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Magnus subgroups of one-relator surface groups

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ABSTRACT

A one-relator surface group is the quotient of an orientable surface group by the normal closure of a single relator. A Magnus subgroup is the fundamental group of a suitable incompressible sub-surface. A number of results are proved about the intersections of such subgroups and their conjugates, analogous to results of Bagherzadeh, Brodskiĭ and Collins in classical one-relator group theory.

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1. Introduction

Recall the Freiheitssatz of Magnus [11,12] for one-relator groups:

Theorem 1.1 *(The Freiheitssatz). Let $G = \langle X : R\rangle$ be a one-relator group where R is cyclically reduced. If Y is a subset of X which omits a generator occurring in R, then the subgroup M_Y generated by Y is freely generated by Y.*

Subgroups of a one-relator group of the form M_Y as in the Freiheitssatz are called *Magnus subgroups*. In [13], Newman proved that the Magnus subgroups of a one-relator group with torsion are malnormal, that is, if M is a Magnus subgroup and $g \notin M$, then $M \cap gMg^{-1}$ is trivial. Bagherzadeh [1] generalized Newman's result in 1976 to ordinary one-relator groups and proved that Magnus subgroups of one-relator groups are cyclonormal. He proved the following

Theorem 1.2. *Let M be a Magnus subgroup of a one-relator group $G = \langle X : R\rangle$. Then M is cyclonormal in G, that is, if $g \notin M$, then $M \cap gMg^{-1}$ is cyclic.*

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Collins [5,6] proved the following results about the intersection of Magnus subgroups of a one-relator group G.

Theorem 1.3. *Let M_Y and M_Z be Magnus subgroups of a one-relator group $G = \langle X : R\rangle$ generated by subsets $Y, Z \subset X$. Then*

$$M_Y \cap_G M_Z = M_{Y\cap Z} * I,$$

where I is a free group of rank 0 *or* 1.

Theorem 1.4. *Let M_Y and M_Z be Magnus subgroups of a one-relator group G as in Theorem* 1.3. *For any $g \in G$, either $M_Y \cap_G gM_Zg^{-1}$ is cyclic* (*possibly trivial*) *or $g \in M_Y M_Z$.*

Here we use the notation $A \cap_G B$ to denote the intersection of two subgroups A, B in the group G, to distinguish it from the intersection in any other group containing them both. For example, in Theorem 1.3, if F is the free group on X, then $M_Y \cap_F M_Z = M_{Y\cap Z}$; the theorem tells us that this may differ from $M_Y \cap_G M_Z$.

When these two intersections do differ, in other words when I has rank 1 in Theorem 1.3, we say that the two Magnus subgroups involved have *exceptional intersection*. The first author [9, Theorem E] has shown that it is algorithmically decidable whether a given pair of Magnus subgroups in a given one-relator group has exceptional intersection.

A *one-relator surface group* is the quotient of the fundamental group of an orientable surface (possibly noncompact, or with boundary) by the normal closure of a single element. These groups were introduced in 1990 by Hempel [7], and have subsequently been studied by Bogopolski and Sviridov [3,4] and by the first author [8].

In this paper we generalize Theorems 1.2, 1.3 and 1.4, as well as [9, Theorem E] from one-relator groups to one-relator surface groups. With an appropriate definition of Magnus subgroup, we prove the following.

Theorem 3.1. *Let G be a one-relator surface group, and let M be a Magnus subgroup of G. Then M is cyclonormal, that is, for any $g \in G \setminus M$, $M \cap gMg^{-1}$ is cyclic.*

Theorem 4.1. *The intersection $M_1 \cap_G M_2$ of two compatible Magnus subgroups M_1 and M_2 of the one-relator surface group $G = \pi_1(S)/\langle\langle R\rangle\rangle$ is the free product of $(M_1 \cap_\Sigma M_2)$ with a cyclic group* (*where $\Sigma = \pi_1(S)$*). *That is,*

$$M_1 \cap_G M_2 = (M_1 \cap_\Sigma M_2) * C.$$

(See Section 2 for the definition of *compatible* Magnus subgroups.)

Theorem 4.2. *There is an algorithm which will decide, given a one-relator surface group and a pair of compatible Magnus subgroups, whether or not the intersection is exceptional* (*that is, $C \neq \{1\}$ in Theorem* 4.1) *and if so will give a generator for C.*

Theorem 5.1. *Let G be a one-relator surface group and let M_1 and M_2 be two compatible Magnus subgroups of G. Let $g \in G$. Then $M_1 \cap_G gM_2g^{-1}$ is cyclic unless $g \in M_1M_2$.*

In Section 2 below we define our notion of Magnus subgroup for one-relator surface groups and present some useful preliminary results. Theorem 3.1 is proved in Section 3, Theorems 4.1 and 4.2 in Section 4, and finally Theorem 5.1 in Section 5.

