

(Clearly, we might even have weakened the hypothesis on I to say that the rings R/P all have p.i. degrees *dividing* n.)

8. Semiprime Azumaya Algebras

Artin's Theorem on Azumaya algebras says that a ring R is an Azumaya algebra of rank n^2 if and only if it satisfies all polynomial identities for p.i. degree n (all identities of $M_n(\mathbf{Z})$), and no homomorphic image of R satisfies the identities of p.i. degree $n - 1$ (M. Artin [2], as generalized in Procesi [17]).

If R is *semiprime*, this is equivalent to saying that for all *maximal* and all *minimal* primes $P \subseteq R$, p.i.deg $R/P = n$. Using the results of this paper, we can show this equivalent to a formally weaker condition. Let us indicate a special case before stating the general result:

Proposition 8.1. *Suppose* R *is a prime* p.i. *ring, of* p.i. *degree* n, *and suppose that for all maximal ideals* $P \subsetneq R$, *one has* p.i.deg $R/P > n/2$. *Then* R *is a rank* n^2 *Azumaya algebra.*

The general result (of which this is the case $M = \{m \mid m > n/2\}$, R prime) is

Proposition 8.2. *Let* n *be a positive integer, and* M *a set of positive integers containing* n, *but such that* n *does not lie in the additive semigroup generated by* $M - \{n\}$.

Suppose R *is a semiprime* p.i. *ring with* p.i.deg $R/P = n$ *for all minimal prime ideals* $P \subsetneq R$, *and* p.i.deg $R/P \in M$ *for all maximal* P. *Then* R *is a rank* n^2 *Azumaya algebra.*

Proof. By our hypothesis on M, we see that if n is to be written as a linear combination with nonnegative coefficients of members of M, the only term occurring with positive coefficient must be n itself. But if P_1 is any maximal ideal of R, choose a minimal prime $P_0 \subseteq P_1$; then by Theorem 6.8, p.i.deg $R/P_0 = n$ can be written as a linear combination of integers p.i.deg R/P for maximal P so that p.i.deg R/P_1 has positive coefficient. Hence p.i.deg $R/P_1 = n$, for any maximal ideal P_1; so R is Azumaya. ∎

As an application of Proposition 8.1, suppose R is a prime p.i. ring of p.i. degree n, $f(X_1, \ldots, X_r) = 0$ is a polynomial identity for p.i. degrees

$\leqslant n/2$, and for some $x_1, \dots, x_r \in R$, $f(x_1, \dots, x_r)$ is invertible. Then R is Azumaya. (This argument with "$n - 1$" in place of "$n/2$" was a known application of Artin's Theorem.)

Let us now turn to Azumaya algebras of not necessarily constant rank, and ask how the above results can be generalized to such rings.

The general Azumaya algebra is a direct product, over a finite family of integers n, of rank n^2 Azumaya algebras. (To see this, let $C =$ center (R) and note that R, being a finitely generated projective C-module, has a rank function locally constant on Spec C, inducing a finite direct product decomposition of C.) Hence it might seem plausible that a semiprime p.i. ring R should be Azumaya if and only if for every pair $P \subseteq P'$ where P is a minimal prime and P' a maximal one, p.i.deg $R/P =$ p.i.deg R/P'. But this condition is too weak. For example, let k be any field, and R the subring of the infinite product ring $M_2(k) \times M_2(k) \times \cdots$, consisting of all sequences of 2×2 matrices which eventually become constant and diagonal:

$$R = \left\{ (x_1, x_2, \dots) \in \prod M_2(k) \mid (\exists n > 0)(\exists a, b \in k)\, x_n = x_{n+1} = \cdots = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\}.$$

(Cf. Pierce [14], Example 11.4.) It is easy to see that R is a von Neumann regular p.i. ring, hence every prime ideal of R is *both* maximal and minimal (also easy to see directly in this case), so the proposed condition holds automatically! But R is not Azumaya. For instance, the ideal P_1 of sequences whose "limit" value has the form $\left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and the ideal P_2 of sequences whose limit value has the form $\left(\begin{smallmatrix} 0 & 0 \\ 0 & b \end{smallmatrix}\right)$ both have the same intersection with the center of R, but a 2-sided ideal of an Azumaya algebra is determined by its intersection with the center (Bourbaki [6, Section 5, Example 15(b), p. 180]).

It turns out that "what is wrong" in the above example is that the maximal prime P_1, which satisfies p.i.deg $R/P_1 = 1$, though it does not contain any minimal prime with residue ring of p.i. degree 2, does contain the intersection of all such primes; for this is the zero ideal of R. To get the "right" criterion for R to be Azumaya, we must take into account such intersections of primes, using the results of the preceeding section; we will also want the following lemma.

LEMMA 8.3. *Let R be a semiprime ring, and J an ideal of R. Let I denote the intersection of all prime ideals of R not containing J. Suppose that every maximal ideal of R containing I also fails to contain J. Then R can be written as a direct product $R' \times R''$ such that $I = \{0\} \times R''$ and $J = R' \times \{0\}$.*

Proof. It suffices to show that $I \cap J = \{0\}$ and $I + J = R$. To get the first statement, suppose $x \in J - \{0\}$. Since R is semiprime, we can find a prime ideal $P \subseteq R$ such that $x \notin P$. Hence $J \nsubseteq P$, hence by construction of I, $I \subseteq P$, so $x \notin I$.

