The Dirac Operator and the Principal Series for Complex Semisimple Lie Groups

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The Dirac operator plays a fundamental role in the geometric construction of the discrete series for semisimple Lie groups. We show that, at the level of K-theory, the Dirac operator also plays a central role in connection with the principal series for complex connected semisimple Lie groups. This proves the Connes–Kasparov conjecture for such groups.

An important problem in C*-algebra theory is to determine the structure of C*-algebras constructed from groups. For a connected Lie group G, we can form the homogeneous space G/H, where H is a maximal compact subgroup. Connes (see [4]) and Kasparov [14, 15] have conjectured that elements constructed from the twisted Dirac operators on G/H form a basis for the K-theory of the reduced C*-algebra C*(G).

The algebra C*(G) is defined as follows. We choose a left-invariant Haar measure on G, and form the Hilbert space L2(G). The left regular representation λ of L1(G) on L2(G) is given by

(λ(f))(h) = f * h,

where f ∈ L1(G), h ∈ L2(G) and * denotes convolution. The C*-algebra generated by the image of λ is the reduced C*-algebra C*(G).

Suppose H acts on the tangent space to G/H at H via a spin representation; the precise condition is given in Section 1. Then to each irreducible complex H-module V we may associate a twisted Dirac operator Dν on
$G/H$, whose "analytical index" Ind $D_\nu$ (as defined in Section 2) is an element of $K_* (C^*_r (G))$.

**Conjecture** (Connes-Kasparov). The map $V \mapsto \text{Ind } D_\nu$ induces an isomorphism of abelian groups

$$R(H) \to K_c (C^*_r (G)),$$

where $\mathcal{R}(H)$ is the complex representation ring of $H$, and $\varepsilon$ is the mod 2 dimension of $G/H$. Further, $K_{\varepsilon + 1} (C^*_r (G)) = 0$.

A minor alteration is required in the case where the action of $H$ is not spin.

The conjecture has been proved for amenable groups $G$ by Kasparov [14], for simply-connected solvable Lie groups by Connes [7] and Kasparov [13], and for nilpotent Lie groups by Rosenberg [20]. Kasparov proves in [14] that the map is a monomorphism in general. The conjecture forms a special case of a much wider conjecture of Baum and Connes [4].

In this paper we prove the conjecture for all connected complex semisimple Lie groups. The techniques used carry over without serious alteration to real semisimple Lie groups with only one conjugacy class of Cartan subgroups, for example $SO(2n + 1, 1)$ (and its quotient the group of isometries of hyperbolic $(2n + 1)$-space) and $SL(n, \mathbb{H})$, the special linear group of $n \times n$ matrices with entries in the quaternions $\mathbb{H}$.

Finally, we remark that the results of this paper can be interpreted in the homology theory of Connes [9].

The general structure of the paper is as follows. In Section 1 we give notation and basic background material on spin representations and the Dirac operator. Section 2 contains a brief description of $K$-theory for $C^*$-algebras, following Connes [8]. At the end of the section, we state the theorem as (2.1). The reduced dual $\hat{\mathcal{G}}_\tau$ of $G$ is studied in Section 3. In the case under consideration $\hat{\mathcal{G}}_\tau$ takes on a particularly simple geometric structure, by the results of Harish-Chandra and others. Using a well-known theorem of Dixmier-Douady [10], we prove in Section 4 that $K_* (C^*_r (G))$ is isomorphic to $K^* (\hat{\mathcal{G}}_\tau)$ (topological $K$-theory). In Section 5, we complete the proof of (2.1) by calculations of the square of the Dirac operator. Section 6 contains a proof of the corresponding result in the nonspin case.

1. **The Dirac Operator on $G/H$**

Let $G$ be a connected, complex semisimple Lie group with $\dim \mathbb{C} G = n$. Let $H$ be a maximal compact subgroup of $G$. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the corresponding Lie algebras. We denote the Killing form on $\mathfrak{g}$ by $B(\ , \ )$. Let $\mathfrak{p} \subseteq \mathfrak{g}$ be the orthogonal complement of $\mathfrak{h}$ under $B$. Then $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ is a Cartan decom-
position, with $B$ positive definite on $p$ and negative definite on $\mathfrak{h}$. We also have Iwasawa decompositions

$$g = \mathfrak{h} + \mathfrak{a} + \mathfrak{n},$$

$$G = HAN,$$

where $\mathfrak{a} \subseteq p$, and $m + \mathfrak{a}$ is a Cartan subalgebra of $g$, with $\text{Int}(m) = M$ a maximal torus in $H$. Let $\dim M = l$. Then $\dim A = l$. We use the Killing form to define a positive definite bilinear form $\langle \cdot, \cdot \rangle$ on $(m + \mathfrak{a})^*$. The subgroup $P = MAN$ is a minimal parabolic subgroup of $G$. We also have

$$n = \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha,$$

where the $\mathfrak{g}_\alpha$, $\alpha > 0$ are the positive root spaces with respect to this Cartan subalgebra.