Theorems 3.1, 4.1 and 5.1 appeared in the second author's thesis [14]. We are grateful to the thesis examiners, Andrew Duncan and Nick Gilbert, for useful comments. We are also grateful to the referee for some perceptive observations and suggestions that have improved our exposition.

The second author is thankful to Heriot–Watt University for awarding him a PhD scholarship which made this research possible.

2. Preliminaries

In order to formulate appropriate generalizations of theorems about Magnus subgroups of one-relator groups, we first need to choose a suitable definition of Magnus subgroup for a one-relator surface group. A minimum requirement for a Magnus subgroup is that it should satisfy an appropriate version of the Freiheitssatz for one-relator surface groups – which turns out to be a somewhat delicate question (see [8,10]). For the purposes of exposition in the present paper we shall restrict our definition of Magnus subgroup to a case where we know that a Freiheitssatz holds.

Definition. Let S be a surface, R an element of $\pi_1(S)$ and $G = \pi_1(S)/\langle\langle R\rangle\rangle$. A subgroup M of G is a *Magnus subgroup* if there is an essential separating simple closed curve α in S that separates S into two components S_1 and S_2, such that:

1. R is not conjugate in $\pi_1(S)$ to an element of $\pi_1(S_1)$ or of $\pi_1(S_2)$; and
2. M is the image in G of the subgroup $\pi_1(S_1)$ of $\pi_1(S)$.

In practice, we can identify the Magnus subgroup M with the subgroup $\pi_1(S_1)$ of $\pi_1(S)$, thanks to the following theorem ([8, Proposition 3.10]; see also [10] for more general versions).

Theorem 2.1 *(Freiheitssatz for one-relator surface groups). If S_1 satisfies the conditions in the definition of Magnus subgroup of the one-relator surface group $G = \pi_1(S)/\langle\langle R\rangle\rangle$, then the inclusion map $\pi_1(S_1) \to G$ is injective.*

In particular, Magnus subgroups of one-relator surface groups are free.

The separating curve α in the above definition is determined by the Magnus subgroup only up to isotopy. Its isotopy type corresponds to a splitting of $\pi_1(S)$ as a free product with amalgamation:

$$\pi_1(S) \cong \pi_1(S_1) *_A \pi_1(S_2),$$

where A is the cyclic subgroup generated by α.

Note that a Magnus subgroup is generated by a subset of some standard generating set for the surface group $\pi_1(S)$ – for example

$$\langle a_1, b_1, \ldots, a_\ell, b_\ell\rangle \subset \bigl\langle a_1, b_1, \ldots, a_k, b_k \colon [a_1, b_1] \cdots [a_k, b_k] = 1\bigr\rangle.$$

We shall also refer to a pair of Magnus subgroups M_1 and M_2 as *compatible* if the corresponding separating curves on S can be chosen to be disjoint. In terms of generators, there exists a standard generating set such that both M_1, M_2 are generated by subsets of the chosen generating set for $\pi_1(S)$.

Remark. Let

$$G = \pi_1(S)/\langle\langle R\rangle\rangle = \bigl\langle a_1, b_1, \ldots, a_k, b_k \colon [a_1, b_1] \cdots [a_k, b_k] = R = 1\bigr\rangle$$

be a one-relator surface group, and $L = \{a_1, b_1, \ldots, a_{k-1}, b_{k-1}, b_k\}$ a proper subset of the generating set of G. Then L generates a subgroup M of $\pi_1(S)$ corresponding to the complement of a nonseparating simple closed curve in S. In [8] it is shown that the Freiheitssatz does not in general hold for such subgroups: the natural map $M \to G$ is not always injective. For this reason, we have excluded such subgroups of $\pi_1(S)$ from our definition of Magnus subgroup.

Moreover, it turns out that the results of this paper do not necessarily extend to groups of this form, even in situations where $M \to G$ is injective. We shall give an example in Section 3 to illustrate this.

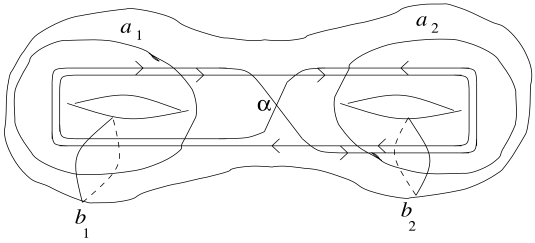

Fig. 1. $\alpha = a_1a_2a_1a_2^{-1}$.

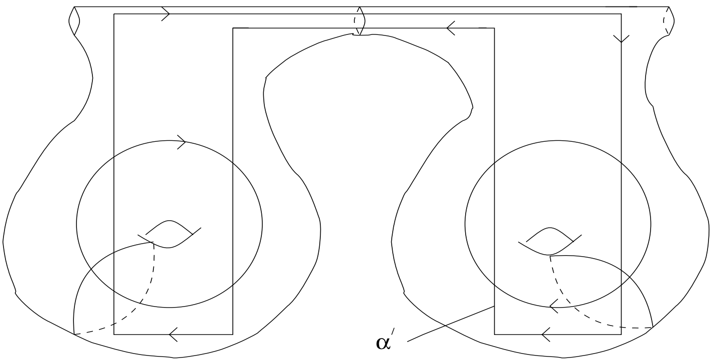

Fig. 2. $\alpha' = (a_1)(a_2a_1a_2^{-1})$.