To get the second equality, note that by our hypothesis on maximal ideals, a maximal ideal containing J cannot contain I, hence no maximal ideal contains $I + J$, hence $I + J = R$.

We can now characterize semiprime Azumaya algebras.

THEOREM 8.4. *Let R be a semiprime* p.i. *ring, and for each integer r, let $I(r)$ denote the intersection of all the minimal prime ideals $P \subseteq R$ satisfying* p.i.deg $R/P = r$. *Then R is Azumaya if and only if for every r, all maximal ideals P of R containing $I(r)$ satisfy* p.i.deg $R/P = r$.

Proof. *If*: Since R is a semiprime p.i. ring, it will satisfy all the identities of some finite p.i. degree. Assume inductively that the desired implication is true for rings satisfying the identities of p.i. degree n, and suppose R satisfies the identities of p.i. degree $n + 1$.

Let J denote the ideal of R generated by all elements $f(x_1, \ldots, x_r)$ such that $f = 0$ is a polynomial identity for p.i. degree n, and $x_1, \ldots, x_r \in R$. Thus a prime factor ring R/P of R has p.i. degree $\leqslant n$ if and only if $J \subseteq P$; otherwise it has p.i. degree $n + 1$. Applying this observation to the case $r = n$ of our hypothesis, we get precisely the hypothesis of the preceding Lemma (with $I(n)$ for I). Hence $R = R' \times R'' = (R/I(n)) \times (R/J)$. The first factor satisfies the hypotheses of Artin's Theorem and so is Azumaya of rank $(n + 1)^2$. The second satisfies the identities of p.i. degree n, and being a direct factor of R, it also inherits the hypotheses of our theorem; so by inductive assumption it is also Azumaya. Hence their direct product, R, is also Azumaya.

Only if: If R is Azumaya, we may write $R = R^{(1)} \times \cdots \times R^{(n)}$, where $R^{(r)}$ is Azumaya of rank r^2. Let us understand $R^{(r)}$ to be the zero ring for $r > n$. For each r, $I(r)$ is easily seen to be the kernel of the projection of R onto the factor $R^{(r)}$, and the maximal primes of R containing $I(r)$ are induced by the maximal primes of $R^{(r)}$ which have residue rings of p.i. degree r. ∎

We can now generalize Proposition 8.2.

PROPOSITION 8.5. *Let N and M be two sets of positive integers, such that no $n \in N$ is contained in the additive semigroup generated by the set $M - \{n\}$.*

Suppose R is a semiprime p.i. *ring, such that for every minimal prime $P \subseteq R$,* p.i.deg $R/P \in N$, *and for every maximal prime P',* p.i.deg $R/P' \in M$. *Then R is Azumaya, a direct product of Azumaya algebras of ranks r^2 for $r \in M \cap N$.*

Proof. Applying Corollary 7.8 for each $r \in N$, and taking for I the ideal $I(r)$, it is easily deduced by the method of Proposition 8.2 that R satisfies the hypothesis of the above theorem, and that the only values of r that can actually occur are those in $M \cap N$. ▮

COROLLARY 8.6. *Let N be a set of positive integers such that no $n \in N$ lies in the additive semigroup generated by $N - \{n\}$. Then any semiprime* p.i. *ring R such that* p.i.deg $R/P \in N$ *for every prime ideal $P \subsetneq R$, is Azumaya.* ▮

For example, a semiprime ring R satisfying the identities of p.i. degree 3, and in which some commutator $xy - yx$ is invertible, will satisfy the above condition with $N = \{2, 3\}$.

In general, the results of this section are not true without the hypothesis of semiprimality. A counterexample to essentially all of them is the upper triangular 2×2 matrix ring over any field k: $\binom{k\ k}{0\ k}$. Here for every prime P, $R/P \cong k$ is commutative, so p.i.deg $R/P = 1$; but R is not Azumaya. However, the following generalization of Artin's Theorem at least seems plausible: If a ring R satisfies all polynomial identities for p.i. degree n, and if for every maximal prime $P \subsetneq R$ one has p.i.deg $R/P > n/2$, then R is Azumaya. We do not know how this might be proved.

We wonder whether various modifications of the conditions considered here might yield interesting generalizations of the concept of an Azumaya algebra. What, for instance, can be said of a prime ring R of p.i. degree n (or perhaps an arbitrary ring R satisfying the identities of p.i. degree n) in which, for every maximal ideal P, p.i.deg $R/P = n$ or $n/2$?

Still another question: Suppose R is a ring such that for every prime P, p.i.deg $R/P = n$. Must the least integer r such that R satisfies all identities of p.i. degree r, if such an r exists, be a multiple of n?

Note added in proof. The second author has recently obtained the following additional consequence of the results of Section 6:

THEOREM 6.10. *If R is a prime* p.i. *ring, and $P \subsetneq R$ a maximal ideal such that $\bigcap_{r \geq 0} p^r = \{\bar{0}\}$, then* p.i.deg R/P *divides* p.i.deg R.

But examples of R and P such that p.i.deg R/P does not divide p.i.deg R are quite common, e.g., in generic matrix rings. Hence the situation $\bigcap_r p^r \neq \{0\}$ is also common.

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