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Penalization for variational inequalities

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ABSTRACT

This work illustrates a penalization mechanism, using the distance function as a tool, for providing necessary and sufficient conditions for the solutions of the nonsmooth variational inequality problems.

1. Introduction

Since it was introduced by Giannessi [5], the theory of variational inequalities has shown many applications in optimization problems and traffic equilibrium problems; see [6,7,14]. Some recent works in optimization theory have shown some relationships between the optimality conditions, the notion of a gap function, and the solutions of variational inequalities. The results established in [1,20] explicitly refer to a relationship between the gap function in optimization theory and a variational view. But it seems that to unify this view some further research is needed.

One way to solve the constrained optimization problems is to approximate the problem with a function which includes a penalty term; see, e.g., [2,3,13]. In this work, we seek a connection between a penalization mechanism and variational inequality problems. To this end, the nonsmooth variational inequality problems in Banach (Hilbert) spaces are dealt with, and a new penalization mechanism will be provided using the distance function in this regard. The established results give the relationships between the solutions of the considered variational inequalities and the minimizers of a penalty function which is defined in terms of the distance function. Section 2 contains some preliminaries; and Section 3 gives the main results of the work.

2. Generalized differentiation in Banach (Hilbert) spaces

The notion of the limiting subdifferential was first introduced by Mordukhovich [10]. A comprehensive theory of limiting normals and subgradients is presented in two books written by Mordukhovich [11,12] as well as a book by Clarke et al. [4]. In the first part of this section, we consider the proximal subgradients of a function which constitute a subclass of limiting subdifferentials studied in [4]. Let H be a real Hilbert space, and D be a nonempty subset of H. The metric projection of a point $u \in H$ to D is given by $\mathcal{M}_D(u) = \{x \in D : \|u - x\| \leq \|u - y\| \ \forall y \in D\}$. The proximal normal cone for D at $x \in D$ is given by

$$N_D^P(x) = \{d \in H : \exists(\lambda \geq 0, u \in H \setminus D) \text{ such that } x \in \mathcal{M}_D(u) \ \& \ d = \lambda(u - x)\}.$$

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$\zeta \in H$ is said to be a proximal subgradient of $f : S \subseteq H \longrightarrow \mathbb{R}$ at $x \in S$ if $(\zeta, -1) \in N^P_{epi f}(x, f(x))$, where $epi f = \{(x, z) : z \geq f(x)\} \subseteq H \times \mathbb{R}$. The set of all proximal subgradients of f at x is denoted by $\partial_P f(x)$. See [4,11,12,15,18,19] for details about this set and its applications.

One of the classes of functions whose set of limiting subdifferentials has been studied is the class of *lower semicontinuous* functions. Considering this class, the following results are obtained and known in nonsmooth analysis (see [4,11,12]).

Lemma 1. *Let $S \subseteq H$ be a convex set and $f : S \longrightarrow \mathbb{R}$ be a lower semicontinuous convex function. Then $\zeta \in \partial_P f(x)$ implies that*

$$f(y) \geq f(x) + \langle \zeta, y - x \rangle \quad \forall y \in S.$$

Dissimilar to the notion of proximal subgradients, the notion of generalized differentiability is developed in the realm of the Banach spaces. Let X be a real Banach space and consider $f : X \longrightarrow \mathbb{R}$. The Clarke generalized directional derivative of f at $\bar{x}$ in the direction d, denoted by $f^\circ(\bar{x}; d)$, is defined as

$$f^\circ(\bar{x}; d) = \limsup_{\substack{x \longrightarrow \bar{x} \\ t \downarrow 0}} (1/t)[f(x + td) - f(x)].$$

The Clarke generalized gradient (see [4,11,12,15,17]) of f at $\bar{x}$ is given by

$$\partial f(\bar{x}) = \{x^* \in X^* : f^\circ(\bar{x}; d) \geq \langle x^*, d \rangle, \ \forall d \in X\}.$$

One of the classes of functions whose Clarke generalized gradient is nonempty is the class of locally Lipschitz functions [4,11,12,17]. Considering this class, the following result is obtained and known in nonsmooth analysis (see [4,11,12]).

Lemma 2. *Let h be a real-valued function defined on a convex subset U of X. If h is convex on U and Lipschitz near x, then $\zeta \in \partial h(x)$ implies that*

$$h(y) \geq h(x) + \langle \zeta, y - x \rangle \quad \forall y \in U.$$

We close this section by definition of distance function. Let E be a nonempty subset of F, a real Banach (Hilbert) space. The distance function of E is a function, $d_E(.) : F \longrightarrow \mathbb{R}$, defined by

$$d_E(x) = \inf\{\|x - e\| : e \in E\}.$$

This function is a useful tool in the rest of the work.

3. Variational inequalities and penalization mechanism

Let S be a closed convex subset of H and $f : S \longrightarrow \mathbb{R}$ be a function. First we define the nonsmooth variational inequality problem in Hilbert spaces as follows [8,9,16]:

Definition 3. A variational inequality problem in Hilbert space (hereafter VIPH) is that of finding a point $y \in S \subseteq H$ such that there exists no $x \in S$ satisfying $\langle \zeta, x - y \rangle < 0$, for each $\zeta \in \partial_P f(y)$.