In Section 3 below we will employ an idea first used by Hempel [7, Lemma 2.1, Theorem 2.2] (see also [8, Proposition 2.1]) to express a one-relator surface group as an HNN extension of a one-relator group. Here the notation $\langle\alpha, \beta\rangle$ denotes the algebraic intersection number of a pair of curves α, β on the surface S.

Proposition 2.2. *Let S be a closed, connected, oriented surface of genus at least 2, and let α be a closed curve in S. Then*

1. *There is a nonseparating simple closed curve β in S such that $\langle\alpha, \beta\rangle = 0$.*
2. *For any such β, there are connected surfaces F, F_0, F_1 and a closed curve α' in F, such that*
   (a) *$F_0 \cong F_1$, $F_0 \subset F$ and $F_1 \subset F$;*
   (b) *$\pi_1(F_0) \to \pi_1(F)/\langle\langle\alpha'\rangle\rangle$ and $\pi_1(F_1) \to \pi_1(F)/\langle\langle\alpha'\rangle\rangle$ are injective;*
   (c) *$\pi_1(S)$ (resp. $\pi_1(S)/\langle\langle\alpha\rangle\rangle$) is an HNN-extension of $\pi_1(F)$ (resp. $\pi_1(F)/\langle\langle\alpha'\rangle\rangle$) with associated subgroups $\pi_1(F_0)$ and $\pi_1(F_1)$;*
   (d) *Each of ∂F, ∂F_0 and ∂F_1 consists of two circles, each of which represents (a conjugate of) $\beta \in \pi_1(S)$.*

This result is proved in [8]. For our purposes, it is sufficient to know that the surface F is formed from a finite number $n \geqslant 1$ of copies of S cut along the curve β, each copy joined to the next at a copy of β. Each of the sub-surfaces F_0, F_1 of F is obtained by removing one of the end copies of $S \setminus \beta$ (except in the case $k = 0$, where we take F_0, F_1 to be copies of a small annular neighbourhood of β).

Example. Figs. 1 and 2 illustrate the case where S has genus 2 and α represents the element $a_1a_2a_1a_2^{-1}$ of $\pi_1(S) = \langle a_1, b_1, a_2, b_2\colon\ [a_1, b_1][a_2, b_2] = 1\rangle$. Here we choose $\beta = b_2$.

In this example, $n = 2$, so F, F_0, F_1 are surfaces of genus 2, 1, 1 respectively, each with two boundary components. We can regard $\pi_1(F)$ as the free group on 5 generators $a_{10} = a_1$, $a_{11} = a_2a_1a_2^{-1}$,

$b_{10} = b_1$, $b_{11} = a_2 b_1 a_2^{-1}$, and b_2. Then $\pi_1(F_0)$ is the subgroup with basis $\{a_{10}, b_{10}, b_2\}$ and $\pi_1(F_1)$ is the subgroup with basis $\{a_{11}, b_{11}, b_2[a_{10}, b_{10}]\}$. The rewrite of $\alpha = a_1 a_2 a_1 a_2^{-1}$ is $\alpha' = a_{10} a_{11}$. We leave it as an exercise for the reader to verify that $\pi_1(S)/\langle\langle\alpha\rangle\rangle$ is indeed isomorphic to the HNN extension of $\pi_1(F)/\langle\langle\alpha'\rangle\rangle$, with stable letter a_2 and associated subgroups $\pi_1(F_0), \pi_1(F_1)$.

In Sections 4 and 5 we shall use a slight variation of Hempel's trick, which we will describe in the course of the proof of Theorem 4.1.

We shall also make extensive use of the fact that there is a lot of freedom in the choice of the curve β. In particular, if $S_0 \subset S$ is a punctured torus, then the restriction of the algebraic intersection map $\langle\alpha, -\rangle$ to S_0 gives a homomorphism $\mathbb{Z}^2 \cong H_1(S_0) \to \mathbb{Z}$ with nonzero kernel; we may choose a simple closed curve $\beta \subset S_0$ to represent a nonzero element of the kernel, and such a curve is automatically nonseparating.

Lemma 2.3 below is an algebraic translation of this observation, applied to the case of the closed orientable surface S of genus g, with a standard generating set $\{a_1, b_1, \ldots, a_k, b_k\}$ for $\pi_1(S)$, where $\pi_1(S_0)$ is generated by $\{a_k, b_k\}$.

If R is an element of a free group F with basis X, and $x \in X$, we denote by $\sigma(R, x)$ the exponent-sum of x in R, in other words the image of R under the homomorphism $F \to \mathbb{Z}$ defined by $x \mapsto 1$, $X \setminus \{x\} \mapsto 0$.