The adjoint action of $H$ on $g$ preserves $p$, and in fact defines a homomorphism

$$\beta: H \to SO(p).$$

If $\beta$ lifts to a homomorphism

$$\bar{\beta}: H \to \text{Spin}(p),$$

then we say that $G$ is acceptable. Equivalently, using the fact that $m = i\alpha$, the group $G$ is acceptable if and only if $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$ is a weight of a representation of $H$. Here $\alpha$ varies over the positive roots of $G$ although in the real semisimple case, one would need to consider $\rho_{\text{nc}} = \frac{1}{2} \sum_{\alpha > 0} \alpha$ where $\alpha$ varies over the noncompact roots. Clearly all simply-connected $G$ are acceptable. Also, all the classical complex simple Lie groups are acceptable, except for $SO(2n + 1, \mathbb{C})$. If $G$ is not acceptable, then there exists a double cover $\tilde{G}$ of $G$ which is acceptable. We shall assume henceforth that $G$ is acceptable, and describe in Section 6 the modifications needed in the nonacceptable case.

Let $s: \text{Spin}(p) \to \text{Aut}(S)$ be the (complex) spin representation. If $\varepsilon = \dim(p) = 2m + 1$ is odd, then $\dim S = 2^m$, and $s$ is irreducible. If $\varepsilon = 2m$, then $s$ splits as the sum of two irreducible representations, $s^+$ and $s^-$, on $S^+$ and $S^-$, where $\dim S^+ = \dim S^- = 2^{m-1}$. Let

$$\chi = s \circ \bar{\beta}$$

be the associated spin representation of $H$, with $\chi^+$ and $\chi^-$ the half-spin representations $\chi^{\pm} = s^{\pm} \circ \bar{\beta}$ if they exist. We consider $S$, $S^+$ and $S^-$ as left $H$-modules under this action.
Associated to any left $H$-module $W$, there exists a vector bundle $\mathcal{W}$ on $G/H$ of the same dimension. Sections of $\mathcal{W}$ are given by functions $f: G \to W$ which satisfy

$$f(gh) = h^{-1} \cdot f(g), \quad h \in H.$$ 

This construction is functorial with respect to direct sums and tensor products.

We give here a brief account of the construction of the Dirac operator. The reader is referred to Atiyah and Singer [2] for details. The Dirac operator is defined on an oriented Riemannian $n$-manifold $X$ with spin structure, that is such that the $SO(n)$ bundle associated to the tangent bundle of $X$ lifts to a $Spin(n)$ bundle. In this case we have a bundle $\mathcal{S}$ on $X$ associated to the spin representation. Let $\mathcal{F}$ be any other complex vector bundle on $X$, let $\nabla$ denote the Riemannian connection on $\mathcal{S} \otimes \mathcal{F}$, and let $\mathcal{S}^*$ denote the dual of the tangent bundle. The twisted Dirac operator $D_\nu$ is defined as the composite

$$C^\infty(X, \mathcal{S} \otimes \mathcal{F}) \xrightarrow{\nabla} C^\infty(X, \mathcal{S} \otimes \mathcal{F} \otimes \mathcal{S}^*) \xrightarrow{c} C^\infty(X, \mathcal{S} \otimes \mathcal{F}) ,$$

where we identify $\mathcal{S}^*$ with $\mathcal{S}$ by the metric and $c$ denotes Clifford multiplication. In terms of an orthonormal basis $\{e_i\}$ for $\mathcal{S}$ at any point of $X$,

$$D_\nu s = \sum_{i=1}^n e_i(\nabla_i s),$$

where $e_i(\;\cdot\;)$ denotes Clifford multiplication by $e_i$, and $\nabla_i$ is the covariant derivative in the $e_i$ direction. $D_\nu$ can be regarded as a formally self-adjoint operator on $L^2$ sections.

If dim $X$ is even, then the Dirac operator naturally maps positive spinors to negative spinors.

$$D_\nu: L^2(X, \mathcal{S}^+ \otimes \mathcal{F}) \to L^2(X, \mathcal{S}^- \otimes \mathcal{F})$$

and vice versa. In the sequel, the Dirac operator on even-dimensional spaces will always mean this restricted operator.

In our case, $X = G/H$, and the bundle $\mathcal{S}$ is precisely the bundle $\mathcal{S}$ already constructed. The tangent space to $G/H$ at any point is canonically isomorphic to $\mathfrak{p}$, and the Riemannian connection coincides with the canonical $G$-invariant connection. To see this, note that the canonical connection on $G$ is torsion-free with holonomy group contained in $H$. 

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Now we let $V$ be an irreducible representation of $H$, with $\mathcal{V}$ the associated vector bundle on $G/H$. If we describe sections of $\mathcal{V} \otimes \mathcal{V}^*$ as maps $f: G \to S \otimes V$, then

$$D_V f = \sum_{i=1}^{n} (c(P_i) \otimes 1) P_i f,$$

where $\{P_i\}_{i=1}^{n}$ is an orthonormal basis for $\mathfrak{p}$, and $c(P_i)$ denotes Clifford multiplication by $P_i$. Here we identify $\mathfrak{p}^* \otimes \mathfrak{p}$ with $\mathfrak{p}$ via the Killing form. Again, if $n$ is even, we really want the operator $D_V$ restricted to sections of $\mathcal{V}^+ \otimes \mathcal{V}^*$ over $G/H$.

2. K-THEORY FOR C*-ALGEBRAS

We present here a brief outline of Kasparov's $K$-theory [13] for C*-algebras, suitably adapted for our purposes.

We start by recalling a few definitions. For details, see [8] or [13]. Let $B$ be a separable C*-algebra. Let $E$ be a complex vector space which is also a right $B$-module. That is, $A \cdot x = (1x)b = x(1b), Vx \in E, A \in B$. $E$ is a pre-Hilbert $B$-module if there exists a map $(,): E \times E \to B$ such that

1. $(x, y) = (y, x)^*$,
2. $(tx, y) = t(x, y)$,
3. $(x, x) > 0$; if $(x, x) = 0$, then $x = 0$.

