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ALMOST COMPLEX STRUCTURES ON EIGHT- AND TEN-DIMENSIONAL MANIFOLDS†

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§0. INTRODUCTION

LET M be a $2n$-dimensional manifold. A manifold will always be assumed to be smooth, closed, connected and oriented. The dimension of M, where it needs to be specified, will be given by a superscript; hence M^{2n}. M is said to have an almost complex structure (acs.) if there exists a complex n-plane bundle ω on M whose underlying real $2n$-plane bundle is isomorphic to τ the tangent bundle of M. One would like to determine necessary and sufficient conditions for the existence of such structures in terms of the cohomology ring and characteristic classes of M. This has been done only for manifolds of dimension <8 by Wu [18] and Ehresmann [9], although various necessary conditions in other cases have been found (for example [9], [12], [14], [8]). For dimension 8 we prove the following theorem.

THEOREM 1. *A manifold M^8 has an almost complex structure iff*

(a) $w_8(M) \in \mathrm{Sq}^2 H^6(M; Z)$

and there exist cohomology classes $u \in H^2(M; Z)$ and $v \in H^6(M; Z)$ such that

(b) $\rho u = w_2(M)$, $\rho v = w_6(M)$

(c) $2\chi(M) + u \cdot v \equiv 0 \pmod 4$

(d) $8\chi(M) = 4p_2(M) + 8u \cdot v - u^4 + 2u^2 \cdot p_1(M) - p_1(M)^2$.

(*In case* $\mathrm{Sq}^2 H^6(M; Z) = 0$, *then* (*a*) *and* (b) *imply* (c) [14].)

Here ρ denotes mod 2 reduction and Sq^i the Steenrod squares. The w_i, p_i, χ denote as usual the Stiefel–Whitney, Pontrjagin and Euler classes.

We take the viewpoint of Thomas [15]. A manifold M^{2n} is said to have a stable acs. (also known in the literature as a weakly acs.) if the tangent bundle is stably isomorphic to the underlying real bundle of some complex vector bundle over M. In the case $n = 4$, condition (a) together with the condition $\delta w_2(M) = 0$ determines when such a structure exists. (δ is the Bockstein coboundary operator from mod 2 to integer coefficients.) This will be shown in §1 as a consequence of a more general theorem which determines the obstruction in the top dimension to M^{8n} having a stable acs. Now if ω is a stable acs. on M^{2n}, then it is known that ω is an actual acs. iff $c_n(\omega) = \chi(M)$ (c_i denotes the ith Chern

† This paper consists of part of the author's thesis submitted at Berkeley (1967).

class). In the 8-dimensional case Thomas (Theorem 1.8 [14]) shows that this is equivalent to condition (d) if u and v are $c_1(\omega)$ and $c_3(\omega)$ respectively. Thus in §2 we show that all possible pairs (u, v) which can occur as the first and third Chern classes of some stable acs. on M are determined by conditions (b) and (c).

We have not been able to give a complete solution in the 10-dimensional case. However, Thomas shows [14] that if M^{10} satisfies the following conditions; $H_1(M; Z_2) = 0$, $\delta w_2(M) = 0$ and $w_4(M) = 0$ then M has a stable acs. In §3, we compute the possible Chern classes for such structures and prove the following.

THEOREM 2. *Suppose M^{10} is a manifold such that $H_1(M; Z_2) = 0$ and $w_4(M) = 0$. Then M has an almost complex structure iff there exist classes $u \in H^2(M; Z)$ and $v \in H^6(M; Z)$ such that*

(a) *if $w_2(M) \neq 0$, then $\rho u = w_2(M)$, $Sq^2 v = 0$ and*
$8\chi(M) \equiv 4u \cdot p_2(M) + 15u^5 + 16u^2 \cdot v - u \cdot p_1(M)^2 + 2u^3 \cdot p_1(M) \pmod{192}$.

