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Stochastic inflation on the brane

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Abstract

Chaotic inflation on the brane is considered in the context of stochastic inflation. It is found that there is a regime in which eternal inflation on the brane takes place. The corresponding probability distributions are found in certain cases. The stationary probability distribution over a comoving volume and the creation probability of a de Sitter braneworld yield the same exponential behaviour.

1. Introduction

The global structure of many four-dimensional inflationary universes is very rich [1,2]. Self-reproduction of inflationary domains leads to a universe consisting of many large domains. In each of these domains there can be different realizations of the inflationary scenario. This property of standard inflation alleviates the problem of fine tuning in inflation. The process of self-reproduction of inflationary domains leads globally to eternal inflation, namely, there is at least one inflating region in the universe. Self-reproduction of inflationary domains can be understood as a branching diffusion process in the space of field values of the inflaton. The probability distributions found in such a way will give the probabilities of finding a certain field value at a certain point in space–time.

There is an interesting connection between the stochastic approach to inflation and quantum cosmology. The probability of finding the universe in a state characterized by certain parameters assuming the Hartle–Hawking no-boundary condition yields the same exponential behaviour as the stationary probability distribution in a comoving volume derived from stochastic inflation. This is the case for standard chaotic inflation. As it turns out this also holds for chaotic inflation on the brane.

The idea that the universe is confined to a brane embedded in a higher-dimensional bulk space–time has received much attention in recent years. In the one brane Randall–Sundrum scenario the brane is embedded in a five-dimensional bulk space–time with negative cosmological constant Λ_5 [3,4]. Chaotic inflation on the brane in this setting has been investigated in [5–7]. The form of the spectrum of perturbations is modified due to the modified dynamics at high energies. Therefore, it seems to be interesting to investigate stochastic inflation on the brane.

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2. Inflation on the brane

In a braneworld scenario 4D Einstein gravity is recovered on the brane with some corrections at high energies. Furthermore, there are corrections due to the gravitational interaction with the bulk space–time. Assuming a fine tuning between the brane tension and the bulk cosmological constant Λ_5 leads to the vanishing of the 4D cosmological constant on the brane. Neglecting contributions from the dark radiation term, this leads to the following Friedmann equation on the brane [5,6]

$$H^2 = \frac{8\pi}{3M_4^2}\rho\left(1+\frac{\rho}{2\lambda}\right), \tag{2.1}$$

where ρ is the energy density on the brane, λ the (positive) brane tension and M_4 the four-dimensional Planck mass. This is related to the five-dimensional Planck mass M_5 by

$$M_4 = \sqrt{\frac{3}{4\pi}}\left(\frac{M_5^2}{\sqrt{\lambda}}\right)M_5. \tag{2.2}$$

Assuming that matter on the brane is dominated by a scalar field ϕ, confined to the brane, with potential $V(\phi)$, its equation of motion is given by

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0. \tag{2.3}$$

In the slow roll approximation,

$$H^2 \simeq \frac{8\pi}{3M_4^2}V\left(1+\frac{V}{2\lambda}\right), \tag{2.4}$$

$$\dot{\phi} \simeq -\frac{V'}{3H}. \tag{2.5}$$

For $\lambda \to \infty$ the usual Friedmann equation is recovered. For $V \gg \lambda$ brane effects dominate.

Inflation takes place if the Hubble parameter satisfies $|\dot{H}| < H^2$. In a braneworld with matter given by a scalar field the condition for inflation yields to [5]

$$\dot{\phi}^2 - V + \frac{\dot{\phi}^2+2V}{8\lambda}\left(5\dot{\phi}^2 - 2V\right) < 0. \tag{2.6}$$

In the following, the potential of the inflaton will be taken to be of the form,

$$V = \frac{1}{2}m^2\phi^2. \tag{2.7}$$

Furthermore, for convenience, everything will be expressed in four-dimensional Planck units, hence $M_4 \equiv 1$.