Then $\|x\| = \|(x, x)\|^{1/2}$ defines a norm on $E$. If $E$ is complete with respect to this norm, we say that $E$ is a Hilbert $B$-module. We shall consider only modules $F$ which are separable. Given two Hilbert $B$-modules $E_1, E_2$, we define $\mathscr{L}(E_1, E_2)$ as the set of linear $B$-module maps $T: E_1 \to E_2$ such that there exists $T^*: E_2 \to E_1$ satisfying

$$(Tx, y) = (x, T^*y), \quad \forall x \in E_1, y \in E_2.$$ 

$T^*$ is called the adjoint of $T$, and is well-defined. Every $T \in \mathscr{L}(E_1, E_2)$ is bounded.

Let $E_1, E_2$ be Hilbert $B$-modules. For $x \in E_2, y \in E_1$, define $\theta_{x,y}(z) = x \cdot (y, z)$. Then $\theta_{x,y} \in \mathscr{L}(E_1, E_2)$. Let $\mathscr{H}(E_1, E_2)$ denote the closure of the linear span of the $\theta_{x,y}$. If $B = C$, then $\mathscr{L}(E_1, E_2)$ and $\mathscr{H}(E_1, E_2)$ are given...
by the bounded linear operators $E_1 \to E_2$ and the compact operators, respectively.

We can now define the $K$-groups of $B$, following the simplified approach of Connes [8]. Let $\mathcal{E}(B)$ denote the set of bounded quasi-isomorphisms $T: E_1 \to E_2$, where

(i) $E_1, E_2$ are Hilbert $B$-modules,

(ii) $T \in \mathcal{L}(E_1, E_2)$ is invertible modulo elements of $\mathcal{H}(E_1, E_1)$ and $\mathcal{H}(E_2, l_2)$.

**Definition.** An analytical index is a map $\text{Ind}: \mathcal{E}(B) \to A$, where $A$ is an abelian group, such that

(a) $\text{Ind}(T) = 0$ if $T$ is an isomorphism,

(b) $\text{Ind}(T_1 \oplus T_2) = \text{Ind}(T_1) + \text{Ind}(T_2)$,

(c) if $T_1$ and $T_2$ are homotopic, then $\text{Ind}(T_1) = \text{Ind}(T_2)$.

(b) and (c) together imply

(d) $\text{Ind}(T_1 \circ T_2) = \text{Ind}(T_1) + \text{Ind}(T_2)$.

$K_0(B)$ is defined as the image of a universal analytical index. In other words, given any analytical index $\text{Ind}: \mathcal{E}(B) \to A$, there exists a group homomorphism $\phi: K_0(B) \to A$ such that

$$K_0(B) \xrightarrow{\text{Ind}} A$$

commutes. If $B = \mathbb{C}$, then $\mathcal{E}(B)$ is just the space of Fredholm operators and $\text{Ind}$ is the usual index. If $B = C(X)$, the continuous functions vanishing at infinity on a locally compact Hausdorff space $X$, then $K_0(B)$ is the usual $K$-theory of $X$ with compact supports, and $\text{Ind}$ is the index of a family of Fredholm operators parametrized by $X$. The index of a family may be nontrivial even if the restriction to any point is an operator with zero index.

Give an unbounded operator $D: E_1 \to E_2$, one defines

$$\text{Ind}(D) = \text{Ind}(D(1 + D^*D)^{-1/2}),$$

by considering $D$ on appropriate Sobolev spaces. In the sequel, it will usually be convenient to use unbounded operators. A recent paper by Baaj and Julg [3] gives details of the Kasparov construction using unbounded operators.

For $p > 0$, we define $C_p$ as the Clifford algebra on $p$ generators $e_1, \ldots, e_p$ satisfying $e_i^2 = 1, e_i e_j = -e_j e_i, i \neq j$. Let $C_p^c = C_p \otimes \mathbb{C}$. Now define $K_p(B) =$
The Bott generator in $K_n(C(\mathbb{R}^n))$ is constructed as follows using the Fourier Transform of the Dirac operator on $\mathbb{R}^n$. (The interested reader should note that this construction is precisely Theorem 2.1 below in the case $G$ is the additive group $\mathbb{R}^n$. The correspondence between the Bott element and the Dirac operator is at the heart of the Atiyah–Singer Index Theorem.)

If $n$ is even, let $S$ be the irreducible complex Clifford module of $C_n$. Then $S$ is naturally $\mathbb{Z}_2$-graded; $S = S^+ \oplus S^-$. We set $E_1 = C(\mathbb{R}^n, S^+)$, $E_2 = C(\mathbb{R}^n, S^-)$ and $T$ the operator

$$ (Tf)(x) = c(x) \cdot f(x), \quad x \in \mathbb{R}^n, f \in E_1, $$

where $c(x)$ is Clifford multiplication by $x$. If $n$ is odd, then $S$ does not split, but the construction is similar. The operator $T$ is precisely the Fourier Transform of the Dirac operator constructed in Section 1. To see this, note that the Fourier Transform of $\partial/\partial x_i$ is multiplication by $ix_i$. However, we have here that $e_i^3 = +1$, as opposed to $e_i^3 = -1$ in Section 1.