(b) *if $w_2(M) = 0$, then $Sq^2 v = 0$ and*
$\chi(M) \equiv 24\pi^1(M) + 4u^2 \cdot v - u^3 \cdot p_1(M) + 2u^5 \pmod{48}$.

Here $\pi^1(M)$ is the first KO-Pontrjagin number of M (see [2]), an invariant of the spin cobordism type of M taking the value of either 0 or 1.

I would like to thank Professor E. Thomas for his help with the preparation of this paper. The essential inspiration came from his work in [14].

§1. EXISTENCE OF STABLE ACS.

Suppose $x \in H^*(M)$ is a cohomology class. The notation $[x]$ will mean the number obtained by taking the Kronecker index of x with the fundamental class of M. We recall the differentiable Riemann–Roch theorem of [3]. The $\mathfrak{A}$-class [5] associates to every real vector bundle a certain rational polynomial in its Pontrjagin classes.

LEMMA 1.1 (Atiyah–Hirzebruch). *If M^{2n} is a manifold, $x \in H^2(M; Z)$ a class such that $\rho x = w_2(M)$ and η a complex vector bundle over M then $[\mathrm{ch}(\eta) \cdot e^{x/2}. \hat{\mathfrak{A}}(M)]$ is an integer* (ch *denotes the Chern character*).

Recall that if ξ is a real vector bundle, the Pontrjagin character of ξ, ph(ξ), is defined to be the Chern character of the complexification of ξ. We prove:

THEOREM 1.2. *Suppose a manifold M^{8n} has a stable acs. over $M - *$. Then M has a stable acs. iff either*

(a) $Sq^2 H^{8n-2}(M; Z) \neq 0$ *or*

(b) $Sq^2 H^{8n-2}(M; Z) = 0$ *and* $[\mathrm{ph}(M) \cdot \hat{\mathfrak{A}}(M)] \equiv 0 \pmod 2$.

Proof. Let ω be a stable acs. over $M - *$. ω may be considered as a vector bundle over M since there is no obstruction to extending. Let ω_R be the underlying real bundle of ω. By hypothesis $\tau - \omega_R$ is trivial over $M - *$ and by Kervaire [11] the obstruction to it being trivial over M is $z = \mathrm{ph}_{2n}(\tau - \omega_R)$ an integral class.

We need the following fact which may be deduced from [1]. If X is a CW-complex of dimension $2n + 1$ and (u, v) is a pair of classes in $H^{2n-2}(X; Z) \times H^{2n}(X; Z)$ such that

$\mathrm{Sq}^2 u = \rho v$, then there exists a complex vector bundle η on X, trivial over the $2n-3$ skeleton, having $\mathrm{ch}_{n-1}(\eta) = u$ and $2\mathrm{ch}_n(\eta) = v$.

(a) $\mathrm{Sq}^2 H^{8n-2}(M; Z) \neq 0$. By hypothesis there exists a class $u \in H^{8n-2}(M; Z)$ such that $\mathrm{Sq}^2 u = \rho z$. Choose η as above for the pair (u, z). Then $\omega + \eta$ is a stable acs. for $M - *$ because η_{R} is trivial over $M - *$ (there is no obstruction to triviality in dimensions $8n-2$ or $8n-1$ for real vector bundles). Moreover $\mathrm{ph}_{2n}(\tau - \omega_{\mathrm{R}} - \eta_{\mathrm{R}}) = z - 2\mathrm{ch}_{4n}(\eta) = 0$. Thus $\tau = (\omega + \eta)_{\mathrm{R}}$ and M has a stable acs.