Lemma 2.3. *Let R be an element of the free group $\langle a_1, b_1, \ldots, a_k, b_k\rangle$. Then there exists a basis $\{a_k', b_k'\}$ of the free group $\langle a_k, b_k\rangle$, such that*

(i) *$[a_k', b_k'] = [a_k, b_k]$; and*
(ii) *as a reduced word in $\{a_1, b_1, \ldots, a_{k-1}, b_{k-1}, a_k', b_k'\}$, R has exponent sum zero in a_k'.*

3. Magnus subgroups are cyclonormal

Theorem 3.1. *Let $G = \pi_1(S)/\langle\langle R\rangle\rangle$ be a one-relator surface group, and let M be a Magnus subgroup of G. Then M is cyclonormal, that is, for any $g \in G \setminus M$, $M \cap gMg^{-1}$ is cyclic.*

Proof. By the definition of Magnus subgroup we may assume that

$$G = \big\langle a_1, b_1, \ldots, a_k, b_k \colon\ [a_1, b_1] \cdots [a_k, b_k] = R = 1\big\rangle$$

and that

$$M = \langle a_1, b_1, \ldots, a_\ell, b_\ell\rangle$$

where $0 < \ell < k$. Let $g \notin M$ be an element of G.

By Lemma 2.3, we may assume that

$$\sigma(R, a_k) = 0,$$

that is, the exponent sum of a_k in R is zero.

Let β denote the simple-closed curve on S representing b_k, such that

$$\sigma(-, a_k) = \langle -, \beta\rangle : \pi_1(S) \to \mathbb{Z}.$$

Now apply Hempel's trick (Proposition 2.2) with this choice of β and with α a closed curve representing $R \in \pi_1(S)$.

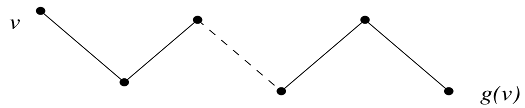

Fig. 3.

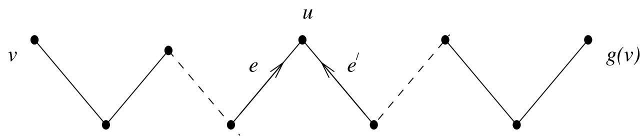

Fig. 4.

By Proposition 2.2(c), $G = \pi_1(S)/\langle\langle\alpha\rangle\rangle$ is an HNN-extension

$$G = \langle H, a_k \colon a_k X a_k^{-1} = Y \rangle$$

where $H = \pi_1(F)/\langle\langle\alpha'\rangle\rangle$, F is a surface with boundary, and $X = \pi_1(F_0)$, $Y = \pi_1(F_1)$ with F_0, F_1 isomorphic sub-surfaces of F. In particular, $\pi_1(F)$ is a free group, so H is a one-relator group. Moreover, since F_0 is a sub-surface of S, X is a free factor of $\pi_1(F)$. Since X embeds as a subgroup of H, it follows that R is not conjugate to an element of X – in other words that X is a Magnus subgroup of H (in the standard sense of one-relator group theory). Similarly, Y is a Magnus subgroup of H. Note also that M is the fundamental group of a sub-surface of S that is disjoint from β, and hence also the fundamental group of a sub-surface of F. Since the composite homomorphism $M \to H \to G$ is injective, the homomorphism $M \to H$ is also injective, and the above argument shows that M is also a Magnus subgroup of H.

Finally, note that $\pi_1(F)$ is free on $a_k b_k a_k^{-1}$ and $a_k^i a_j ak^{-i}$, $a_k^i b_j a_k^{-i}$ for $1 \leqslant j \leqslant k-1$ and for $0 \leqslant i \leqslant n$ (for some $n \geqslant 1$), while M is free on a_j, b_j for $1 \leqslant j \leqslant \ell$ and Y is free on $a_k b_k a_k^{-1}$ and $a_k^i a_j a_k^{-i}$, $a_k^i b_j a_k^{-i}$ for $1 \leqslant j \leqslant k-1$ and for $1 \leqslant i \leqslant n$. In particular, there is a free group $L \supset Y$ such that $\pi_1(F) = M * L$, and hence H is a one-relator product of the two free groups M, L.

The Bass–Serre tree for our HNN-extension has vertex-stabilizers the conjugates of H and edge-stabilizers the conjugates of X (or the conjugates of Y). Let T be the Bass–Serre tree for this HNN-extension and suppose $g \in G \setminus M$. Then G acts on T and there exists a vertex v such that

$$H = Stab(v),$$
$$gHg^{-1} = Stab\big(g(v)\big).$$

Moreover, X and Y are the stabilisers of two edges of T, which have v as source and target respectively.

Now $M \subset H$ stabilizes v and $gMg^{-1} \subset gHg^{-1}$ stabilizes $g(v)$ so that $M \cap gMg^{-1}$ stabilizes both v and $g(v)$ and hence stabilizes the path P in T from v to $g(v)$ (see Fig. 3). Here three different cases arise.