The following theorem is one of the main results of the work. This theorem provides a penalty function and obtains a relationship between the VIPH and the minimization of the provided penalty function.

Theorem 4. *Suppose that f is Lipschitz of rank K on an open set U that contains S. Assume that f is convex and $y \in S$ solves VIPH. Then the function $h_{\bar{K}}(x) = f(x) + \bar{K} d_S(x)$ attains its minimum over U at $x = y$, for each $\bar{K} \geq K$.*

Proof. If there exists an $x \in S$ such that $f(x) < f(y)$, then by Lemma 1 we have $\langle \zeta, x - y \rangle < 0$ for each $\zeta \in \partial_P f(y)$. Therefore y cannot be a solution to VIPH. This contradicts the assumption of the theorem, and hence we have $f(y) \leq f(x)$ for each $x \in S$.

Now consider an arbitrary $x \in U$ and $\epsilon > 0$. Let $s \in S$ be such that $\|x - s\| \leq d_S(x) + \epsilon$. Regarding the above result and the Lipschitz property, we have

$$\begin{aligned} f(y) &\leq f(s) \\ &\leq f(x) + K\|s - x\| \\ &\leq f(x) + \bar{K}\|s - x\| \\ &\leq f(x) + \bar{K} d_S(x) + \epsilon \bar{K}. \end{aligned}$$

Now by $\epsilon \longrightarrow 0^+$, we have

$$f(y) \leq f(x) + \bar{K} d_S(x) = h_{\bar{K}}(x),$$

for each $x \in U$. Since $y \in S$, then $h_{\bar{K}}(y) = f(y)$. Thus we have

$$h_{\bar{K}}(y) \leq h_{\bar{K}}(x)$$

for each $x \in U$, and the proof is complete. □

Definition 3 and Theorem 4 deal with the variational inequality problems in Hilbert spaces considering the notion of limiting subdifferentials. The following definition and theorem consider these problems in Banach spaces, considering the Clarke generalized gradient notion. In the rest of this section C is a nonempty subset of X, a real Banach space.

Definition 5. A variational inequality problem in Banach space (hereafter VIPB) is that of finding a point $y \in C \subseteq X$ such that there exists no $x \in C$ satisfying $\langle \zeta, x - y \rangle < 0$, for each $\zeta \in \partial f(y)$.

Theorem 6. *Suppose that f is Lipschitz of rank K on a set $Y \subseteq X$ that contains C.*

(i) *Assume that f is convex, y belongs to $C \subseteq Y$, and y solves VIPB. Then the function $h_{\bar{K}}(x) = f(x) + \bar{K} d_C(x)$ attains its minimum over Y at $x = y$, for each $\bar{K} \geq K$.*

(ii) *Conversely, if C is closed, $\bar{K} > K$ and $h_{\bar{K}}$ attains its minimum over Y at $x = y$, then $y \in C$. Furthermore if f is concave, then y solves VIPB.*

Proof. The proof of part (i) is similar to that of Theorem 4. To prove part (ii), by contradiction suppose that $\bar{K} > K$ and $y \notin C$ minimizes $h_{\bar{K}}$ over Y. Considering $\epsilon = \frac{\bar{K}}{K} - 1$, there exists an $s \in C$ such that $\|y - s\| < \frac{\bar{K}}{K} d_C(y)$. Since y minimizes $h_{\bar{K}}$ over Y and $s \in C \subseteq Y$, by the Lipschitz property we have

$$\begin{aligned} h_{\bar{K}}(y) &= f(y) + \bar{K} d_C(y) \\ &\leq h_{\bar{K}}(s) = f(s) \\ &\leq f(y) + K\|y - s\| \\ &< f(y) + \bar{K} d_C(y) \\ &= h_{\bar{K}}(y). \end{aligned}$$

Thus $h_{\bar{K}}(y) < h_{\bar{K}}(y)$, and this clear contradiction shows that $y \in C$.

Since y minimizes $h_{\bar{K}}$ over Y and $C \subseteq Y$, we have

$$h_{\bar{K}}(y) \leq h_{\bar{K}}(x),$$

for each $x \in C$. Also for each $x \in C$ we have $h_{\bar{K}}(x) = f(x)$. Thus

$$f(y) \leq f(x), \tag{$*$}$$

for each $x \in C$. By contradiction suppose that y cannot solve VIPB; then there exists an $\bar{x} \in C$ such that $\langle \zeta, \bar{x} - y \rangle < 0$, for each $\zeta \in \partial f(y)$. Since f is concave, $-f$ is convex. Therefore by Lemma 2 and considering the fact that $\partial(-f)(.) = -\partial f(.)$, we have

$$f(\bar{x}) - f(y) \leq \langle \zeta, \bar{x} - y \rangle < 0.$$

This contradicts $(*)$ and completes the proof. □

Theorem 6 provides an equivalence between the minimization of the function $h_{\bar{K}}$ and the VIPB. The results of this work can be updated for other types of differentiations; see [4,11,12] for details about different types of differentiation. Also, note that the proof of part (ii) of Theorem 6 is not valid when we consider the variational problem in Hilbert spaces using the proximal subgradients, because $\partial_P(-f)(.) = -\partial_P f(.)$ is necessarily not valid. Finally, it is clear that if we define VIPH considering the Clarke generalized gradient under Hilbert spaces, then both parts of Theorem 6 will still be valid for VIPH.

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