(b) $\mathrm{Sq}^2 H^{8n-2}(M; Z) = 0$. Let $u \in H^{8n-2}(M; Z)$. Then $\mathrm{Sq}^2 u = u \cdot w_2(M)$ [17]. Thus $w_2(M)$ annihilates $H^{8n-2}(M; Z)$ under the operation of cupproduct. By Massey [13], this implies that there exists a torsion class $t \in H^2(M; Z)$ such that $\rho t = w_2(M)$. Suppose ω is a stable acs. on M. Then by Lemma 1.1 $[\mathrm{ch}(\omega) \cdot e^{t/2} \cdot \hat{\mathfrak{A}}(M)] = [\mathrm{ch}(\omega) \cdot \hat{\mathfrak{A}}(M)]$ is an integer. But $\mathrm{ph}_k(M) = 2\mathrm{ch}_{2k}(\omega)$ and $\hat{\mathfrak{A}}(M)$ has components only in dimensions congruent to 0 (mod 4). Thus

$$[\mathrm{ph}(M) \cdot \hat{\mathfrak{A}}(M)] = [2\mathrm{ch}(\omega) \cdot \hat{\mathfrak{A}}(M)] \equiv 0 \pmod 2.$$

Conversely suppose $[\mathrm{ph}(M) \cdot \mathfrak{A}(M)] \equiv 0 \pmod 2$. If ω is the stable acs, over $M - *$ described above then

$$z = \mathrm{ph}_{2n}(M) - 2\mathrm{ch}_{4n}(\omega) = \mathrm{ph}(M) \cdot \mathfrak{A}(M) - 2\mathrm{ch}(\omega) \cdot \mathfrak{A}(M).$$

Thus $z = 2u$ by Lemma 1.1. We may choose η to be trivial over the $8n-1$ skeleton and so that $\mathrm{ch}_{4n}(\eta) = u$. Then $\omega + \eta$ is a stable acs. over M since $\mathrm{ph}_{2n}(\tau - (\omega + \eta)_{\mathrm{R}}) = z - 2u = 0$.

We specialize to the 8-dimensional case. Massey [12] shows that $M^8 - *$ has a stable acs. iff $\delta w_2(M) = \delta w_6(M) = 0$. In [13] he shows that $\delta w_6(M) = 0$.

COROLLARY 1.3. *M^8 has a stable acs. iff $\delta w_2(M) = 0$ and $w_8(M) \in \mathrm{Sq}^2 H^6(M; Z)$.*

Proof. We need only consider the case $\mathrm{Sq}^2 H^6(M; Z) = 0$. Let L be the complex line bundle over M such that $c_1(L) = t$ where t is the torsion class such that $\rho t = w_2(M)$. Then $\tau' = \tau + L$ is a spin bundle and spin characteristic classes q_1 and q_2 are defined for it (see [15]) such that

$$2q_1 = p_1(\tau + L_{\mathrm{R}}) = p_1(M) + t^2$$

$$2q_2 + q_1^2 = p_2(\tau + L_{\mathrm{R}}) = p_2(M) + t^2 \cdot p_1(M) = p_2(M).$$

We have

$$\mathfrak{A}_1(M) = -\tfrac{1}{24} p_1(M)$$

$$\mathrm{ph}_1(M) = p_1(M)$$

$$\mathrm{ph}_2(M) = -\tfrac{1}{12}(2p_2(M) - p_1(M)^2).$$

Moreover by Lemma 1.1 $[\hat{\mathfrak{A}}_2(M)]$ is an integer. Thus since $\mathrm{ph}_0(M) = 8$

$$[\mathrm{ph}(M) \cdot \mathfrak{A}(M)] \equiv \tfrac{1}{24}[p_1(M)^2 - 4p_2(M)] \pmod 2$$

$$\equiv \tfrac{1}{24}[4q_1^2 - 4q_1^2 - 8q_2] \pmod 2$$

$$\equiv \tfrac{1}{3}[q_2] \pmod 2.$$

But $\rho q_2 = w_8(\tau') = w_8(M) + t \cdot w_6(M) = w_8(M)$ so

$$[\mathrm{ph}(M) \cdot \hat{\mathfrak{U}}(M)] \equiv 0 \pmod 2 \text{ iff } w_8(M) = 0.$$

This result was first suggested to me by E. Thomas and D. W. Anderson.

§2. ENUMERATION OF STABLE ACS.