The Dirac operator as constructed in Section 1 is an operator on the $C^\infty(G)$-module $L^2(G/H, \mathcal{F} \otimes \mathcal{F}')$. We prove here that it has an analytical index in $K_*(C^*_p(G))$. The Dirac operator has been shown by Atiyah-Schmid [1], Schmid [23], and Parthasarathy [19] to be of fundamental importance in the determination of the discrete series representations of real semisimple Lie groups. The aim of this paper is to show the corresponding importance at the level of $K$-theory for the principal series of complex semisimple Lie groups, thereby proving the Connes–Kasparov conjecture in this case.

**Theorem 2.1.** Let $G$ be an acceptable, connected, complex semisimple Lie group, and $H$ its maximal compact subgroup. Let $l = \text{rank } H$. Let $R(H)$ denote the free abelian group generated by the isomorphism classes of irreducible representations $V$ of $H$. The map $V \rightarrow \text{Ind } D_V$ determines an isomorphism of abelian groups

$$ R(H) \rightarrow K_*(C^*_p(G)). $$

Further, $K_{l+1}(C^*_p(G)) = 0$.

**3. The Structure of the Reduced Dual**

In order to prove (2.1) we need detailed information on the representation theory of $G$. In fact, the classes of irreducible unitary representations which occur in the Plancherel formula are all induced from unitary representations of the minimal parabolic subgroup chosen in Section 1.

The dual of $G$, denoted $\hat{G}$, is defined as the space of equivalence classes of irreducible unitary representations of $G$, equipped with the hull-kernel
topology. The reduced dual $\hat{G}$ is the support in $\hat{G}$ of the measure $\mu$ in the Abstract Plancherel Theorem

\[ L^2(G) \cong \int V_\pi \otimes V_\pi^* \, d\mu(\pi). \]

For connected complex semisimple Lie groups, the important representations for our purposes are the principal series representations. Given elements $\sigma$ of $\mathbb{Z}^l$ and $\lambda$ of $\mathbb{R}^l$, we construct unitary representations $\Sigma$ of $M$ and $\Lambda$ of $A$ by

\[ \Sigma(m) = e^{i\sigma(\log m)}, \]
\[ \Lambda(a) = e^{i\lambda(\log a)}. \]

The principal series representation is given by

\[ \pi_{\sigma, \lambda} = \text{Ind}_{MAN}^G(\Sigma \otimes \Lambda \otimes 1). \]

These representations may be realized on subspaces of $L^2(H)$ as follows. Let

\[ V_\pi = \{ f : G \to \mathbb{C} \text{ such that } f|_H \in L^2(H) \text{ and } \}
\[ f(\text{man}) = \sum_{m \in M, a \in A, n \in N, h \in H} \]
\[ \text{where } \Delta(a) = e^{i\hat{\rho}(\log a)}. \text{ (See Lipsman [17, p. 16].) Here } \hat{\rho} = \frac{1}{2} \sum_{a > 0} (\dim_{\mathbb{R}} g_a) a = 2\rho, \text{ and so } \Delta(a) = e^{2\rho(\log a)}. \]

The action of $G$ on $V_\pi$ is the obvious one:

\[ g f(g_1) = f(g^{-1}g_1). \]

**Lemma 3.1 (Wallach [21]).** $\pi_{\sigma, \lambda}$ is irreducible for all $\sigma, \lambda$.

$M$ is the centralizer of $a$ in $H$. Denote by $M^*$ the normalizer of $a$ in $H$. Then the group

\[ W = M^*/M \]

is called the Weyl group associated to $a$. Since $a = im$, we have $N_H(M) = M^*$ and so $W$ agrees with the usual Weyl group of $H$. By construction, $W$ acts on both $\hat{M} \cong \mathbb{Z}^l$ and $\hat{A} \cong \mathbb{R}^l$. The action of $W$ on $\hat{M}$ is the usual one, so that $\hat{M}/W = \hat{H}$. Similarly the action of $W$ on $\hat{A}$ is by reflections in hyperplanes.

**Lemma 3.2 (Bruhat [6]).** $\pi_{\sigma, \lambda} \cong \pi_{\sigma', \lambda'}$, if and only if there exists $w \in W$ such that $\sigma' = w\sigma$, $\lambda' = w\lambda$.

Let $\hat{G}_\mu$ denote the subset of $\hat{G}$ determined by the principal series in the relative hull-kernel topology. According to a basic result of Lipsman [18,
p. 411] the orbit space $(\tilde{M} \times \tilde{A})/W$ in its natural topology is homeomorphic to $\hat{G}_p$:

$$(\tilde{M} \times \tilde{A})/W \cong \hat{G}_p, \quad (\sigma, \lambda) \to \pi_{\sigma, \lambda}.$$ 

According to the Harish-Chandra Plancherel Theorem for complex semisimple groups [12],

$$\hat{G}_r = \hat{G}_p.$$ 

Therefore, we have

**Lemma 3.3.** $\hat{G}_r$ is homeomorphic to $(\tilde{M} \times \tilde{A})/W$.

Suppose $\sigma, \sigma_2, \ldots$ chosen, one in each $W$-orbit in $\tilde{A}$. Recall that $W$ acts on $\tilde{A}$, hence any stability subgroup of $W$ also acts on $\tilde{A}$.

**Definition.** Let $E_j$ be a fundamental domain in $\tilde{A}$ for the action of the stabilizer of $\sigma_j$.

We thus have a homeomorphism of $\hat{G}_r$ with the disjoint union

$$\hat{G}_r \cong \bigcup_j (\sigma_j, E_j) \subseteq \mathbb{Z}^l \times \mathbb{R}^l.$$ 

There are two cases.