3. Eternal inflation on the brane

The evolution of a scalar field in an inflationary universe is determined by two contributions. On the one hand, there is the classical rolling down of the scalar field down its potential. On the other hand, there are quantum fluctuations of the inflaton which become classical outside the horizon. This latter contribution can have either positive or negative sign. The classical rolling down is given by, $\Delta\phi \simeq \dot{\phi}\Delta t$, where in the slow roll approximation $\dot{\phi}$ is given by $\dot{\phi} \simeq -V'/3H$. The amplitude of a quantum fluctuation is given by $\delta\phi = H/2\pi$. In a typical time interval H^{-1}, e^3 new domains appear each containing an almost homogeneous field $\phi - \Delta\phi + \delta\phi$ [2]. There is a critical value ϕ_s for which for all $\phi \geqslant \phi_s$ quantum fluctuations dominate over the classical evolution towards smaller field values. This is the regime of self-reproduction of inflationary domains. ϕ_s is determined by [8]

$$\left.\frac{2\pi}{3}\frac{V'}{H^3}\right|_{\phi=\phi_s} = 1. \tag{3.1}$$

For field values $\phi \geqslant \phi_s$ there will be domains in which quantum jumps lead to an increase in the field value of the inflaton. In a small percentage of domains this will lead to the maximal field value at which inflation takes place. The upper boundary ϕ_{5D} in this braneworld model is determined by the five-dimensional Planck boundary. For energies higher than $V(\phi_{5D}) = M_5^4$ the scalar field becomes deconfined and flows off the brane into the bulk [6]. As in the four-dimensional case inflation will stop at this boundary. Here one might argue that inflation on the brane stops since the scalar field is flowing off the brane and thus four-dimensional inflation can no longer be sustained on the brane. Thus the upper boundary is given by

$$\phi_{5D} = \sqrt{2}\left(\frac{4\pi}{3}\right)^{1/3}\frac{\lambda^{1/3}}{m}. \tag{3.2}$$

In the low energy regime, $V \ll \lambda$ the Friedmann equation on the brane reduces to the standard one,

$$H^2 \simeq \frac{8\pi}{3}V. \tag{3.3}$$

It will be assumed that the whole period of inflation takes place in the low energy regime. The end of inflation ϕ_e is determined by the first two terms

in (2.6), $\dot{\phi}^2 = V$, which yields, $\phi_e = 1/\sqrt{6\pi}$. Furthermore, $\phi_e < \phi_{5D}$ implies $\lambda > (12\pi)^{-3/2}(3/4\pi)m^3$. The lower boundary for self-reproduction ϕ_s is given by

$$\phi_s = \left(\frac{3}{16\pi}\right)^{1/4} m^{-1/2}.$$

The requirement $V/2\lambda < 1$ at ϕ_{5D} implies $\lambda > 2\pi^2/9$. Eternal inflation takes place if $\phi_s < \phi_{5D}$ which yields the condition, $\lambda > (1/8)(3/4\pi)^{7/4}m^{3/2}$. In the standard inflationary scenario observations require that $m \simeq 10^{-13}$ GeV $= 10^{-6}M_4$. Thus for realistic values of m eternal inflation takes place on the brane in the low energy regime, as long as $\lambda > 2\pi^2/9$. However, note that inflation as well as self-reproduction takes place at field values larger than the four-dimensional Planck scale.

In the high energy limit, $V \gg \lambda$, the quadratic term in the Friedmann equation dominates,

$$H^2 \simeq \frac{8\pi}{3}\frac{V^2}{2\lambda}. \tag{3.4}$$

Inflation is assumed only to take place in the high energy regime. The end of inflation is determined by the last two terms in the condition (2.6), $5\dot{\phi}^2 = 2V$, implying, $\phi_e = (5/3\pi)^{1/4}\lambda^{1/4}m^{-1/2}$. The lower boundary for eternal inflation ϕ_s is given by

$$\phi_s = \left(\frac{12}{\pi}\right)^{1/10} \lambda^{3/10}m^{-4/5}.$$

Out of the two inequalities $\phi_e < \phi_{5D}$ and $\phi_s < \phi_{5D}$ it is found that the first one provides the stronger bound on the brane tension λ, namely, $\lambda > 8 \times 10^{-6}m^6$. An upper bound on λ is found by requiring that $V(\phi_e)/2\lambda > 1$, implying $\lambda < 3 \times 10^{-3}m^2$. As shown in [5] in the limit of strong brane corrections the COBE normalization of the curvature perturbations put a bound on m, which can be written in units of M_4 as, $m \sim 6 \times 10^{-5}\lambda^{1/6}$. This implies $\phi_e \sim 10^2\lambda^{1/6}$, $\phi_s \sim 3 \times 10^3\lambda^{1/6}$, and $\phi_{5D} \sim 4 \times 10^4\lambda^{1/6}$. Thus $\phi_e < \phi_s < \phi_{5D}$ and eternal inflation takes place for values of m derived from observations. As already pointed out in [5], the 5D Planck boundary ϕ_{5D} is below the 4D Planck scale, M_4. Thus eternal inflation takes place at field values below M_4.