In this section we will determine which classes can appear as the Chern classes of some stable acs. on a 8-dimensional manifold.

LEMMA 2.1. *Let M be a manifold with a stable acs. and $u \in H^2(M; Z)$ a class such that $\rho u = w_2(M)$. Then there exists a stable acs. ω on M such that $c_1(\omega) = u$.*

Proof. Let ω' be a stable acs. on M. Then there exists a class $x \in H^2(M; Z)$ such that $u = c_1(\omega') + 2x$. Let L be the complex line bundle over M such that $c_1(L) = x$. Then if L^* is the conjugate bundle to L, $L - L^*$ has trivial underlying real bundle. Thus $\omega = \omega' + L - L^*$ is a stable acs. on M and $c_1(\omega) = u$.

To deal with the third Chern class we recall that Milnor has produced certain congruence relations which are satisfied by the Chern numbers of all manifolds with stable acs. In particular, in the 8-dimensional case the following is true [10]

$$2c_4(\omega) + c_1(\omega) \cdot c_3(\omega) \equiv 0 \pmod 4.$$

Since $\rho\chi(M) = \rho c_4(\omega) = w_8(M)$ this may be written as

$$2_x(\mathrm{M}) + \mathrm{c}_1(\omega) \cdot \mathrm{c}_3(\omega) \equiv 0 \pmod 4.$$

PROPOSITION 2.2. *Let M^8 be a manifold with a stable acs. Let (u, v) be a pair of classes in $H^2(M; Z) \times H^6(M; Z)$ such that $\rho u = w_2(M)$ and $\rho v = w_6(M)$. Then M has a stable acs. ω such that $c_1(\omega) = u$ and $c_3(\omega) = v$ iff $2\chi(M) + u \cdot v \equiv 0 \pmod 4$.*

Proof. Let ω' be a stable acs. of Lemma 2.1 such that $c_1(\omega') = u$. Then there exists $x \in H^6(M; Z)$ such that $v = c_3(\omega') + 2x$. We have

$$2\chi(M) + u \cdot c_3(\omega') \equiv 0 \pmod 4$$

$$2\chi(M) + u \cdot c_3(\omega') + 2u \cdot x \equiv 0 \pmod 4.$$

Thus $u \cdot x \equiv 0 \pmod 2$. Since $\rho u = w_2(M)$, this implies by [17] that $\mathrm{Sq}^2 x = 0$. The pair $(x, 0)$ is suitable then for the construction of a complex vector bundle η over M with $\mathrm{ch}_3(\eta) = x$, $\mathrm{ch}_4(\eta) = 0$ and η trivial over the 5 skeleton. Moreover η_{R} is trivial since it is trivial over the 7 skeleton and $\mathrm{ph}_2(\eta_{\mathrm{R}}) = 0$. Thus $\omega = \omega' + \eta$ is a stable acs. on M and $c_3(\omega) = c_3(\omega') + 2x = v$.

This proposition completes the proof of Theorem 1.

§3. THE TEN-DIMENSIONAL CASE

As stated in the introduction, we have not been able to completely extend the results of §1 and §2 in the 10-dimensional case. However, the following result has been obtained by Thomas [14].

THEOREM 3.1. *If M^{10} is a manifold such that $H_1(M; Z_2) = 0$ and $w_4(M) = 0$, then M has a stable acs. iff $\delta w_2(M) = 0$.*

This section will develop the analogue of §2 in this special case. We will need the following formula also to be found in [14]. If M is as in Theorem 3.1 and $x \in H^4(M; Z)$, then

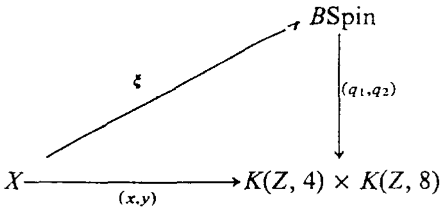
$$(3.1) \qquad \mathrm{Sq}^4 x = \mathrm{Sq}^2(x \cdot w_2(M)).$$

THEOREM 3.2. *Suppose M^{10} is a manifold such that $H_1(M; Z_2) = 0$ and $w_4(M) = 0$. Then if $u \in H^2(M; Z)$ is a class such that $\rho u = w_2(M)$, there exists a stable acs. ω on M such that $c_1(\omega) = u$ and $c_3(\omega) = 2u^3$.*

The key piece in the construction of ω will be obtained from the next proposition which will be proved first.