(i) $\sigma_j$ is a regular weight. This means that its stabilizer $W_j$ is trivial. Hence $E_j = \tilde{A}$. Note that $\sigma$ is regular if and only if $\sigma = \tau + \rho$ for some positive weight $\tau$.

(ii) $\sigma_j$ is a singular weight. By Chevalley's Theorem [22, p. 14, volume I], the stabilizer $W_j$ is generated by the reflections which fix $\sigma_j$. These are reflections in hyperplanes. In fact the pair $(W_j, \tilde{A})$ satisfies the conditions required on p. 72 of Bourbaki [5], so, as proved on p. 75 of [5], the closure $\overline{C}$ of a chamber $C$ is a fundamental domain for the action of $W_j$ on $\tilde{A}$. Therefore $E_j \cong \overline{C}$, a closed simplicial cone with vertex at the origin of $\tilde{A}$, and has the topological type of a half-space in $\mathbb{R}^l$. Hence the $K$-theory of $E_j$ is trivial.

We may summarize this discussion as follows:

- (i) $\sigma_j$ regular $\Rightarrow K^*(E_j) = \mathbb{Z}$,
- (ii) $\sigma_j$ singular $\Rightarrow K^*(E_j) = 0$. 

EXAMPLE.

\[ G = \text{SL}(3, \mathbb{C}). \]
\[ H = \text{SU}(3), \]
\[ M \cong S^1 \times S^1 \subset \text{SU}(3), \]
\[ A \cong \mathbb{R}^2, \]
\[ \tilde{M} \times \tilde{A} \cong \mathbb{Z}^2 \times \mathbb{R}^2. \]

There are three cases:

(i) \( \sigma_j \) is a regular weight; \( W_j = 1; E_j = \mathbb{R}^2 \).

(ii) \( \sigma_j \) is a singular weight and \( W_j = \mathbb{Z}/2; E_j \) is the closed upper half-plane in \( \mathbb{R}^2 \).

(iii) \( \sigma_j = 0; W_j = W; E_j \) is the closed simplicial cone in \( \mathbb{R}^2 \) bounded by two infinite half-lines which intersect in the origin at angle \( \pi/3 \).

Let \( \tilde{\mathcal{C}} \) be the closed upper half-plane in \( \mathbb{R}^2 \), and let \( \tilde{C}_0 \) be the closed simplicial cone described in (iii). With the dominant weights of \( \text{SU}(3) \) in their standard configuration, the reduced dual is the following topological subspace of \( \mathbb{R}^4 \):

\[
\begin{array}{cccc}
\tilde{C} & \mathbb{R}^2 \\
\tilde{C} & \mathbb{R}^2 \\
\tilde{C} & \mathbb{R}^2 \\
\tilde{C}_0 & \tilde{C} & \tilde{C} & \tilde{C}
\end{array}
\]

The space \( \tilde{C}_0 \) is the space of the class-one principal series for \( \text{SL}(3, \mathbb{C}) \). Note that the singular weights, those in the walls of the fundamental dual Weyl chamber, do not contribute generators to the \( K \)-theory of the reduced dual. Each other weight contributes one generator to \( K^0 \) of the reduced dual.

4. THE REDUCED \( C^* \) ALGEBRA

The aim of this section is to prove that \( K_*(C^*_r(G)) \) is given precisely by the \( K \)-theory of the reduced dual \( \tilde{G}_r \). This will follow from a characterization by Dixmier–Douady [10] of certain \( C^* \)-algebras by their duals, once we have established some results about \( C^*_r(G) \) and \( \tilde{G}_r \). We remark that the isomorphism is described in a natural way.
It is important for our purposes to realize the members of the principal series on a fixed Hilbert space. On a fixed connected component of $\hat{G}$, this is easily done on a subspace $V_\lambda$ of $L^2(H)$.

$$V_\lambda = \{ f \in L^2(H) \mid f(hm) = \Sigma(m)^{-1} f(h), \forall m \in M \}.$$ 

The $G$-action on $V_\lambda$ for each $\lambda \in \mathbb{R}^l$ can easily be computed from the description of the principal series given in Section 3. We denote by $\mathcal{H}$ the $C^*$-algebra of compact operators on $V_\lambda$.

Let $\mathcal{A} = C^*_r(G)$. Then $\mathcal{A}$, the dual of $\mathcal{A}$, is given by $\hat{G}$. We need to know that the topologies coincide. Denote by $D$ the set of all minimal central functions in $L^*(H)$, i.e.,

$$D = \{ (\dim \delta) \chi_\delta : \delta \in \hat{H} \}, \quad \chi_\delta = \text{character of } \delta.$$

Since $C^*_c(G)$, the space of $C^\infty$ functions $G \to \mathbb{C}$ with compact support, is dense in $\mathcal{A}$, the linear span of $D \ast C^\infty_c(G) \ast D$ is dense in $\mathcal{A}$. We denote this linear span by $\mathcal{B}$, and note that $\mathcal{B} \subset C^*_c(G)$. Now $\mathcal{B}$ is a dense self-adjoint subalgebra of $\mathcal{A}$, all of whose elements are boundedly represented in $\mathcal{B}$.