For a certain range of parameters eternal inflation takes place inside the low energy and inside the high energy regime on the brane. In this case domains reproduce themselves. Since it was assumed that the inflationary period is either in the low or in the high energy regime the dynamics of each of them will be determined by the characteristics of the regime that they are originating from. However, it could also be considered that a domain starts in the low energy regime and then due to the process of stochastic inflation, field values in successive domains reach such high values that strong brane corrections become important. Thus, in this case the Friedmann equation on the brane changes from Eq. (3.3) to Eq. (3.4). In order for this to happen, one has to require that eternal inflation takes place in the low energy regime. Furthermore, the 5D Planck boundary has to be in the high energy regime. This implies that the brane tension has to satisfy, $\lambda < 2\pi^2/9$. In this case, out of a low energy domain, domains with the low energy characteristics emerge as well as those with the high energy dynamics. This picture is similar to the model proposed in [9] where the space–time dimension can change locally in chaotic eternal inflation. In this case, on the brane there are regions in which the dynamics are determined by the high energy Friedmann equation (3.4) and there are domains in which the Friedmann equation is the low energy one (3.3).

4. Stochastic description

The stochastic nature of the effect of the competition between the classical rolling down and the quantum perturbations is captured by a Fokker–Planck equation [10]. The (classical) field ϕ is performing a Brownian motion described by a Langevin equation, [1,2]

$$\frac{d}{dt}\phi = -\frac{V'(\phi)}{3H(\phi)} + \frac{H^{3/2}(\phi)}{2\pi}\xi(t), \tag{4.1}$$

where $\xi(t)$ describes the white noise due to the quantum fluctuations, which causes the Brownian motion of the classical field ϕ.

The probability distribution $P_c(\phi, t)$ determines the probability to find a given value of the field ϕ at a given time at a given point. This is the probability distribution over a comoving coordinate volume, i.e., over a physical volume at some initial moment of

inflation. $P_c(\phi,t)$ is determined by the equation [2],

$$\frac{\partial P_c}{\partial t} = \frac{\partial}{\partial\phi}\left[\frac{H^{3(1-\beta)}(\phi)}{8\pi^2}\frac{\partial}{\partial\phi}\big(H^{3\beta}(\phi)P_c\big) + \frac{V'}{3H(\phi)}P_c\right]. \tag{4.2}$$

The parameter β encodes an ambiguity in the derivation of this equation for systems for which the diffusion coefficient depends on ϕ. $\beta = 1$ corresponds to the Itô version of stochastic analysis and $\beta = 1/2$ to the Stratonovich version [10]. An exact stationary solution, for which $\partial P_c/\partial t = 0$, can be found for the Hubble parameter given by Eq. (2.4), namely,

$$P_c(\phi) = \left(\frac{8\pi}{3}\right)^{-3\beta/2} V^{-3\beta/2}\left[1+\frac{V}{2\lambda}\right]^{-3\beta/2} \times \left[1+\frac{2\lambda}{V}\right]^{-3/8\lambda} \exp\left[\frac{3}{8V}\frac{1+\frac{V}{\lambda}}{1+\frac{V}{2\lambda}}\right]. \tag{4.3}$$

In the low energy regime, $V \ll \lambda$, this reduces to the well-known expression [2]

$$P_c \sim V^{-3\beta/2}(\phi)\exp\left(\frac{3}{8V(\phi)}\right). \tag{4.4}$$

Considering the high energy regime, $V \gg \lambda$, the stationary distribution $P_c(\phi)$ approaches,

$$P_c(\phi) \simeq \left(\frac{4\pi}{3}\right)^{-3\beta/2}\left(\frac{V^2}{\lambda}\right)^{-3\beta/2}\exp\left[\frac{\lambda^2}{2V^3}\right]. \tag{4.5}$$

It is interesting to compare the expression for the stationary probability distribution (4.3) with the probability for creation of a braneworld from nothing. This is described by the de Sitter brane instanton [11]. The probability for creation of a universe in the Hartle–Hawking no-boundary proposal [12] is given by, $\mathcal{P} \sim \exp(-S_E)$, with S_E the Euclidean action. In the case of the creation of a braneworld containing just one de Sitter brane in an AdS bulk, the Euclidean action, in the notation used here, is given by [11],

$$S_E = -\frac{\pi}{H^2}\left(1+\frac{V}{\lambda}\right). \tag{4.6}$$

Using the expression for the Hubble parameter on the brane (2.4) the nucleation probability of a de Sitter braneworld is given by

$$\mathcal{P} \sim \exp\left(\frac{3}{8V}\frac{1+\frac{V}{\lambda}}{1+\frac{V}{2\lambda}}\right). \tag{4.7}$$

Thus comparing the exponentials in the stationary probability distribution P_c (4.3) and in the probability distribution $\mathcal{P}$ (4.7) it is found that they are exactly the same. Therefore, the same coincidence between these two probability distributions appears as is the case in standard four-dimensional inflation [2].