Case 1. If $g(v) = v$, then $g \in Stab(v) = H$ and the result follows from Bagherzadeh's Theorem 1.2.

Case 2. If the path P is not coherently oriented, then there is an intermediate vertex $u = g'(v)$ of P that is either the source of each incident edge of P or the target of each incident edge of P. We treat the latter case (Fig. 4); the former is entirely analogous.

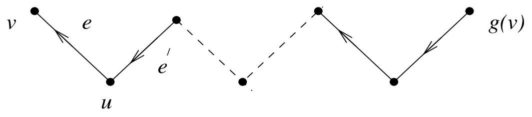

Fig. 5.

If e, e' are the edges of P incident at u, then

$$Stab(e) = g'Yg'^{-1},$$
$$Stab(e') = (g'h)Y(g'h)^{-1}$$

for some $h \in H$. Now

$$M \cap gMg^{-1} \subseteq Stab(v) \cap Stab(g(v)) \subseteq Stab(e) \cap Stab(e').$$

But

$$Stab(e) \cap Stab(e') = g'Yg'^{-1} \cap g'hYh^{-1}g'^{-1} = g'(Y \cap hYh^{-1})g'^{-1}.$$

Therefore

$$M \cap gMg^{-1} \subseteq g'(Y \cap hYh^{-1})g'^{-1}$$

is cyclic by Bagherzadeh's Theorem 1.2.

Case 3. If the path P is coherently oriented, then we will assume that the orientation is from $g(v)$ to v – see Fig. 5. (The argument for the opposite orientation is analogous.)

The edge e of P incident at v has target v and so has stabilizer hYh^{-1} for some $h \in H$. Recall that H is a one-relator product of two free groups M, L with $Y \subset L$. Hence $M \cap hLh^{-1}$ is cyclic by a result of Brodskiĭ [2, Teorema 6(в)]. But

$$M \cap gMg^{-1} \subseteq M \cap hYh^{-1} \subseteq M \cap hLh^{-1},$$

so $M \cap gMg^{-1}$ is also cyclic.

In all cases we have shown that $M \cap gMg^{-1}$ is cyclic. Hence M is cyclonormal in G. $\square$

Below we give an example to show that Theorem 3.1 does not extend to a subgroup M of G generated by $2k-1$ of the $2k$ generators of G, even in cases where M is free on those generators.

Example. Let

$$G = \langle a_1, b_1, a_2, b_2 \colon [a_1, b_1][a_2, b_2] = b_1^4 a_2^{-1} b_1^3 a_2 b_1^2 a_2^{-1} b_1^3 a_2 = 1 \rangle.$$

Then the second relator $R \equiv b_1^4 a_2 b_1^3 a_2^{-1} b_1^2 a_2 b_1^3 a_2^{-1}$ has exponent-sum 0 in a_2, so the Freiheitssatz for one-relator surface groups [8, Proposition 3.10] implies that $M = \langle a_1, b_1, b_2 \rangle$ embeds in G via the natural map. On the other hand, Collins [5] shows that $R = 1 \Rightarrow b_1^6 = a_2^{-1} b_1^6 a_2$. Note also that $[a_1, b_1][a_2, b_2] = 1 \Rightarrow a_2^{-1} b_2 a_2 = b_2[a_1, b_1]$, so that the nonabelian free subgroup $\langle b_1^6, b_2[a_1, b_1] \rangle$ of M is identified in G with the subgroup $\langle a_2^{-1} b_1^6 a_2, a_2^{-1} b_2 a_2 \rangle$ of $a_2 M a_2^{-1}$. Hence $M \cap_G a_2 M a_2^{-1}$ is not cyclic, so M is not cyclonormal in G.

4. Intersections of Magnus subgroups in one-relator surface groups

Our aim in this section is to prove the analogue of Theorem 1.3 of Collins for one-relator surface groups.

Theorem 4.1. *The intersection $M_1 \cap_G M_2$ of two compatible Magnus subgroups M_1 and M_2 of the one-relator surface group $G = \pi_1(S)/\langle\langle R\rangle\rangle$ is the free product of $(M_1 \cap_\Sigma M_2)$ with a cyclic group (where $\Sigma = \pi_1(S)$). That is,*

$$M_1 \cap_G M_2 = (M_1 \cap_\Sigma M_2) * C.$$

Proof. By the definition of compatible Magnus subgroups, we have $M_1 = \pi_1(S_1)$ and $M_2 = \pi_1(S_2)$, where S_1, S_2 are sub-surfaces of S, each with a single boundary component, and $\partial S_1 \cap \partial S_2 = \emptyset$.

There are four possibilities to consider:

1. $S_1 \subset S_2$;
2. $S_2 \subset S_1$;
3. $S_1 \cap S_2 = \emptyset$;
4. $S_1 \cup S_2 = S$.