Given $b \in \mathcal{B}$, there exists $n = n(b)$ such that rank $n(b) \leq n$ for all $\pi \subset \mathcal{A}$. Let $b \in \mathcal{B}$. Suppose $q_n \to q$ in $\mathcal{B}$. Let $\pi_n, \pi$ be the corresponding elements of $\hat{G}$; let $\theta_n, \theta$ be their characters, as locally integrable functions on $G$. As in Lipsman [18, p. 411], we have

$$\lim_{n \to \infty} \text{Tr} \pi_n(b) = \lim_{n \to \infty} \int b(g) \theta_n(g) \, dg$$

$$= \int b(g) \theta(g) \, dg$$

$$= \text{Tr} \pi(b).$$

We have used the Lebesgue Dominated Convergence Theorem: the dominating function is written out by Lipsman [18, p. 406] and depends critically on very precise results of Harish-Chandra [12]. Since $\pi \in \hat{G}$, we may conclude, by a classic result of Fell [11, p. 391], that $\pi_n \to \pi$ in the hull-kernel topology.

We may also conclude that the map

$$\hat{G} \to \mathbb{C}, \quad \pi \mapsto \text{Tr} \pi(b)$$

is continuous, for each $b$ in $\mathcal{B}$. Following Dixmier [10, p. 93], let $\mathcal{Q}$ be the set of $x$ in $\mathcal{A}^+$ such that the map $\pi \mapsto \text{Tr} \pi(x)$ is finite and continuous on $\mathcal{A}$. Let $[\mathcal{Q}]$ be the linear span of $\mathcal{Q}$. Then $[\mathcal{Q}]$ is a two-sided self-adjoint ideal in $\mathcal{A}$, and, if $x \in [\mathcal{Q}]$, the map $\pi \mapsto \text{Tr} \pi(x)$ is continuous on $\mathcal{A}$. Now
\( \mathcal{A} = \hat{G}_i = \mathcal{G}_p \), and it is clear that \([Q]\) contains \( \mathcal{D} \). Therefore \([Q]\) is dense in \( \mathcal{A} \). Therefore \( \mathcal{A} \) is, by definition, a continuous trace algebra.

We already know that \( \mathcal{A} \) is locally compact, Hausdorff, separable, finite-dimensional and has zero Cech cohomology in dimension 3. The Dixmier–Douady invariant \( \delta(\mathcal{A}) \) vanishes, and we have the Dixmier–Douady isomorphism of \( C^* \)-algebras [10, 10.9.6]:

**Proposition 4.1.** \( C^*_r(G) \cong C(\hat{G}_r; \mathcal{H}) \), the \( C^* \)-algebra of continuous maps from \( \hat{G}_r \) to \( \mathcal{H} \) which vanish at infinity.

**Remarks.** (1) The trace on \( C^*_r(G) \) above is given canonically by the Harish–Chandra character.

(2) The isomorphism above can be made explicit as a Fourier Transform in the following way. Let \( a \in C^*_r(G) \), and let \( \pi \) be an element of the principal series. Then the isomorphism above is given by

\[
\hat{a} \mapsto \hat{a}
\]

with

\[
\hat{a}(\pi) = \pi(a).
\]

Here \( \pi(a) \in \mathcal{H}(V_\pi) \), and \( V_\pi \) may be taken to be a constant subspace of \( L^2(\mathcal{H}) \) on each connected component of \( \hat{G}_r \).

5. **Proof of Theorem 2.1**

We have from Sections 3 and 4 that \( K_{i+1}(C^*_r(G)) = 0 \) and that \( K_i(C^*_r(G)) \) is a free abelian group with one generator for each principal series \( \pi_{\sigma, \lambda} \) such that \( \sigma = \tau + \rho \) for \( \tau \) the highest weight of some irreducible representation \( V \) of \( H \). Here \( \rho = \frac{1}{2} \sum_{i=1}^p \alpha_i \) is half the sum of the positive roots of \( G \). It remains to show that \( D_\nu \) gives precisely this element of \( K_i(C^*_r(G)) = K_i(C(\hat{G}_r); \mathcal{H}) \).

We apply the Plancherel theorem to get a direct integral decomposition of \( L^2(G/H, \mathcal{P} \otimes \mathcal{V}) \).

\[
L^2(G/H, \mathcal{P} \otimes \mathcal{V}) \cong \int V_\pi \otimes (V_\pi^* \otimes S \otimes V)^H \, d\mu(\pi).
\]

Here \( V_\pi^H \) refers to the \( H \)-invariant elements of \( W \).

**Lemma 5.1.** \( D_\nu \) decomposes as

\[
D_\nu = \int 1 \otimes \phi(\pi) \, d\mu(\pi)
\]
where

\[
\phi(\pi) \in \text{End}(V_\chi^* \otimes S \otimes V)^H, \quad \text{odd}
\]

\[
\in \text{Hom}((V_\chi^* \otimes S^+ \otimes V)^H, (V_\chi^* \otimes S^- \otimes V)^H), \quad \text{even.}
\]

**Proof.** $V_\chi$ is always irreducible. $D_\chi$ is a $G$-invariant operator. The result thus follows from Schur's lemma.

We need some information about the structure of $\chi$.

**Lemma 5.2.** Let $\alpha_1, \ldots, \alpha_p$ denote the positive roots of $G$. The highest weight of $\chi$ is given by $\rho = \frac{1}{2} \sum_{i=1}^{p} \alpha_i$. It has multiplicity $2^{\lfloor l/2 \rfloor}$, where $\lfloor l/2 \rfloor$ is the greatest integer not exceeding $l/2$. Furthermore, $\chi = 2^{\lfloor l/2 \rfloor} V_\rho$, where $V_\rho$ is irreducible with highest weight $\rho$.