PROPOSITION 3.3. *Let X be a CW-complex of dimension <9 and (x, y) a pair of classes in $H^4(X; Z) \times H^8(X; Z)$. Then there exists a spin bundle ξ over X such that $q_1(\xi) = x$ and $q_2(\xi) = y$ iff $\mathscr{P}_3{}^1 x = -\rho_3(x^2 + y)$. ($\mathscr{P}_3{}^i$ denotes the mod 3 Steenrod reduced power, ρ_3 is mod 3 reduction.)*

Proof. A spin bundle ξ over X may be considered as a map $\xi: X \to B\mathrm{Spin}$ where BSpin is the classifying space. There are universal spin characteristic classes $(q_1, q_2) \in H^4 \times (B\mathrm{Spin}; Z) \times H^8(B\mathrm{Spin}; Z)$ and $q_1(\xi) = \xi^* q_1$ etc. Thus our problem may be considered as a homotopy lifting problem as the diagram indicates:



The only possible obstruction to lifting for complexes of dimension <9 is in $H^8(X; Z_3)$ and is in fact $\mathscr{P}_3{}^1 x + \rho_3(x^2 + y)$. This is a simple exercise in the theory of Postnikov resolutions as given in [16], using the fact that $\mathscr{P}_3{}^1 q_1 + \rho_3(q_1{}^2 + q_2) = 0$ in $H^8(B\mathrm{Spin}; Z_3)$.

Proof of Theorem 3.2. Let X be the 9 skeleton of M. Any stable complex structure for τ/X extends to a complex vector bundle over M, in fact to a stable acs. for M since any real vector bundle over M which is trivial over X has a stable complex structure trivial over X. The following construction takes place over X. Let L be the complex line bundle over X such that $c_1(L) = u$. If we consider L as a stable vector bundle, then L^3 is trivial over the 5 skeleton of X and since $\mathrm{ch}(L^3) = (e^u - 1)^3$ we see that $c_3(L^3) = 2u^3$ and $c_4(L^3) = -9u^4$.

Let $\tau' = \tau - (L + L^3)_{\mathbf{R}}$. τ' is a spin bundle and $w_4(\tau') = w_4(M) = 0$. Thus there exists a class $x \in H^4(X; Z)$ such that $2x = q_1(\tau')$ since $\rho q_1(\tau') = w_4(M) = 0$. Next we observe that by (3.1) $\mathrm{Sq}^2(u \cdot x) = \mathrm{Sq}^2(w_2(M) \cdot x) = \rho(-x^2)$ and so there exists a complex vector bundle η over X such that $\mathrm{ch}_3(\eta) = u \cdot x$ and $2\mathrm{ch}_4(\eta) = -x^2$. This may be written as $c_3(\eta) = 2u \cdot x$ and $c_4(\eta) = 3x^2$.

Let $\tau'' = \tau' - \eta_R$. A computation shows that $w_8(\tau'') = \rho x^2$. Thus there exists a class $y \in H^8(X; Z)$ such that $2y = q_2(\tau'') - x^2$. We can apply Prop. 3.3 to the pair (x, y) because

$$\mathcal{P}_3{}^1 x = -\mathcal{P}_3{}^1 q_1(\tau'') = \rho_3(q_1(\tau'')^2 + q_2(\tau''))$$
$$= \rho_3(x^2 + 2y + x^2) = -\rho_3(x^2 + y).$$