If $S_1 \subset S_2$ then M_1 is a subgroup of M_2 and there is nothing to prove. Similarly there is nothing to prove if $S_2 \subset S_1$. If $S_1 \cap S_2 = \emptyset$ then we can form a new sub-surface S_3 by taking the union of S_1, S_2 and a regular neighbourhood of an arc γ connecting ∂S_1 to ∂S_2. The result follows easily from the Freiheitssatz (Theorem 2.1) if $M_3 = \pi_1(S_3)$ is a Magnus subgroup of G, for then

$$M_1 \cap_G M_2 = M_1 \cap_{M_3} M_2 = M_1 \cap_\Sigma M_2.$$

If M_3 is not a Magnus subgroup, and S_3 has genus less than that of S, then R is conjugate in Σ to an element of $M_3 = M_1 * M_2$. In this case, replacing γ by another arc gives a different choice of S_3 such that the corresponding M_3 is a Magnus subgroup and the above argument applies. If M_3 fails to be a Magnus subgroup because S_3 has the same genus as S, then $S \setminus (S_1 \cup S_2)$ is an open annulus A. In this case, adjoining the closure of A to each of S_1, S_2 reduces us to the fourth possibility, where

$$S_1 \cup S_2 = S.$$

Thus we are reduced to the situation in which

$$G = \left\langle a_1, b_1, \ldots, a_k, b_k\colon\ [a_1, b_1] \cdots [a_k, b_k] = R = 1\right\rangle,$$

$M_1 = \langle a_1, b_1, \ldots, a_\ell, b_\ell\rangle$ and $M_2 = \langle a_{j+1}, b_{j+1}, \ldots, a_k, b_k\rangle$ with $1 \leqslant j \leqslant \ell < k$.

By Lemma 2.3, we may assume that a_1, a_k appear in R with exponent sum zero, that is,

$$\sigma(R, a_1) = 0 = \sigma(R, a_k).$$

Note that

$$\Sigma = M_1 *_{M_0} M_2$$

where $M_0 = \pi_1(S_1 \cap S_2) = M_1 \cap_\Sigma M_2$.

By definition of Magnus subgroup, R is not conjugate to an element of M_1 or of M_2. Hence we may assume that $R \in \Sigma = M_1 *_{M_0} M_2$ is a cyclically reduced word of length greater than 1 (with respect to the amalgamated free product length function).

We apply an amended form of Hempel's trick as follows. The kernel K of $\sigma(-, a_k) : \Sigma \to \mathbb{Z}$ has an induced graph-of-groups decomposition as an infinite amalgamated free product of $\widetilde{M}_2 = K \cap M_2$ and the groups $a_k^n M_1 a_k^{-n}$ for $n \in \mathbb{Z}$, amalgamating the copy of $a_k^n M_0 a_k^{-n}$ in $a_k^n M_1 a_k^{-n}$ with that in $\widetilde{M}_2$. Choose a conjugate $\widetilde{R}$ of R that belongs to the subgroup K_0 of K generated by $\widetilde{M}_2$ and $a_k^n M_1 a_k^{-n}$ for $0 \leqslant n \leqslant m$, and assume that all the choices have been made to minimize m. Then G is an HNN-extension of the one-relator group $\widetilde{G} = K_0/\langle\langle \widetilde{R} \rangle\rangle$, with stable letter a_k and associated subgroups $K_1 = K_0 \cap a_k K_0 a_k^{-1}$, $K_2 = K_0 \cap a_k^{-1} K_0 a_k$.

Clearly $M_1 \cap_G M_2 \subset M_1 \cap_G \widetilde{M}_2$. Note also that $M_1 \subset K_0$. If $m > 0$ in the above construction, then the join of M_1 and $\widetilde{M}_2$ in K_0 is a Magnus subgroup of the one-relator group $\widetilde{G}$, from which it follows that

$$M_1 \cap_G M_2 = M_1 \cap_{K_0} \widetilde{M}_2 = M_0.$$

Hence we are reduced to the situation where $m = 0$ in the HNN construction. Now M_0 is a free factor of $\widetilde{M}_2$, so we can write $\widetilde{M}_2 = M_0 * F$ for some free group F, and then we also have $K_0 = M_1 * F$.

We now essentially repeat the above argument, with a_1 replacing a_k. Specifically, let N be the kernel of $\sigma(-, a_1) : K_0 \to \mathbb{Z}$. Then N is the free product of $\widetilde{M}_1 = M_1 \cap N$ and the groups $F_n := a_1^n F a_1^{-n}$ for $n \in \mathbb{Z}$. Choosing a suitable conjugate $\widehat{R}$ of $\widetilde{R}$, we may assume that

$$\widehat{R} \in \widetilde{M}_1 * F_0 * F_1 * \cdots * F_p$$

with all choices made to minimize p. Then $\widetilde{G}$ is an HNN-extension of the one-relator group $\widehat{G} = (\widetilde{M}_1 * F_0 * F_1 * \cdots * F_p)/\langle\langle \widehat{R} \rangle\rangle$ with stable letter a_1.