$P_c(\phi,t)$ is the probability distribution in a comoving volume, neglecting the expansion of the universe. The probability distribution $P_p(\phi,t)$ in a proper volume takes into account that during a small time interval dt the total number of points associated with the field ϕ is additionally increased by a factor $3H(\phi)\,dt$. Thus this leads to the following equation [1,8]

$$\frac{\partial P_p}{\partial t} = \frac{\partial}{\partial\phi}\left[\frac{H^{3(1-\beta)}(\phi)}{8\pi^2}\frac{\partial}{\partial\phi}\big(H^{3\beta}(\phi)P_p\big) + \frac{V'}{3H(\phi)}P_p\right] + 3H(\phi)P_p(\phi,t). \tag{4.8}$$

In order to solve this equation it is convenient to make the ansatz [2]

$$P_p(\phi,t) = \sum_{s=1}^{\infty} e^{\lambda_s t}\pi_s(\phi) \sim e^{\lambda_1 t}\pi_1(\phi) \quad \text{for } t\to\infty. \tag{4.9}$$

For large times $t\to\infty$ only the largest eigenvalue is kept. In the high energy regime brane effects are dominant and the deviations from standard 4D inflation are the largest. Therefore, in the following, the probability distribution P_p will be discussed for an inflationary period entirely in the high energy regime. Thus the Hubble parameter is given by (3.4). Together with this and (4.9) equation (4.8) yields to

$$\pi_1'' + \left[\frac{9}{\phi} + 24\frac{\lambda^2}{m^6}\frac{1}{\phi^7}\right]\pi_1' + \left[\frac{15}{\phi^2} + 72\pi\frac{\lambda}{m^4}\frac{1}{\phi^4} - 8\pi^2\left(\frac{\pi}{3}\right)^{-3/2}\lambda_1\frac{\lambda^{3/2}}{m^6}\frac{1}{\phi^6} - 24\frac{\lambda^2}{m^6}\frac{1}{\phi^8}\right]\pi_1 = 0, \tag{4.10}$$

where $\beta = 1/2$. The boundary conditions on P_p are equivalent to those in the standard four-dimensional case [2]. There is no diffusion below the end of inflation, which implies

$$\frac{\partial}{\partial\phi}\big(H^{3/2}(\phi)P_p\big)_{\phi_e} = 0,$$

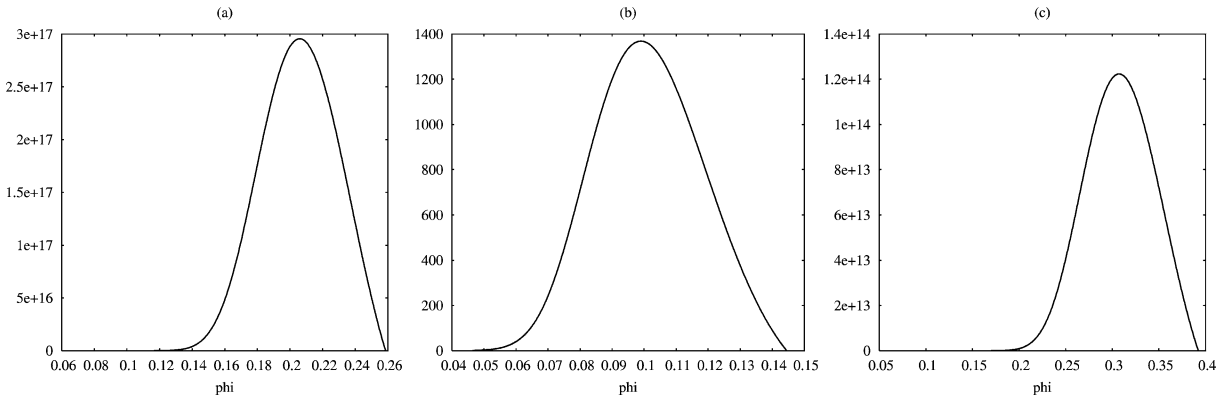

Fig. 1. (a) shows the numerical solution for $\pi_1(\phi)$ for $\lambda = 10^{-8}$, $m = 1.9 \times 10^{-2}$, $\lambda_1 = 0.417$. (b) shows $\pi_1(\phi)$ for $\lambda = 10^{-8}$, $m = 3.4 \times 10^{-2}$, $\lambda_1 = 0.316$. (c) shows $\pi_1(\phi)$ for $\lambda = 2 \times 10^{-7}$, $m = 3.4 \times 10^{-2}$, $\lambda_1 = 0.666$.