Let ξ be the real vector bundle such that $q_1(\xi) = x$ and $q_2(\xi) = y$. (ξ extends over X because $H^9(X; Z_2) = H^9(M; Z_2) = 0$.) Then $\xi' = \tau'' - 2\xi$ is a spin bundle and $q_1(\xi') = 0$, $q_2(\xi') = 0$. It follows from [11] that ξ' is 3 torsion over the 8 skeleton of X and hence is 3 torsion over X, again since $H^9(X; Z_2) = 0$. Let $\omega = L + L^3 + \eta + (\xi + 2\xi')_C$ where $(\ )_C$ denotes complexification. We have shown that ω is a stable complex structure for τ/X. Moreover $c_1(\omega) = u$ and

$$c_3(\omega) = c_3(L^3) + c_3(\eta) + c_1(L) \cdot c_2(\xi_C)$$
$$= 2u^3 + 2u \cdot x + u \cdot (-2x) = 2u^3.$$

COROLLARY 3.4. *Let (u, v) be a pair of classes in $H^2(M; Z) \times H^6(M; Z)$. Then there exists a stable acs. ω on M such that $c_1(\omega) = u$ and $c_3(\omega) = v$ iff $\rho u = w_2(M)$ and $v = 2u^3 + 2x$ where $\mathrm{Sq}^2 x = 0$.*

Proof. Suppose u, v and x are as above. Let ω' be the stable acs. of Theorem 3.2 such that $c_1(\omega') = u$ and $c_3(\omega') = 2u^3$. Since $\mathrm{Sq}^2 x = 0$ we can find a complex vector bundle η over X trivial over the 5 skeleton such that $\mathrm{ch}_3(\eta) = x$ and $\mathrm{ch}_4(\eta) = 0$. The obstruction to η_R being trivial is $\mathrm{ph}_2(\eta_R) = 2\mathrm{ch}_4(\eta) = 0$ in dimension 8 [11] and is 0 in dimension 9 since $H^9(M; Z_2) = 0$. Thus η_R is trivial and $\omega = \omega' + \eta$ is a stable complex structure for τ/X. Moreover $c_1(\omega) = u$ and $c_3(\omega) = 2u^3 + 2x = v$ so the required stable acs. for M is any extension of ω.

Conversely if ω is a stable acs. on M such that $c_1(\omega) = u$ and ω' is the stable acs. for which $c_1(\omega') = u$ and $c_3(\omega') = 2u^3$, then $\eta = \omega - \omega'$ has trivial underlying real bundle and $c_1(\eta) = 0$. This implies η is trivial over the 5 skeleton, $\mathrm{ch}_3(\eta) = x$ an integral class and $\mathrm{ch}_4(\eta) = 0$. Thus $\mathrm{Sq}^2 x = 0$ [1] and $c_3(\omega) = 2u^3 + 2x$.

We will use this to determine all possible values of the fifth Chern class of a stable acs. on M^{10}. As tools, we need the following two relations among the Chern numbers of 10-dimensional manifolds with stable acs. The first of these is the Todd genus. The second comes from observing that $[\mathrm{ch}(\omega - L) \cdot T(\omega^*)]$ must be an integer by the complex Riemann–Roch theorem. Here L is the complex line bundle such that $c_1(L) = c_1(\omega)$.

(3.2) (a) $-c_1 \cdot c_4 + c_3 \cdot c_1{}^2 + 3c_2{}^2 \cdot c_1 - c_2 \cdot c_1{}^3 \equiv 0 \pmod{1440}$

(b) $c_1 \cdot c_4 + c_5 \equiv 0 \pmod{24}$.