**Proof.** It is well known that the weights of the spin representation $s$ of $\text{Spin}(p)$ are given by

\[
\frac{1}{2}(\pm \gamma_1 \pm \gamma_2 \pm \cdots \pm \gamma_m),
\]

where $\gamma_1, \ldots, \gamma_m$ are the positive weights of the standard representation of $\text{SO}(p)$. Here $\dim(p) = 2m$ or $2m + 1$. Since $\beta$ composed with the standard representation of $\text{SO}(p)$ is precisely the adjoint representation of $H$, we may assume that $\beta^* \gamma_i = \alpha_i$, $i = 1, \ldots, p$. Hence the highest weight of $\chi = s \circ \beta$ is given by

\[
\rho = \frac{1}{2}(\alpha_1 + \cdots + \alpha_p)
\]

with multiplicity $2^{m-p}$. Here $p$ is the number of positive roots of $G$, and so $p = \frac{1}{2}(n-l)$. Noting that $n-l$ is even and that $n = 2m$ or $2m + 1$, we get the required multiplicity. The fact that $\chi$ is a multiple of the unique irreducible representation with that highest weight follows by a comparison of dimensions.

**Lemma 5.3.**

\[
\dim_c(V_\chi^* \otimes S \otimes V)^H = 2^{\lfloor l/2 \rfloor} \quad \text{if} \quad \sigma = \tau + \rho
\]

\[
= 0 \quad \text{if} \quad \tau + \rho - \sigma \text{ is not in the fundamental Weyl chamber.}
\]

**Proof.** $V_\chi|_H = \text{Ind}_M^H \sigma$ by Frobenius reciprocity (see Lipsman [17, p. 48]). Hence as $H$-modules

\[
V_\chi|_H \cong W_\sigma \oplus W',
\]
where $W_\sigma$ is the irreducible $H$-module with highest weight $\sigma$, and $W'$ is the sum of irreducible $H$-modules with $\sigma$ as a weight which is not the highest weight.

Decomposing $S$ and $V$ into weight spaces,

$$S = 2^{1/2}C_\rho \oplus \text{lower weights},$$

$$V = C_\tau \oplus \text{lower weights}$$

$$\Rightarrow S \otimes V = 2^{1/2}C_{\rho + \tau} \oplus \text{lower weights}.$$ 

Hence

$$S \otimes V = 2^{1/2}W_{\rho + \tau} \oplus W'',$$

where $W''$ is a sum of irreducible $H$-modules each of whose highest weights is strictly less than $\rho + \tau$.

The result now follows by Schur's lemma.

Let $\Omega_G$ and $\Omega_H$ denote the Casimir elements of $G$ and $H$. In terms of bases $Y_1, \ldots, Y_n$ for $\mathfrak{h}$ and $X_1, \ldots, X_n$ for $\mathfrak{p}$,

$$\Omega_H = -Y_1^2 - \cdots - Y_n^2,$$

$$\Omega_G = -Y_1^2 - \cdots - Y_n^2 + X_1^2 + \cdots + X_n^2.$$ 

Note that the Casimir $\Omega_G$ lies in the centre of the universal enveloping algebra $U(\mathfrak{g})$.

**Lemma 5.4** (Parthasarathy [19, p. 13]).

$$D^2 = -\Omega_G + \langle \tau + 2\rho, \tau \rangle - \langle \rho + 2\rho, \rho \rangle.$$ 

**Remarks.** (1) Parthasarathy restricts his attention to the case of real groups such that $\text{rank } G = \text{rank } H$. However the proof of this result does not depend on that assumption. Lemma 5.2 above replaces Lemma 2.2 of [19].

(2) Twisted spinors are given by functions on $G$ with a particular variance with respect to $H$. The action of $\Omega_G$ on $C_c^\infty(G)$ extends naturally to twisted spinors.

We are now in a position to calculate $\phi^2$ in terms of $\sigma$ and $\lambda$. Given that $\Omega_G$ is in the centre of the universal enveloping algebra $U(\mathfrak{g})$, it can be chosen to act on either the right or left on $L^2(G)$, and similarly on twisted spinors. We choose it to act on the left, as right invariant vector fields, so that it acts infinitesimally on $V_\sigma$ under the decomposition of the Plancherel theorem. Following Warner [22, p. 168], we choose a basis for $\mathfrak{g}$ as follows. Let $\{H_1, \ldots, H_i\}$ and $\{H_{i+1}, \ldots, H_{2i}\}$ be orthonormal bases for $\mathfrak{m}$ and $\mathfrak{a}$, respec-
tively. Choose, for each positive root $\alpha$, elements $X_\alpha$ and $X_{-\alpha}$ of $g_\alpha$ and $g_{-\alpha}$ such that $B(X_\alpha, X_{-\alpha}) = 1$. Then $X_\alpha$ and $iX_\alpha$ form a basis for $g_\alpha$. Further, 

$$B(iX_\alpha, iX_{-\alpha}) = -1 \quad \text{and} \quad B(X_\alpha, iX_\alpha) = B(X_\alpha, X_\alpha) = 0.$$ 

Thus 

$$\Omega_g = -\sum_{i=1}^l H_i^2 + \sum_{i=l+1}^{2l} H_i^2 + 2 \sum_{\alpha > 0} (X_\alpha X_{-\alpha} + X_{-\alpha} X_\alpha)$$ 

$$= -\sum_{i=1}^l H_i^2 + \sum_{i=l+1}^{2l} H_i^2 + 2 \sum_{\alpha > 0} [X_\alpha, X_{-\alpha}] + Y$$ 

for some element $Y$ of $U(g)\mathfrak{n}$.