Arguing as before, $M_1 \cap_G \widetilde{M}_2 \subset \widetilde{M}_1 \cap_G \widetilde{M}_2$. Moreover, $\widetilde{M}_2 \subset F_0$. If $p > 0$, then the join of $\widetilde{M}_1$ and F_0 in N is a Magnus subgroup of $\widehat{G}$, from which it follows that

$$\widetilde{M}_1 \cap_G \widetilde{M}_2 = \widetilde{M}_1 \cap_N \widetilde{M}_2 = M_0.$$

Hence we are reduced to the case where $p = 0$.

Now

$$\widetilde{M}_1 = M_0 * L$$

where L is a free group. Also

$$\widetilde{M}_2 = M_0 * F_0,$$

and

$$\widehat{G} = (M_0 * F_0 * L)/\langle\langle \widehat{R} \rangle\rangle.$$

Therefore

$$M_1 \cap_G M_2 = \widetilde{M}_1 \cap \widetilde{M}_2 = (M_0 * F_0) \cap_{\widehat{G}} (M_0 * L).$$

Since $\widehat{G}$ is a one-relator group, Collins' Theorem 1.3 applies, and so

$$M_1 \cap_G M_2 = M_0 * C$$

with C cyclic, as required. □

The proof of Theorem 4.1 shows that Magnus subgroups can have exceptional intersection only in very restricted circumstances – where $R \in M_0 * F_0 * L$ in the notation of the proof. Moreover, in that case it is equivalent to a pair of Magnus subgroups in a one-relator group having exceptional intersection. We can use this to generate examples of exceptional intersections of Magnus subgroups in one-relator surface groups.

Example. Let G be the one-relator surface group

$$\left\langle a_1, b_1, a_2, b_2\colon [a_1, b_1][a_2, b_2] = a_1^{-2} b_1^4 a_1^2 a_2^{-2} b_2^{-3} a_2^2 a_1^{-2} b_1^2 a_1^2 a_2^{-2} b_2^{-3} a_2^2 = 1\right\rangle.$$

If $x = a_1^{-2} b_1 a_1^2$ and $y = a_2^{-2} b_2 a_2$, then the second relation is $x^4 y^{-3} x^2 y^{-3} = 1$. Collins [5] shows that $x^6 = y^6$ is a consequence of that relation. If $M_1 = \langle a_1, b_1\rangle$ and $M_2 = \langle a_2, b_2\rangle$, then $x^6 \in M_1$ and $y^6 \in M_2$. Hence M_1 and M_2 have exceptional intersection in G.

The strong restrictions on exceptional intersection that arise in the proof of Theorem 4.1 also give rise to a proof of Theorem 4.2, which we sketch below.

Theorem 4.2. *There is an algorithm which will decide, given a one-relator surface group G and two Magnus subgroups M_1, M_2, whether or not M_1 and M_2 have exceptional intersection in G. If the intersection is exceptional, the algorithm will provide a generator for the free factor C in the statement of Theorem* 4.1.

Sketch proof. The theorem is proved by noting that each step in the proof of Theorem 4.1 can be carried out algorithmically.

We may assume that the one-relator surface group has the form

$$\left\langle a_1, b_1, \dots, a_k, b_k\colon [a_1, b_1] \cdots [a_k, b_k] = R = 1\right\rangle,$$

where R is a word in the generators.

The first step is a basis-change in the free group $\langle a_1, b_1\rangle$ to allow us to assume that $\sigma(R, a_1) = 0$. The Euclidean algorithm transforms the vector $(\sigma(R, a_1), \sigma(R, b_1)) \in \mathbb{Z}^2$ to a vector of the form $(0, \ell)$ using integer elementary column operations, which can be lifted to Nielsen operations on $\langle a_1, b_1\rangle$ in the standard way. Thus the basis-change operation of Lemma 2.3 can be performed algorithmically, and so we may assume without further ado that $\sigma(R, a_1) = 0$, and similarly that $\sigma(R, a_k) = 0$.

The rewrites $R \to \widetilde{R} \to \widehat{R}$ in the proof of Theorem 4.1 are entirely mechanical processes, as is the choice of a suitable cyclic conjugate in each case. Thus the non-negative integers m, p occurring in the proof can be algorithmically computed. Should either be strictly positive, then we can stop, declaring the intersection to be non-exceptional.

Hence we may assume that $m = p = 0$, so that (up to conjugation), $R \in M_0 * F_0 * L$ in the notation of the proof of Theorem 4.1. Now F_0 and L are free groups of infinite rank, so in order to handle this situation algorithmically we must replace them by appropriate finite rank free groups. In practice, one can algorithmically generate finite sets B_1, B_2 that are subsets of bases of F_0, L respectively, and such that R can be expressed (up to conjugacy) as a word in $M_0 * \langle B_1\rangle * \langle B_2\rangle$.