Proof of Theorem 2(*a*). We assume $w_2(M) \neq 0$. It is possible to construct complex vector bundles over M which are trivial over the 9 skeleton and such that c_5 is any desired multiple of 24. In this case the underlying real bundles of such complex vector bundles are all trivial. This is because if ξ is a real vector bundle trivial over the 9 skeleton then the obstruction to ξ being trivial is in $H^{10}(M; Z_2)$ and has indeterminacy at least

$\mathrm{Sq}^2 H^8(M; Z_2) = H^{10}(M; Z_2)$ so ξ is trivial. Thus (3.2)(b) determines all possible values of $c_5(\omega)$ for fixed $c_1(\omega)$ and $c_4(\omega)$. As in the 8-dimensional case (see Theorem 1(d))

$$8c_4(\omega) = 4p_2(M) + 8c_1(\omega) \cdot c_3(\omega) - c_1(\omega)^4 + 2c_1(\omega)^2 \cdot p_1(M) - p_1(M)^2.$$

Thus (3.2)(b) together with Corollary 3.4 determine all possible values of $c_5(\omega)$ and a simple substitution gives the formula of Theorem 2(a).

Proof of Theorem 2(*b*). We assume $w_2(M) = 0$. The possible values of $c_5(\omega)$ for fixed $c_1(\omega)$ and $c_4(\omega)$ are no longer determined by (3.2)(b) but are instead a coset of $48Z$. This is because there is a real vector bundle trivial over the 9 skeleton but not trivial over M, a fact which is a consequence of the reducibility of the Thom complex of the normal bundle and the Thom isomorphism in real K-theory. (In fact, the complete set of relations among Chern numbers does not determine $c_5(\omega)$.)

Let ω' be a stable acs. on M. We may write $\omega' = \omega + L - L^*$ where ω is a SU structure on M (i.e. ω is a stable acs. on M such that $c_1(\omega) = 0$) and L is the complex line bundle over M such that $2c_1(L) = c_1(\omega')$. Then by (3.2)(b); $c_5(\omega) = 24d$. We will prove at the end of this section that $[d] \equiv \pi^1(M)$ (mod 2). The total Chern class of ω is

$$c(\omega) = 1 - q_1(M) + 2v + q_2(M) + 24d$$

where $v \in H^6(M; Z)$ is a class such that $\mathrm{Sq}^2 v = 0$.

The total Chern class of ω' is thus

$$\begin{aligned} c(\omega') = 1 &+ 2u + (2u^2 - q_1(M)) + (2u^3 - 2u \cdot q_1(M) + 2v) \\ &+ (2u^4 - 2u^2 \cdot q_1(M) + 4u \cdot v + q_2(M)) \\ &+ (2u^5 - 2u^3 \cdot q_1(M) + 4u^2 \cdot v + 2u \cdot q_2(M) + 24d). \end{aligned}$$

Let (u, v) be a pair of classes in $H^2(M; Z) \times H^6(M; Z)$ such that $\mathrm{Sq}^2 v = 0$ and d a class in $H^{10}(M; Z)$ such that $[d] \equiv \pi^1(M)$ (mod 2). Then

$$\begin{aligned} \mathrm{Sq}^2(u^3 - u \cdot q_1(M) + v) &= \rho u^4 + \mathrm{Sq}^2(u \cdot w_4(M)) \\ &= \mathrm{Sq}^2(u^2 \cdot w_2(M)) \text{ by (3.1)} \\ &= 0 \end{aligned}$$

so by Theorem 3.2 and remarks above we can find ω' whose Chern classes are of the form above for this u, v and d. The proof will be complete therefore when we show that $u \cdot q_2(M) \equiv 0$ (mod 24) for all u. Substitute the Chern classes of ω' into (3.2)(a) to obtain

$$6u^5 - 10u^3 \cdot q_1(M) + 3q_1(M)^2 \cdot u - u \cdot q_2(M) \equiv 0 \pmod{720}$$

and into (3.2)(b) to obtain

$$6u^5 + 12u^2 \cdot v - 6u^3 \cdot q_1(M) + 4u \cdot q_2(M) \equiv 0 \pmod{24}.$$

It follows from the second relation that $u \cdot q_2(M) \equiv 0$ (mod 3). Subtract the second relation from the first to get

$$-4u^3 \cdot q_1(M) - 5u \cdot q_2(M) + 3u \cdot q_1(M)^2 - 12u^2 \cdot v \equiv 0 \pmod{24}.$$