and inflation stops at the (5D) Planck boundary, thus $P_p(\phi_{5D}) = 0$. This imposes the following conditions on $\pi_1(\phi)$,

$$\pi_1'(\phi_e) = -\frac{3}{\phi_e}\pi_1(\phi_e), \qquad \pi_1(\phi_{5D}) = 0. \tag{4.11}$$

Numerical solutions have been plotted in Fig. 1. In Fig. 1, the probability distribution shows a maximum in all cases. For larger values of m at constant brane tension λ it is shifted towards smaller field values ϕ. For larger values of λ at constant m it is shifted towards higher field values ϕ, concentrated very close to the 5D Planck boundary. The eigenvalue λ_1 can be related to the fractal dimension of the universe, $d_{\rm fr}$, defined as $d_{\rm fr} = \lambda_1/H_{\rm max}$ [2,13], where $H_{\rm max}$ is the maximal Hubble parameter at the five-dimensional Planck boundary. The fractal dimension is motivated by the observation that at the Planck boundary the total volume of inflationary domains does not grow as e^3 during a time interval $H_{\rm max}^{-1}$ but only as $\exp(\lambda_1/H_{\rm max})$. Some domains will reach energies beyond the Planck scale and thus drop out of the total volume. On the brane in the high energy regime the maximal Hubble parameter at the 5D Planck boundary is given by

$$H_{\rm max} = \left(\frac{4\pi}{3}\right)^{7/6} \lambda^{1/6}.$$

Thus the numerical examples in Fig. 1 have fractal dimensions much below 3.

Before closing this section some comments on possible non-perturbative effects in eternal inflation will be made. Linde et al. argued in [8] that in eternal inflation fluctuations with amplitudes $\delta\phi$ much larger than $H/2\pi$ could occur. They calculated the probability distribution for $\delta\phi$ using proper time as time coordinate and found that a typical fluctuation could have an amplitude much larger than the standard value of $H/2\pi$.

However, as was pointed out by Vilenkin [14] (see also [15]) this result depends critically on the choice of time parametrization, namely, on the use of proper time. Using the scale factor as time parameter leads to the standard result. In this case the probability distribution for $\delta\phi$ yields a nonzero mean value $\langle\delta\phi\rangle$ which is much smaller than $H/2\pi$.

In the case of eternal inflation on the brane one is faced with the same problem of choice of time parametrization. As can be shown easily calculating the probability distribution for $\delta\phi$ using proper time as time coordinate following [8] favours an amplitude $\delta\phi$ much larger than $H/2\pi$.

However, calculating the probability distribution with the scale factor as time coordinate following [14] leads to $\langle\delta\phi\rangle/\sigma \ll 1$, with $\sigma = H/2\pi$, as is the case in standard 4D eternal inflation.

5. Conclusions

The stochastic approach to standard 4D inflation and its variations opened the way to a rich global structure of an inflating universe. Here the stochastic approach to inflation has been applied to a braneworld model, namely, to chaotic inflation on the brane. It

has been shown that eternal inflation takes place for a certain range of parameters, and in particular, for those satisfying observational bounds.

The competition between the evolution towards smaller field values due to classical dynamics and the evolution towards either even smaller or higher field values can be described as a Brownian motion. There exists a well defined procedure to obtain a Fokker–Planck equation determining the probability distribution to find a certain value of the scalar field at a given point in space–time. Furthermore, there are two types of probability distributions. Firstly, the probability distribution P_c in a given comoving volume. Secondly, the probability distribution P_p in a given physical volume, which takes into account the expansion of the universe. In standard 4D chaotic inflation, apart from some prefactors, the dominant behaviour is determined by an exponential function, which is exactly the square of the Hartle–Hawking no-boundary wavefunction of the universe. In the braneworld scenario discussed here, a similar result was found. Comparing the expression for P_c found in the stochastic approach to chaotic inflation on the brane with the de Sitter brane instanton for a one brane system as calculated in [11], apart from some prefactors, the same exponential function was found in the two cases.

The probability distribution in a given physical volume, P_p, was calculated numerically in the high energy regime where brane effects dominate. The results are similar to the ones in standard 4D inflation with the distribution concentrated near the 5D Planck boundary.

Finally, possible non-perturbative effects and the issue of determining probability distributions in eternal inflation on the brane were briefly discussed.

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