**Lemma 5.5.** $\Omega_g$ acts as multiplication by the scalar 

$$\Omega_g = \langle \sigma, \sigma \rangle + \langle i + 2\rho, i\lambda + 2\rho \rangle - 2\langle i\lambda + 2\rho, 2\rho \rangle.$$

**Proof.** $\Omega_g$ is central in $U(g)$, and so it acts as multiplication by a scalar on $\mathfrak{v}_\alpha$. Using distributions centered at 1, we calculate $\Omega_g f$ for $f \in \mathfrak{v}_\alpha$ simply by the infinitesimal action at 1.

$$Yf(1) = 0 \quad \text{for all } Y \in \mathfrak{u},$$

$$H_i f(1) = -i\sigma(H_i)f(1) \quad \text{for } i = 1, \ldots, l,$$

$$H_i f(1) = -(i\lambda + 2\rho)(H_i)f(1) \quad \text{for } i = l + 1, \ldots, 2l.$$

Also, $[X_\alpha, X_{-\alpha}] = H_\alpha$, the image of $\alpha$ under the natural correspondence $\alpha^* \to a$ given by $\langle , \rangle$. Thus 

$$[X_\alpha, X_{-\alpha}] f(1) = -(i\lambda + 2\rho)(H_\alpha)f(1).$$

Hence $\Omega_g f = (\langle \sigma, \sigma \rangle + \langle i\lambda + 2\rho, i\lambda + 2\rho \rangle - 2\langle i\lambda + 2\rho, 2\rho \rangle)f$ as required.

Combining Lemma 5.5 with Lemma 5.4, we see that $D^2$, and thus $\phi^2$, acts as the scalar $\square \in \mathbb{R}$ where 

$$\square = -\langle \sigma, \sigma \rangle - \langle i\lambda + 2\rho, i\lambda + 2\rho \rangle + 2\langle i\lambda + 2\rho, 2\rho \rangle$$ 

$$+ \langle \tau + 2\rho, \tau \rangle - \langle \rho + 2\rho, \rho \rangle.$$ 

Thus if we partially order weights by $\tau_1 \leq \tau_2$ if $\tau_2 - \tau_1$ is in the fundamental Weyl chamber,

$$\square = \langle \lambda, \lambda \rangle \quad \text{if } \sigma = \tau + \rho,$$

$$\square > 0 \forall \lambda \quad \text{if } \sigma < \tau + \rho.$$ 

(5.6)
Now we look at the $K$-theory elements we have constructed. If $\sigma < \tau + \rho$, then $D_\lambda$ is invertible for all $\lambda$, and thus defines the zero element in the $K$-theory of the $\sigma$-principal series. We see, then, by Lemma 5.3, that the $K$-theory element determined by $D_\nu$ is in the image of the map

$$K^*(FG) \to K^*(C_r^*(G))$$

defined by the $\tau + \rho$ principal series. It remains to prove that it is the image of the Bott element. By Lemma 5.1, $D_\nu$ acting on $L^2(G/H, \mathcal{S} \otimes \mathcal{S})$ gives the same $K$-theory element as $\phi$ acting on $(V^*_\chi \otimes \mathcal{S} \otimes V)^H$. We observe now that $\phi$ depends linearly on $\lambda$, and so by (5.6) $\phi$ gives a nontrivial Clifford algebra representation. Since the dimension (5.3) is the smallest possible, we see that $\phi$ gives precisely the generator required. Alternatively, one notes by Lemma 5.3 and (5.6) that $\text{Ind}(D^2)$ is precisely twice the Bott generator. Since $\text{Ir} \ d(D^2) = 2 \text{Ind}(D)$, the result follows.

6. The Nonacceptable Case

If we drop the assumption that $G$ be acceptable, then there exists a double cover $\pi: \tilde{G} \to G$, with $\pi^{-1}(H) = \tilde{H}$, and a map

$$\tilde{\beta}: \tilde{H} \to \text{Spin}(\mathfrak{p})$$

covering $\beta$. If $G$ is acceptable, then the covering is trivial. If $1 \neq u \in \pi^{-1}(1)$, then $u$ is central in $\tilde{H}$. Hence $\varepsilon(u) = \pm I$ for every irreducible representation of $\tilde{H}$. Thus $R(\tilde{H})$ splits as a direct sum

$$R(\tilde{H}) = R(\tilde{H})^+ \oplus R(\tilde{H})^-,$$

where $\varepsilon \in R(\tilde{H})^\pm$ as $\varepsilon(u) = \pm I$. If $G$ is acceptable, $R(\tilde{H})^+ \cong R(\tilde{H})^- \cong R(H)$. In general we have

**Theorem 6.1.** The result (2.1) holds for any connected complex semisimple Lie group with $R(H)$ replaced by $R(\tilde{H})^\pm$.

**Proof:** Apply Theorem 2.1 to $\tilde{G}$. A principal series representation of $\tilde{G}$ gives one of $G$ if and only if the restriction to $\tilde{H}$ gives a representation of $H$. Thus if $V$ is an irreducible representation of $\tilde{H}$, $D_\nu$ gives an element of $K^*(C_r^*(\tilde{G}))$ if and only if the maximal weight $\tau$ of $V$ is such that $\tau + \rho$ is a weight of $H$. Since $\rho$ is a weight of $H$ if and only if $G$ is acceptable, the result follows.
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Note added in proof. The diagram at the end of Section 3 is infinitely extended to the right. The Connes–Kasparov conjecture is a special case of Conjecture 1 in Kasparov [24].

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