$\rho q_1(M) = w_4(M) = 0$ so $-4u^3 \cdot q_1(M) \equiv 0$ (mod 8). Moreover $q_1(M)^2 = 4x^2$ some x and by (3.1) $\rho x^4 = \mathrm{Sq}^4 x = \mathrm{Sq}^2(x \cdot w_2(M)) = 0$. Thus $3u \cdot q_1(M)^2 \equiv 0 \pmod 8$. Finally $\rho(u^2 \cdot v) = \mathrm{Sq}^2 u \cdot v = \mathrm{Sq}^2(u \cdot v) + u \cdot \mathrm{Sq}^2 v = 0$ so $12u^2 \cdot v \equiv 0 \pmod 8$. Thus $u \cdot q_2(M) \equiv 0 \pmod 8$. There remains to prove that $[d] \equiv \pi^1(M) \pmod 2$.

PROPOSITION 3.5. *Suppose that M^{8n+2} is a manifold with a stable acs. ω such that $c_1(\omega) = 0$. Then $\pi^1(M)$ is defined and* $[\mathrm{ch}(\omega) \cdot \hat{\mathfrak{A}}(M)] \equiv \pi^1(M)$ *(mod 2).*

Proof. The normal bundle v of M may be considered to be a complex $4n + 4$ plane bundle. Then Thom isomorphisms ϕ_u, ϕ_0 are defined for the complex and real K-theory of M, in the latter case since v_{R} is a spin $8n + 8$ plane bundle. We show that the following diagram commutes

$$\begin{array}{ccc} \phi_u\colon KU(M) & \longrightarrow & \tilde{K}U(M^v) \\ \big\downarrow R & & \big\downarrow R \\ \phi_0\colon KO(M) & \longrightarrow & \tilde{K}O(M^v) \end{array}$$

The isomorphisms ϕ_u and ϕ_0 are given by the formulas $\phi_u(\eta) = \eta \cdot \phi_u(1)$ and $\phi_0(\xi) = \xi \cdot \phi_0(1)$. Products in complex and real K-theory are related by the formula $(\eta \cdot \xi_{\mathrm{C}})_{\mathrm{R}} = \eta_{\mathrm{R}} \cdot \xi$. Now Conner and Floyd show in [7] that $\phi_u(1) = \phi_0(1)_{\mathrm{C}}$. Thus

$$(\eta \cdot \phi_u(1))_{\mathrm{R}} = (\eta \cdot \phi_0(1)_{\mathrm{C}})_{\mathrm{R}} = \eta_{\mathrm{R}} \cdot \phi_0(1).$$

Let $g\colon S^{16n+10} \to M^v$ be the map which induces an isomorphism in $16n + 10$ dimensional cohomology. Then $\pi^1(M) = g^*\phi_0(\omega_{\mathrm{R}})$ so by the commutativity of the diagram above $\pi_1(M) = \rho g^*\phi_u(\omega)$. Let ϕ be the Thom isomorphism for v in ordinary cohomology. Then (p. 32 [6])

$$\mathrm{ch}(\phi_u(\omega)) = \phi(\mathrm{ch}(\omega) \cdot \hat{\mathfrak{A}}(M)) \text{ which implies}$$

$$\mathrm{ch}(g^*\phi_u(\omega)) = [\mathrm{ch}(\omega) \cdot \hat{\mathfrak{A}}(M)].$$

Since $\mathrm{ch}\colon \tilde{K}U(S^{2m}) \to H^{2m}(S^{2m}; Z)$ is an isomorphism the proposition follows.

This proves that $[d] \equiv \pi^1(M) \pmod 2$ for if ω is a SU-structure on M^{10} a calculation shows that $[\mathrm{ch}(\omega) \cdot \hat{\mathfrak{A}}(M)] = [d]$. There do exist spin manifolds for which $\pi^1(M) \neq 0$. However I do not know of an example for which $w_4(M) = 0$ as well.

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