Now apply the algorithm of [9, Theorem E] to the one-relator group $(M_0 * \langle B_1\rangle * \langle B_2\rangle)/\langle\langle R\rangle\rangle$ to decide whether or not the intersection is exceptional. If so, the algorithm provides a generator γ for the exceptional free factor, in terms of our chosen basis for M_0 together with $B_1 \cup B_2$. Finally, we translate γ into a word in the original generators $a_1, b_1, \dots, a_k, b_k$ of G to complete the algorithm. □

5. Intersections of conjugates of Magnus subgroups of one-relator surface groups

In this section we prove the analogue of Theorem 1.4.

Theorem 5.1. *Let G be a one-relator surface group and let M_1 and M_2 be two compatible Magnus subgroups of G. Let $g \in G$. Then $M_1 \cap_G gM_2g^{-1}$ is cyclic unless $g \in M_1M_2$.*

Proof. Arguing as in the proof of Theorem 4.1, we either have M_1, M_2 free factors of a Magnus subgroup M_3 with $M_3 = M_1$, $M_3 = M_2$ or $M_3 = M_1 * M_2$; or $M_1 = \pi_1(S_1)$, $M_2 = \pi_1(S_2)$ where $S_1 \cup S_2 = S$. In the first case, the result follows from Theorem 3.1 unless $g \in M_3$, in which case it follows from the Freiheitssatz (Theorem 2.1).

Hence we may assume that

$$G = \langle a_1, b_1, \ldots, a_k, b_k \colon [a_1, b_1] \cdots [a_k, b_k] = R = 1 \rangle,$$

$M_1 = \langle a_1, b_1, \ldots, a_\ell, b_\ell \rangle$ and $M_2 = \langle a_{j+1}, b_{j+1}, \ldots, a_k, b_k \rangle$, where $1 \leqslant j \leqslant \ell < k$. We also assume, by virtue of Lemma 2.3, that $\sigma(R, a_1) = \sigma(R, a_k) = 0$.

Let $g \in G$. Note that for any $m, n \in \mathbb{Z}$ we may replace g by $g' = a_1^m g a_k^n$, since $M_1 \cap g' M_2 (g')^{-1} = M_1 \cap gM_2g^{-1}$. Hence we may assume that $\sigma(g, a_1) = 0 = \sigma(g, a_k)$.

With the same notation as in the proof of Theorem 4.1, we express G as an HNN extension of a one-relator group $\widetilde{G} = K_0/\langle\langle \widetilde{R} \rangle\rangle$, with stable letter a_k and associated subgroups K_1, K_2, where K_0 is generated by $\widetilde{M}_2 = M_2 \cap \operatorname{Ker}(\sigma(-, a_k))$ together with $a_k^n M_1 a_k^{-n}$ for $0 \leqslant n \leqslant m$, for some $m \geqslant 0$. In particular $M_1 \subset K_0$.

Note that, since $M_1 \subset \operatorname{Ker}(\sigma(-, a_k))$, we have

$$M_1 \cap_G gM_2g^{-1} = M_1 \cap_G g\widetilde{M}_2g^{-1} \subset \widetilde{G} \cap_G g\widetilde{G}g^{-1}.$$

Now G acts on the Bass–Serre tree T arising from this HNN description. The stabilizers of the vertices are conjugates of $\widetilde{G}$ and the stabilizers of the edges are conjugates of K_1 (and hence also of K_2). Let u be a vertex of T such that $\widetilde{G} = Stab(u)$, and let e_1, e_2 be two edges of T incident at u such that $K_1 = Stab(e_1)$ and $K_2 = Stab(e_2)$.

Now suppose that $g \notin \widetilde{G}$. Then $M_1 \cap_G gM_2g^{-1} \subset \widetilde{G} \cap_G g\widetilde{G}g^{-1}$ stabilises the (nonempty) geodesic path P in T from u to $g(u)$. Moreover, since $\sigma(g, a_k) = 0$, this path has even length and contains the same number of forward-pointing and backward-pointing edges. In particular, there is an intermediate vertex v in P which is either the source of both the incident edges of P or the target of both the incident edges of P. We assume the latter. (The analysis of the former case is analogous.)

If $v = h(u)$, then the stabilisers of the edges of P incident at v have the form $hsK_2s^{-1}h^{-1}$ and $htK_2t^{-1}h^{-1}$ for some $s, t \in \widetilde{G}$ with $s^{-1}t \notin K_2$. By Bagherzadeh's Theorem 1.2, $sK_2s^{-1} \cap_{\widetilde{G}} tK_2t^{-1}$ is cyclic, and hence the stabiliser of P is cyclic, and the result follows.

Thus we are reduced to the case where $g \in \widetilde{G}$. But in that case M_1 and $\widetilde{M}_2$ are Magnus subgroups of the one-relator group $\widetilde{G}$, and

$$M_1 \cap_G gM_2g^{-1} = M_1 \cap_{\widetilde{G}} g\widetilde{M}_2g^{-1},$$

which is cyclic by Collins' Theorem 1.4, unless

$$g \in M_1 \cdot \widetilde{M}_2 \subseteq M_1 \cdot M_2. \qquad \square$$

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